Non-Quantitative Modeling of Service-Oriented Architectures, Refactorings, and Performance

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Abstract. Service-oriented architecture has become a popular architectural model to design applications with. Once implemented and deployed, however, service-oriented architectures often deteriorate into hardly comprehensible spaghettis of dependencies among services, until the point where implementing new performance requirements has become prohibitively complex for non-experts. At this point, an architecture threatens business growth and architecture specialists need to be hired to take corrective measures. Such measures include architectural refactorings, which improve the performance of an architecture, while preserving its functional behavior. Even for architecture specialists, however, reasoning about refactorings constitutes a complex intellectual enterprise, currently undertaken manually. Moreover, predicting performance impact of a given refactoring is often nothing more than guesswork.
In this paper, we present a formalism of service-oriented architecture that supports reasoning about refactorings and predicting their performance impact, by non-quantitatively analyzing dependencies among services. The aim of this formalism is to provide a foundation for a tool that aids architecture specialists in their refactoring activities. We also present such a proof-of-concept tool.

1 Introduction

Background. *Service-orientation* [7,15,14] is a software engineering paradigm centered around the concept of loosely-coupled, reusable, autonomous software units called *services*. Key to service-orientation is the practice of *composing* small primitive services, with simple capabilities, into larger compound services, with complex capabilities. As service-orientation has become widespread, *service-oriented architecture* has become a popular architectural model. In this paper, as part of a research project with industrial partner First8, we present a formalism of service-oriented architecture to simplify service-oriented software engineering.

First8 is a software company that specializes in business-critical applications. Among other activities, First8 provides consultancy to improve clients' existing service-oriented architectures. For instance, clients ask First8 to *refactor* problematic architectures—architectures that have deteriorated into a hardly

comprehensible spaghetti of dependencies among services—that (are starting to) pose a threat to their business. Typically in such cases, clients need to implement new performance requirements to sustain business growth, but lack in-house expertise to make the necessary changes.

To improve a client's existing service-oriented architecture, the client and First8 first engage in an exploratory phase. In this phase, a high-level model of the architecture is made, and based on intuition and experience of the expert architects involved, candidate refactorings are proposed. After the exploratory phase, two activities remain, namely checking that the candidate refactorings:

1. preserve functional behavior (i.e., *preservation-checking*)
2. improve performance (i.e., *improvement-checking*)

Intentionally, and notwithstanding a substantial body of scientific literature (e.g., [1,3,6,9,17]), First8 architects carry out improvement-checking (i.e., activity (2)) using *non-quantitative* techniques. Although quantitative techniques are powerful in theory, experience at First8 suggests that their usage is prohibitively difficult for them *and for their customers* in practice. One issue is gathering the necessary measurements to instantiate a quantitative formal model with (e.g., arrival rates of requests). Another issue is that measurements are not only very implementation-specific but also very deployment-specific: changes in a service implementation or deployment can greatly impact performance and immediately render previous measurements—painfully obtained—obsolete.

Problem. The reasoning involved in preservation-checking (i.e., activity (1)) and improvement-checking (i.e., activities (2)) constitutes a complex intellectual enterprise, currently undertaken manually. To assist its architects with such reasoning, First8 aims to develop *computer-aided software engineering* (CASE) tools that can (semi)automatically reason about service-oriented architectures, through the use of formal methods.

To develop such CASE tools, a formalism of service-oriented architecture that supports reasoning about refactorings and performance, non-quantitatively, is needed. Moreover, such a formalism must be close to what architects and without training in formal methods can comfortably use. Such a formalism does, to our knowledge, not exist. This makes its development not only practically relevant but also a novel scientific challenge. In this paper, we present such a formalism.

Contribution. In Sect. 2, we present the core of our formalism: its syntax and its semantics. This formalism formalizes the informal diagrams that First8 architects and clients currently use as high-level architectural models. In Sect. 3, we present a refactoring framework and example refactorings. This refactoring framework, which extends the core of our formalism with a composition operation and a congruence relation, supports automation of preservation-checking (i.e., activity (1)). In Sect. 4, we present non-quantitative performance indicators to evaluate the effectiveness of refactorings. These performance indicators, which

complete our formalism, support automation of improvement-checking (i.e., activity (2)). In Sect. 5, we describe the current version of our proof-of-concept tool, founded on our formalism. In Sect. 6, we discuss related work. Section 7 concludes this paper.

Running example. Throughout this paper, to illustrate various elements of our formalism, we use a service-oriented webshop as a running example. The webshop consists of the following services.

First, there is a central database service, called *db*, that contains information about products and orders. Then, there is a checkout service, called *chkout*, where customers can order products. The checkout service uses the database service to read product information and to store order information. The checkout service also uses a pricing service, called *price*, for calculating prices (including additional fees and transport costs). The pricing service, in turn, reads product information from the database service to get product prices.

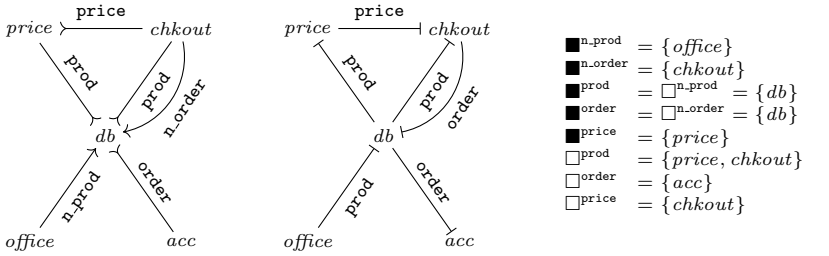
There is also an accounting service, called *acc*, that is responsible for checking if orders have been paid for. The accounting service uses the database service to read orders. Finally, there is a back-office service, called *office*, that is responsible for maintaining the product catalog of the webshop. The back-office service uses the database service to store (update) product information.

2 Modeling Architectures

Overview. In this section, we present the core of our formalism of service-oriented architecture. It consists of two main parts: a *syntax* to model the structure of architectures and a *semantics* to model their functional behavior in terms of information flows. The idea behind this division is that whenever an architecture is refactored by changing its syntax (i.e., structure), its semantics (i.e., functional behavior) should remain the same. For the example refactorings that we introduce in Sect. 3, we formally prove this property.

Syntax. In the exploratory phase, First8 architects sit together with their client to make a high-level model of the architecture-to-be-refactored. These models are graphical diagrams with services ("boxes") and calls between services ("arrows"). Services are annotated with the *types* of information they *produce* and *consume*. Calls come in two flavors: *pushes* and *pulls*. A push from service s_1 to service s_2 means that s_1 sends information to s_2, while a pull by s_1 from s_2 means that s_1 requests and receives information from s_2. A push is a "fire-and-forget" operation. This means that, at a conceptual level, service s_1 does not wait for an acknowledgment from service s_2 after the push.[5]

[5] In terms of the OSI transport layer, of course, TCP/IP packet(s) involved in a push *are* acknowledged (as part of the TCP/IP protocol), but this is at a lower level of abstraction than the pushes in terms of which we model architectures.



(a) Call graph (b) Flow graph (c) Producers+consumers

Fig. 1: Architecture model in the webshop running example

These informal diagrams are straightforwardly formalized as digraphs with type-labeled vertices and arcs. We call such digraphs *call graphs*. As experience at First8 suggests, call graphs provide an intuitive syntax for both architects and clients to work with. Let $\mathbb{S}$ denote the set of all services, ranged over by s, and let $\mathbb{T}$ denote the set of all types of information, ranged over by t.

Definition 1. *A call graph is a tuple* $(S, \blacksquare, \square, \longrightarrow, \longrightarrow\!\!\!\!\prec)$*, where:*

- $S \subseteq \mathbb{S}$ *denotes a set of services*
- $\blacksquare, \square : \mathbb{T} \to 2^S$ *denote sets of producers and consumers (per type)*
- $\longrightarrow, \longrightarrow\!\!\!\!\prec : \mathbb{T} \to 2^{S \times S}$ *denote typed push and pull relations*

$\mathbb{C}$all *denotes the set of all call graphs, ranged over by* γ*.*

In words, $s \in \blacksquare(t)$ and $s \in \square(t)$ mean that service s produces/consumes information of type t; in that case, s is called a *t-producer*/*t-consumer*. We write $\blacksquare^t$ and $\square^t$ instead of $\blacksquare(t)$ and $\square(t)$. In words, $(s_1, s_2) \in \longrightarrow(t)$ and $(s_1, s_2) \in \longrightarrow\!\!\!\!\prec(t)$ mean that service s_1 pushes/pulls information of type t to/from service s_2. We write $s_1 \xrightarrow{t} s_2$ and $s_1 \xrightarrow{t}\!\!\!\!\prec\ s_2$ instead of $(s_1, s_2) \in \longrightarrow(t)$ and $(s_1, s_2) \in \longrightarrow\!\!\!\!\prec(t)$.

We call a type t *relevant* in a call graph $\gamma = (S, \blacksquare, \square, \longrightarrow, \longrightarrow\!\!\!\!\prec)$ whenever $t \in \mathrm{Dom}(f)$ for some $f \in \{\blacksquare, \square, \longrightarrow, \longrightarrow\!\!\!\!\prec\}$; otherwise, we call t *irrelevant*. In practice, call graphs have only few relevant types; we omit irrelevant types from examples.[6]

Figure 1a shows a call graph for the webshop running example in Sect. 1. As in practice, we make a distinction between new order/product information (`n_prod` and `n_order`), produced by *chkout*/*office*, and existing order/product information (`prod` and `order`), "produced"—"owned" or "maintained" would perhaps be a better description in this case—by *db*. This distinction allows for more fine-grained reasoning about producership and consumership.

Call graphs model only the *direction* and the *initiative* of communication; they do not model quantitative aspects of communication (e.g., frequency) or transport characteristics (e.g., synchronous vs. asynchronous, reliable vs. lossy, unordered vs. order-preserving).

[6] Alternatively, we could have (i) added a set of relevant types $T \subseteq \mathbb{T}$ to Def. 1 and (ii) defined every $f \in \{\blacksquare, \square, \longrightarrow, \longrightarrow\!\!\!\!\prec\}$ as a function over domain T instead of $\mathbb{T}$.

Semantics. First8 architects reason about (preservation of) functional behavior in terms of (preservation of) information *flows* between services. At this level of abstraction, the semantics of an architecture can be expressed as a graph whose arcs represent information flows. We call such graphs *flow graphs*.[7]

Definition 2. *A flow graph is a tuple* $(S, \blacksquare, \square, \multimap)$*, where:*

- $S \subseteq \mathbb{S}$ *denotes a set of services*
- $\blacksquare, \square : \mathbb{T} \to 2^S$ *denote sets of producers and consumers (per type)*
- $\multimap : \mathbb{T} \to 2^{S \times S}$ *denotes a typed flow relation*

$\mathbb{F}\text{low}$ *denotes the set of all call graphs, ranged over by* φ*.*

In words, $(s_1, s_2) \in \multimap(t)$ means that information of type t flows from service s_1 to service s_2. We write $s_1 \overset{t}{\multimap} s_2$ instead of $(s_1, s_2) \in \multimap(t)$.

Figure 1b shows a flow graph for the webshop running example in Sect. 1.

Call graphs (i.e., structure, i.e., syntax) and flow graphs (i.e., functional behavior, i.e., semantics) are related by an *interpretation function*. Let R^{-1} denote the inverse of a binary relation R.

Definition 3. $[\![\cdot]\!] : \mathbb{C}\text{all} \to \mathbb{F}\text{low}$ *denotes the function defined by the following equation:*

$$[\![(S, \blacksquare, \square, \longrightarrow, \multimap)]\!] = \left(S, \blacksquare, \square, \left\{ t \mapsto \overset{t}{\longrightarrow} \cup \left(\overset{t}{\multimap}\right)^{-1} \;\middle|\; t \in \mathbb{T} \right\}\right)$$

The interpretation of the call graph in Fig. 1a is the flow graph in Fig. 1b.

3 Modeling Refactorings

Overview. The syntax (i.e., call graphs) and semantics (i.e., flow graphs) in Sect. 2 formalize the informal diagrams that First8 architects and clients currently use as high-level architectural models. Our next step, in this section, is to extend this core of our formalism with a refactoring framework, to support automation of preservation-checking (i.e., activity (1) in Sect. 1).

To refactor an architecture, First8 architects replace a part of an architecture (the "substituted part") for an equivalent part (the "substitute part") such that all information flows are preserved and no spurious new ones are introduced. To formally model such refactorings, we introduce a formal framework consisting of a binary *composition operation* on call/flow graphs, denoted by $\oplus$, and an *equivalence relation* on call/flows graphs, denoted by $\sim$.

The idea is that a full architecture (e.g., call graph γ) can be represented as the composition of the substituted part (e.g., call graph γ_1) and the remaining

[7] In this paper, we present only the "tip" of a "semantics iceberg" for call graphs. Below the surface, more advanced and detailed semantics can be associated with call graphs, such as a structural operational semantics (where every step corresponds to an information flow between two services). Such semantics can be seen as refinements of flow graphs. In this paper, however, this level of detail is unnecessary.

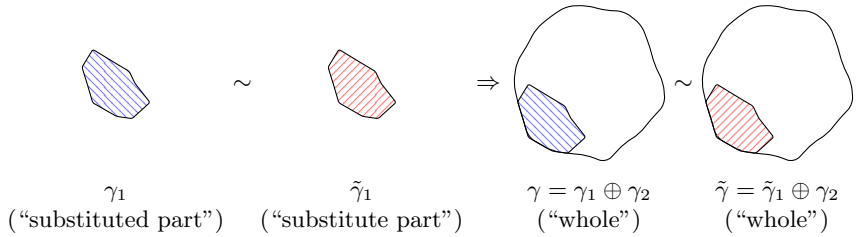


Fig. 2: Refactoring framework

part (e.g., call graph γ_2, such that $\gamma = \gamma_1 \oplus \gamma_2$). Refactoring, then, amounts to replacing the substituted part with an equivalent substitute part (e.g., call graph $\tilde{\gamma}_1$, such that $\gamma_1 \sim \tilde{\gamma}_1$). To guarantee that substitution of equivalent parts yields equivalent "wholes" (i.e., that $\gamma_1 \sim \tilde{\gamma}_1$ implies $\gamma_1 \oplus \gamma_2 \sim \tilde{\gamma}_1 \oplus \gamma_2$), our equivalence relation needs to be a *congruence relation*; shortly, we prove this. Figure 2 graphically shows the idea behind our refactoring framework.

Composition. We start with defining the composition operation. For functions $f_1, f_2 : X \to 2^Y$, let $f_1 \uplus f_2$ denote the pointwise union of f_1 and f_2.[8]

Definition 4. $\oplus : (\mathbb{C}\text{all} \times \mathbb{C}\text{all} \to \mathbb{C}\text{all}) \cup (\mathbb{F}\text{low} \times \mathbb{F}\text{low} \to \mathbb{F}\text{low} \times \mathbb{F}\text{low})$ *denotes the function defined by the following equations:*

$$\begin{aligned} \gamma_1 \oplus \gamma_2 &= (S_1 \cup S_2, \blacksquare_1 \uplus \blacksquare_2, \square_1 \uplus \square_2, \longrightarrow_1 \uplus \longrightarrow_2, \longrightarrow\!\!\!\!\prec_1 \uplus \longrightarrow\!\!\!\!\prec_2) \\ \varphi_1 \oplus \varphi_2 &= (S_1 \cup S_2, \blacksquare_1 \uplus \blacksquare_2, \square_1 \uplus \square_2, \longrightarrow\!\!\!\!\shortmid_1 \uplus \longrightarrow\!\!\!\!\shortmid_2) \end{aligned}$$

where $\gamma_i = (S_i, \blacksquare_i, \square_i, \longrightarrow_i, \longrightarrow\!\!\!\!\prec_i)$ *and* $\varphi_i = (S_i, \blacksquare_i, \square_i, \longrightarrow\!\!\!\!\shortmid_i)$ *for* $i \in \{1, 2\}$.

Note that when composing call/flow graphs, some of the services (i.e., vertices) are typically shared between the operands, whereas calls (i.e., arcs) are not.

The following theorem states that the interpretation function $[\![\cdot]\!]$ (Def. 3) is a homomorphism for the composition operation $\oplus$ (Def. 4).

Theorem 1. $[\![\gamma_1 \oplus \gamma_2]\!] = [\![\gamma_1]\!] \oplus [\![\gamma_2]\!]$

Equivalence. Intuitively, architectures are equivalent if they induce "the same" information flows. To formally define such equivalence, we draw inspiration from concurrency theory (e.g., [12,13]). In concurrency theory, a *simulation relation* is a binary relation on the state spaces of two concurrent processes P_1 and P_2, modeled as labeled transition systems. If a pair of states (q_1, q_2) is in a simulation relation, intuitively, P_2 in q_2 can perform at least the same actions as P_1 in q_1 (i.e., P_2 can "mimic" P_1). Thus, simulation is an inherently asymmetric concept; mathematically, it gives rise to a *preorder* on processes. If two processes can simulate each other under the same simulation relation, we call that

[8] Formally: $f_1 \uplus f_2 = \{x \mapsto f_1(x) \cup f_2(x) \mid x \in X\}$

simulation relation a *bisimulation relation* instead. Thus, bisimulation is an inherently symmetric concept; mathematically, it gives rise to an *equivalence* on processes. Below, instead of defining simulation and bisimulation on states in labeled transitions systems of processes, we define it on services in flow graphs of architectures. Let $\mathrm{Dom}(R)$ denote the domain of binary relation R, and let R^+ denote its transitive closure.

Definition 5. $\preceq \subseteq \mathbb{F}\mathrm{low} \times \mathbb{F}\mathrm{low} \times (1 + \mathbb{S}^2)$ *denotes the smallest relation induced by the following rules:*

$$\frac{\exists R \,.\, \varphi \preceq_R \tilde{\varphi}}{\varphi \preceq \tilde{\varphi}} \qquad \frac{\begin{array}{l} \forall t, s, s' \,.\, \left[s \xrightarrow{t}_{\mathtt{I}}{}^{+}\ s' \ \Rightarrow\ \exists \tilde{s}.\tilde{s}' \,.\, \left[\tilde{s} \xrightarrow{\sim t}_{\mathtt{I}}{}^{+}\ \tilde{s}' \ \wedge\ s\ R\ \tilde{s} \ \wedge\ s'\ R\ \tilde{s}'\right]\right] \\ \wedge\ \forall t, s \,.\, \left[s \in \blacksquare^t \ \Rightarrow\ \exists \tilde{s} \,.\, \left[\tilde{s} \in \tilde{\blacksquare}^t \ \wedge\ s\ R\ \tilde{s}\right]\right] \\ \wedge\ \forall t, s \,.\, \left[s \in \square^t \ \Rightarrow\ \exists \tilde{s} \,.\, \left[\tilde{s} \in \tilde{\square}^t \ \wedge\ s\ R\ \tilde{s}\right]\right] \\ \wedge\ R \subseteq S \times \tilde{S} \ \wedge\ S \subseteq \mathrm{Dom}(R) \end{array}}{\varphi \preceq_R \tilde{\varphi}}$$

where $\varphi = (S, \blacksquare, \square, \xrightarrow{}_{\mathtt{I}})$ *and* $\tilde{\varphi} = (\tilde{S}, \tilde{\blacksquare}, \tilde{\square}, \xrightarrow{\sim}_{\mathtt{I}})$.

In words, a flow graph φ is simulated by a flow graph $\tilde{\varphi}$ whenever there exists a (left-total) simulation relation R on the services in φ and $\tilde{\varphi}$ such that every information flow between services in φ can be mimicked with a corresponding information flow between services in $\tilde{\varphi}$, where "mimicked" is defined in terms of reachability.[9] At our current level of abstraction, thus, the specific path that information flows along is unimportant; only reachability matters.

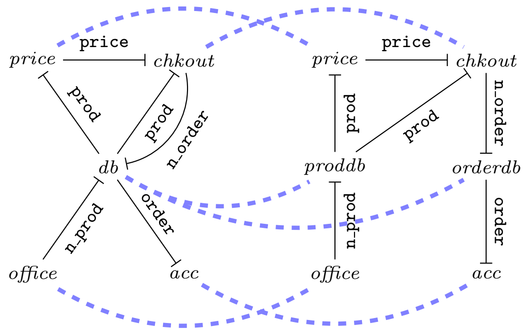
Definition 6. $\sim \subseteq \mathbb{F}\mathrm{low} \times \mathbb{F}\mathrm{low} \times (1 + \mathbb{S}^2)$ *denotes the smallest relation induced by the following rules:*

$$\frac{\exists R \,.\, \varphi \sim_R \tilde{\varphi}}{\varphi \sim \tilde{\varphi}} \qquad \frac{\varphi \preceq_R \tilde{\varphi} \ \wedge\ \tilde{\varphi} \preceq_{R^{-1}} \varphi}{\varphi \sim_R \tilde{\varphi}}$$

Figure 3 shows two equivalent flow graphs. The left flow graph φ is the same as the one in Fig. 1b. The right flow graph $\tilde{\varphi}$ models an architecture in which the database service is split into two services: one that stores only product information, and one that stores only order information. To see that φ is simulated by $\tilde{\varphi}$, note that `prod` information flows from *db* to *chkout* and *price* in φ, while `prod` information flows from *proddb* to *chkout* and *price* in $\tilde{\varphi}$. Thus, *db* in the left flow graph is (partially) simulated by *proddb* in the right flow graph. Similarly, with respect to `order` information flows, *db* in φ is (partially) simulated by *orderdb* in $\tilde{\varphi}$. Consequently, *db* in φ is simulated by the combination of *proddb* and *orderdb* in $\tilde{\varphi}$. In the same way, we can also derive that $\tilde{\varphi}$ is simulated by φ.

The following theorem states that equivalence relation $\sim$ (Def. 6) is a congruence relation for the composition operation $\oplus$ (Def. 4). This means that any part of an architecture can safely be substituted for an equivalent part. To prove

[9] This is where our approach differs significantly from concurrency theory, where the branching structure of the underlying graphs also plays a major role.



(a) Flow graphs φ (left) and $\tilde{\varphi}$ (right)

$$
\begin{aligned}
\blacksquare^{\texttt{n\_prod}} &= \tilde{\blacksquare}^{\texttt{prod}} &&= \{\mathit{office}\}\\
\blacksquare^{\texttt{n\_order}} &= \tilde{\blacksquare}^{\texttt{order}} &&= \{\mathit{chkout}\}\\
\blacksquare^{\texttt{price}} &= \tilde{\blacksquare}^{\texttt{price}} &&= \{\mathit{price}\}\\
\square^{\texttt{prod}} &= \tilde{\square}^{\texttt{prod}} &&= \{\mathit{price}, \mathit{chkout}\}\\
\square^{\texttt{order}} &= \tilde{\square}^{\texttt{order}} &&= \{\mathit{acc}\}\\
\square^{\texttt{price}} &= \tilde{\square}^{\texttt{price}} &&= \{\mathit{chkout}\}\\
\\
\blacksquare^{\texttt{prod}} &= \square^{\texttt{n\_prod}} &&= \{\mathit{db}\}\\
\blacksquare^{\texttt{order}} &= \square^{\texttt{n\_order}} &&= \{\mathit{db}\}\\
\\
\tilde{\blacksquare}^{\texttt{prod}} &= \tilde{\square}^{\texttt{n\_prod}} &&= \{\mathit{dbprod}\}\\
\tilde{\blacksquare}^{\texttt{order}} &= \tilde{\square}^{\texttt{n\_order}} &&= \{\mathit{dborder}\}
\end{aligned}
$$

(b) Producers+consumers

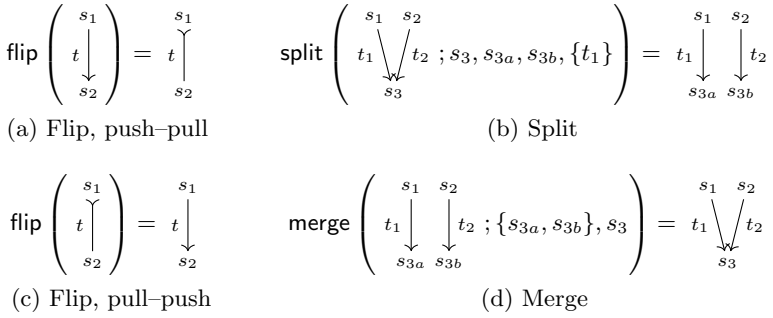
Fig. 3: Equivalent flow graphs in the webshop running example (blue dashed lines indicate bisimulation relation R in Def. 6)

the theorem, we need three additional assumptions beside equivalence of the parts. These additional assumptions essentially state that *after substitution*, the services on the "boundary" between the substituted/substitute part and the remaining part must be indistinguishable from those *before substitution* (in terms of both their names and information flows). In other words, the interface between the substituted/substitute and the remainder must stay the same; services on the boundary may not be renamed, added, or removed by a refactoring.

Theorem 2.
$$
\begin{bmatrix}
(S_1, \blacksquare_1, \square_1, \dashv_1) \sim_{R_1} (\tilde{S}_1, \tilde{\blacksquare}_1, \tilde{\square}_1, \tilde{\dashv}_1)\\
\wedge\ S_B = S_1 \cap S_2 = \tilde{S}_1 \cap S_2\\
\wedge\ \forall s, \tilde{s}\,.\left[\left[s\ R_1\ \tilde{s}\ \wedge\ s \in S_B\right] \ \Rightarrow\ s = \tilde{s}\right]\\
\wedge\ \forall s, \tilde{s}\,.\left[\left[s\ R_1\ \tilde{s}\ \wedge\ \tilde{s} \in S_B\right] \ \Rightarrow\ s = \tilde{s}\right]
\end{bmatrix}
\Rightarrow \underbrace{\varphi_1}_{(S_1, \blacksquare_1, \square_1, \dashv_1)} \oplus \varphi_2 \sim \overbrace{\tilde{\varphi}_1}^{(\tilde{S}_1, \tilde{\blacksquare}_1, \tilde{\square}_1, \tilde{\dashv}_1)} \oplus \varphi_2
$$

Refactorings. The definition of every refactoring in our refactoring framework (Fig. 2) has two parts: a *condition* that identifies architectures that can take on the role of γ_1 (the substituted part) and an *instruction* that describes the transformation of γ into $\tilde{\gamma}_1$ (the substitute part). The condition is formally modeled as a relation R; the instruction as a function f. An *instance* of a refactoring, then, is the transformation of a concrete γ_1 that satisfies R into $\tilde{\gamma}_1$ according to f. We call a refactoring (R, f) *safe* whenever satisfaction of R by γ_1 (a syntactic property) implies that $[\![\gamma_1]\!]$ and $[\![f(\gamma_1)]\!] = [\![\tilde{\gamma}_1]\!]$ are equivalent (a semantic property). Subsequently, Thm. 2 ensures that a safe refactoring for γ_1 can, indeed, safely be applied in any architecture that contains γ_1 (provided that also the additional assumptions about the boundary services hold; see Thm. 2).

We proceed with three example refactorings: *Flip*, *Split*, and *Merge*. Refactoring Flip converts pushes between services to corresponding "reverse-pulls" and vice versa. Refactoring Split divides the responsibilities of a single service over multiple services. Refactoring Merge combines the responsibilities of multiple services in a single service. Figure 4 shows simple instances of these refactorings.



(a) Flip, push–pull (b) Split

(c) Flip, pull–push (d) Merge

Fig. 4: Instances of example refactorings

CanFlip, CanSplit, CanMerge are the smallest relations induced by the following rules:

$$\frac{}{\mathsf{CanFlip}(\gamma)} \qquad \frac{\overline{s} \in S \ \wedge \ \overline{s}_1, \overline{s}_2 \notin S}{\mathsf{CanSplit}(\gamma; \overline{s}, \overline{s}_1, \overline{s}_2, T_1)} \qquad \frac{\overline{S} \subseteq S \ \wedge \ \overline{s} \notin S}{\mathsf{CanMerge}(\gamma; \overline{S}, \overline{s})}$$

where $\gamma = (S, \blacksquare, \square, \longrightarrow, \longrightarrow\!\!\!\!\prec)$. These are sufficient conditions for Flip, Split, and Merge to be safe (Thms. 3, 4, and 5). Extra size conditions can be imposed on γ to ensure that refactorings involving γ (i.e., where γ is the substituted part) are minimal: $\sum\{|\xrightarrow{t}| + |\xrightarrow{t}\!\!\!\!\prec| \mid t \in \mathbb{T}\} = 1$ for Flip, $\longrightarrow, \longrightarrow\!\!\!\!\prec : \mathbb{T} \to 2^{(\{\overline{s}\} \times S) \cup (S \times \{\overline{s}\})}$ for Split, and $\longrightarrow, \longrightarrow\!\!\!\!\prec : \mathbb{T} \to 2^{(\overline{S} \times S) \cup (S \times \overline{S})}$ for Merge.

Fig. 5: Conditions of example refactorings, modeled as relations

Let $|X|$ denote the cardinality of a set X, let $X[y/Y]$ denote the substitution in X of element y for every element from set Y,[10] and let $\circ : 2^{X \times Y} \times 2^{Y \times Z} \to 2^{X \times Z}$ denote relational composition. Figures 5/6 show the conditions/instructions of our example refactorings, modeled as relations/functions. In these relations/functions, the γ left of the semicolon represents the substituted part (i.e., γ_1 in Fig. 2), while the additional variables right of the semicolon represent other elements that play a role in the refactoring.

The condition of Flip, modeled by relation CanFlip, is empty, meaning that Flip is applicable to any γ. The instruction of Flip, modeled by function flip, subsequently states that every push is transformed into a reverse-pull, and vice versa. Figures 4a and 4c show examples.

The condition of Split, modeled by relation CanSplit, states that for Split to be applicable to γ, it *must* contain service $\overline{s}$ (the existing service to split), while it *may not* contain services $\overline{s}_1, \overline{s}_2$ (the new services after splitting). The instruction of Split, modeled by function split, subsequently states that the responsibilities of $\overline{s}$ are divided over $\overline{s}_1$ and $\overline{s}_2$. Service $\overline{s}_1$ becomes producer/consumer of information of all the types in T_1 that $\overline{s}$ used to produce/consume; service $\overline{s}_2$ gets the

[10] Formally: $X[y/Y] = (X \setminus Y) \cup \begin{cases} \emptyset & \textbf{if } X \cap Y = \emptyset \\ \{y\} & \textbf{otherwise} \end{cases}$

$\mathsf{flip}, \mathsf{split}, \mathsf{merge}$ are the functions defined by the following equations:

$$\mathsf{flip}(\gamma) = (S, \blacksquare, \square, \{t \mapsto (\overset{t}{\longrightarrow\!\!\!\!\prec})^{-1} \mid t \in \mathbb{T}\}, \{t \mapsto (\overset{t}{\longrightarrow})^{-1} \mid t \in \mathbb{T}\})$$

$$\begin{aligned}
\mathsf{split}(\gamma; \overline{s}, \overline{s}_1, \overline{s}_2, T_1) &= ((S \setminus \{\overline{s}\}) \cup \{\overline{s}_1, \overline{s}_2\}, \tilde{\blacksquare}, \tilde{\square}, \overset{\sim}{\longrightarrow}, \overset{\sim}{\longrightarrow\!\!\!\!\prec}) \\
\tilde{\blacksquare} &= \{t \mapsto \blacksquare^t[\overline{s}_1/\{\overline{s}\}] \mid t \in T_1\} \cup \{t \mapsto \blacksquare^t[\overline{s}_2/\{\overline{s}\}] \mid t \notin T_1\} \\
\tilde{\square} &= \{t \mapsto \square^t[\overline{s}_1/\{\overline{s}\}] \mid t \in T_1\} \cup \{t \mapsto \square^t[\overline{s}_2/\{\overline{s}\}] \mid t \notin T_1\} \\
\overset{\sim}{\longrightarrow} &= \quad \{t \mapsto \{(\overline{s}_1, \overline{s})\} \circ \overset{t}{\longrightarrow} \circ \{(\overline{s}, \overline{s}_1)\} \mid t \in T_1\} \\
&\quad \cup \{t \mapsto \{(\overline{s}_2, \overline{s})\} \circ \overset{t}{\longrightarrow} \circ \{(\overline{s}, \overline{s}_2)\} \mid t \notin T_1\} \\
\overset{\sim}{\longrightarrow\!\!\!\!\prec} &= \quad \{t \mapsto \{(\overline{s}_1, \overline{s})\} \circ \overset{t}{\longrightarrow\!\!\!\!\prec} \circ \{(\overline{s}, \overline{s}_1)\} \mid t \in T_1\} \\
&\quad \cup \{t \mapsto \{(\overline{s}_2, \overline{s})\} \circ \overset{t}{\longrightarrow\!\!\!\!\prec} \circ \{(\overline{s}, \overline{s}_2)\} \mid t \notin T_1\}
\end{aligned}$$

$$\begin{aligned}
\mathsf{merge}(\gamma; \overline{S}, \overline{s}) &= ((S \setminus \overline{S}) \cup \{\overline{s}\}, \tilde{\blacksquare}, \tilde{\square}, \overset{\sim}{\longrightarrow}, \overset{\sim}{\longrightarrow\!\!\!\!\prec}) \\
\tilde{\blacksquare} &= \{t \mapsto \blacksquare^t[\overline{s}/\overline{S}] \mid t \in \mathbb{T}\} \\
\tilde{\square} &= \{t \mapsto \square^t[\overline{s}/\overline{S}] \mid t \in \mathbb{T}\} \\
\overset{\sim}{\longrightarrow} &= \{t \mapsto (\overline{s} \times \overline{S}) \circ \overset{t}{\longrightarrow} \circ (\overline{S} \times \overline{s}) \mid t \in \mathbb{T}\} \\
\overset{\sim}{\longrightarrow\!\!\!\!\prec} &= \{t \mapsto (\overline{s} \times \overline{S}) \circ \overset{t}{\longrightarrow\!\!\!\!\prec} \circ (\overline{S} \times \overline{s}) \mid t \in \mathbb{T}\}
\end{aligned}$$

where $\gamma = (S, \blacksquare, \square, \longrightarrow, \longrightarrow\!\!\!\!\prec)$.

Fig. 6: Instructions of example refactorings, modeled as functions

remaining production/consumption responsibilities. Accordingly, calls for information of a type in T_1 involve $\overline{s}_1$ instead of $\overline{s}$; other calls involve $\overline{s}_2$. (Relational compositions act as service renaming operations.) Figure 4b shows an example.

The condition of Merge, modeled by relation $\mathsf{CanMerge}$, states that for Merge to be applicable to γ, it *must* contain the services in $\overline{S}$ (the existing services to merge), while it *may not* contain service $\overline{s}$ (the new service after merging). The instruction of Merge, modeled by function merge, subsequently states that the responsibilities of the services in $\overline{S}$ are combined in $\overline{s}$. Service $\overline{s}$ becomes producer/consumer of information of all the types that services in $\overline{S}$ used to produce/consume. Accordingly, calls involve $\overline{s}$ instead of any service in $\overline{S}$. Figure 4d shows an example.

The following theorems state the safeness of our example refactorings.

Theorem 3. $\mathsf{CanFlip}(\gamma)$ **implies** $[\![\gamma]\!] \sim [\![\mathsf{flip}(\gamma)]\!]$

Theorem 4. $\mathsf{CanSplit}(\gamma; \overline{s}, \overline{s}_1, \overline{s}_2, T_1)$ **implies** $[\![\gamma]\!] \sim [\![\mathsf{split}(\gamma; \overline{s}, \overline{s}_1, \overline{s}_2, T_1)]\!]$

Theorem 5. $\mathsf{CanMerge}(\gamma; \overline{S}, \overline{s})$ **implies** $[\![\gamma]\!] \sim [\![\mathsf{merge}(\gamma; \overline{S}, \overline{s})]\!]$

4 Modeling Performance, Non-Quantitatively

Overview. Our refactoring framework in Sect. 3 offers a formal means of defining and reasoning about architectural refactorings, three examples of which we presented. Moreover, the framework supports automatically checking that

a refactoring is applicable and actually carrying out its corresponding changes, while guaranteeing that all information flows are preserved and no spurious new ones are introduced (i.e., activity (1) in Sect. 1). What is still missing, though, is a mechanism to evaluate the *effectiveness* of a refactoring; our tool should also assist architects in selecting a "good" refactoring among all applicable ones. To fill this gap, in this section, we present non-quantitative performance indicators based on which architects can make informed decisions about refactorings.

(In)sensitivity. An important optimization criterion for First8 architects is the extent to which one service s_1 affects the performance of another service s_2. We call this property the *(in)sensitivity* of s_1 to s_2.

Typically, First8 architects refactor architectures to minimize service sensitivities. For instance, in the webshop running example, services *chkout* and *price* are sensitive to service *acc*: once *acc* starts checking whether orders have been paid for, the performance of *chkout* and *price* may decrease, because service *db* may be unable to process the additional calls from *acc* without affecting the calls from *chkout* and *price*. Checking payment statuses is, however, only a low-priority task—it does not matter whether it happens immediately or in a few hours—and it should definitely not hinder the high-priority front end of the system (which directly affects business). Refactoring the architecture to make *chkout* and *price* insensitive to *acc* is therefore an important improvement.

First8 architects distinguish three levels of (in)sensitivity. If services s_1 and s_2 cannot affect each other's performance whatsoever, they are *insensitive* to each other. If the performance of s_1 is affected by s_2 because s_1 requires information from s_2 (by means of a pull), then s_1 is *voluntarily sensitive* to s_2. If the performance of s_1 is affected by s_2 regardless of s_1's calls to s_2, then s_1 is *forcibly sensitive* to s_2. Insensitivity is symmetric, but (voluntary/forcible) sensitivity is not: service s_1 may be sensitive to service s_2, while s_2 may not be sensitive to s_1.

To formalize these levels of (in)sensitivity, we introduce two auxiliary properties: *stress* and *delay*. The stress of a service is a non-quantitative abstraction of the number of incoming calls that it needs to process. The higher the number of calls, the higher the stress of the service and the lower its performance. A service s_1 is therefore forcibly sensitive to another service s_2 if s_2 stresses s_1. The delay of a service is a non-quantitative abstraction of the number of outgoing pulls whose processing (by other services) it needs to await. The higher the number of pulls, the higher the delay of the service and the lower its performance. A service s_1 is therefore voluntarily sensitive to another service s_2 if s_2 delays s_1.

We formalize stress and delay graph-theoretically over call graphs, as sets of services. The *stress set* of a service s contains the services that affect the stress of s: if the stress of a service in its stress set increases, then so does the stress of s. By convention, we also include s in its own stress set (s may stress itself). The *delay set* of a service s contains the services that affect the delay of s.

	Stress	Delay	
db	S	$\emptyset$	
chkout	$\{chkout\}$	$S = S \cup \emptyset \cup \{chkout, price\} \cup S$	
		$= \mathsf{Stress}(\gamma, db) \cup \mathsf{Delay}(\gamma, db) \cup \mathsf{Stress}(\gamma, price) \cup \mathsf{Delay}(\gamma, price)$	
price	$\left\{ \begin{matrix} chkout, \\ price \end{matrix} \right\}$	$S = S \cup \emptyset = \mathsf{Stress}(\gamma, db) \cup \mathsf{Delay}(\gamma, db)$	
acc	$\{acc\}$	$S = S \cup \emptyset = \mathsf{Stress}(\gamma, db) \cup \mathsf{Delay}(\gamma, db)$	
office	$\{office\}$	$\emptyset$	

Fig. 7: Stress sets and delay sets in the webshop running example, where γ denotes the call graph in Fig. 1a and $S = \{db, chkout, price, acc, office\}$

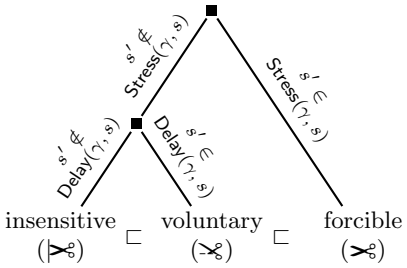


Fig. 8: (In)sensitivity levels, where $x \sqsubset y$ means "x is preferred over y"

$s \setminus s'$	*db*	*chkout*	*price*	*acc*	*office*
db	–	$\not\rightarrow$	$\not\rightarrow$	$\not\rightarrow$	$\not\rightarrow$
chkout	$\not\rightarrow$	–	$\not\rightarrow$	$\boxed{\not\rightarrow}$	$\boxed{\not\rightarrow}$
price	$\not\rightarrow$	$\not\rightarrow$	–	$\boxed{\not\rightarrow}$	$\boxed{\not\rightarrow}$
acc	$\not\rightarrow$	$\not\rightarrow$	$\not\rightarrow$	–	$\not\rightarrow$
office	$\not\not\rightarrow$	$\not\not\rightarrow$	$\not\not\rightarrow$	$\not\not\rightarrow$	–

Fig. 9: (In)sensitivities in the webshop running example ($s \ R \ s'$ for $R \in \{\not\not\rightarrow, \not\rightarrow, \rightarrow\}$), based on the stress sets and delay sets in Fig. 7. Undesirable sensitivities are framed.

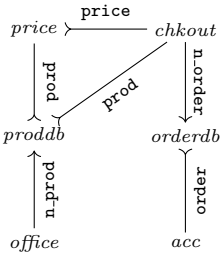
Definition 7. $\mathsf{Stress}, \mathsf{Delay} : \mathbb{C}\mathrm{all} \times \mathbb{S} \to 2^{\mathbb{S}}$ *denote the functions defined by the following equations:*

$$\mathsf{Stress}(\gamma, s) = \{s\} \cup \bigcup \{\mathsf{Stress}(\gamma, s') \mid s' \xrightarrow{t} s \ \vee \ s' \xrightarrow{t} s\}$$
$$\mathsf{Delay}(\gamma, s) = \bigcup \{\mathsf{Stress}(\gamma, s') \cup \mathsf{Delay}(\gamma, s') \mid s \xrightarrow{t} s'\}$$

where $\gamma = (S, \blacksquare, \square, \longrightarrow, \longrightarrow)$.

Note that the delay set of a service s contains the stress set of every service s' from which s pulls information. This is because the services in the stress set of s' may negatively affect the rate at which s' can process pulls by s: if the services in the stress set of s' heavily stress s', then this rate goes down. For instance, in the webshop running example, if service *office* makes many pushes to service *db* (increasing the stress of *db*), then the rate at which *db* can process pulls by service *chkout* may be negatively affected (increasing the delay of *chkout*). Therefore, *office* is in the delay set of *chkout* (i.e., *chkout* pulls from *db*, which has *office* in its stress set). Figure 7 shows a complete overview of the stress sets and delay sets in the webshop running example.

We define insensitivity, voluntary sensitivity, and forcible sensitivity in terms of stress and delay; see also Fig. 8.



(a) Call graph

$s \setminus s'$	*proddb*	*orderdb*	*chkout*	*price*	*acc*	*office*
proddb	–	⊭	⋊	⋊	⊭	⋊
orderdb	⊭	–	⋊	⊭	⋊	⊭
chkout	⋊	⊭	–	⋊	⊭	⋊
price	⋊	⊭	⋊	–	⊭	⋊
acc	⊭	⋊	⋊	⊭	–	⊭
office	⊭	⊭	⊭	⊭	⊭	–

(b) (In)sensitivities. Undesirable sensitivities are framed.

Fig. 10: Webshop running example after applying Split (cf. Figs. 1 and 9)

Definition 8. $\models\!\!\!\!\circ, \not\rightarrow\!\!\!\!\circ, \not\models\!\!\!\!\circ \subseteq \mathbb{C}\text{all} \times \mathbb{S} \times \mathbb{S}$ *denote the smallest relations induced by the following rules:*

$$\underbrace{\frac{s' \notin \mathsf{Stress}(\gamma, s) \;\wedge\; s' \notin \mathsf{Delay}(\gamma, s)}{s \not\models\!\!\!\!\circ_{\gamma}\; s'}}_{\text{insensitivity}} \qquad \underbrace{\frac{s' \notin \mathsf{Stress}(\gamma, s) \;\wedge\; s' \in \mathsf{Delay}(\gamma, s)}{s \not\rightarrow\!\!\!\!\circ_{\gamma}\; s'}}_{\text{voluntary}} \qquad \underbrace{\frac{s' \in \mathsf{Stress}(\gamma, s)}{s \not\models\!\!\!\!\circ_{\gamma}\; s'}}_{\text{forcible}}$$

Figure 9 shows the (in)sensitivities in the webshop running example.

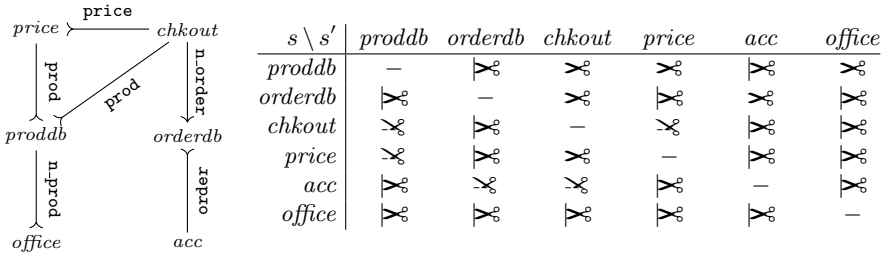
Running example. To illustrate reasoning based on (in)sensitivities of services, we end this section by applying two refactorings in the webshop running example.

Figure 9 already showed that services *chkout* and *price* are voluntarily sensitive to services *acc* and *office*. As stated before, however, the former two high-priority services should not be hindered by the latter two low-priority services.

The first refactoring that we apply is Split, to divide the responsibilities of the existing service *db* over new services *proddb* and *orderdb*. The former becomes responsible for product information; the latter for order information. Figure 10 shows the result of this refactoring. By the theorems in Sect. 3, we already know that this refactoring is correct. Figure 10b moreover shows that services *chkout* and *price* are insensitive to service *acc* after the refactoring.

To make services *chkout* and *price* insensitive also to service *office*, we apply refactoring Flip to the part consisting of service *proddb*, *office*, and the `n_prod`-call between them. This basically means that instead of letting *office* take the initiative of pushing new product information to *db*, the initiative lies now with *db* (e.g., pulling via some ETL process). Figure 11 shows the result of this refactoring. After this refactoring, *chkout* and *price* are insensitive to both *acc* and *office*.

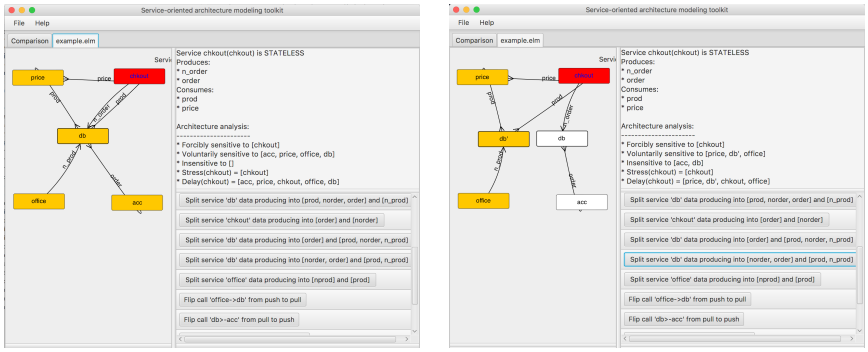
Reasoning in terms of sensitivities in this way thus provides a formal justification for applying refactorings. Note also that the computation of sensitivities can be fully automated, both before and after refactoring.



$s \setminus s'$	*proddb*	*orderdb*	*chkout*	*price*	*acc*	*office*
proddb	–	✂	✂	✂	✂	✂
orderdb	✂	–	✂	✂	✂	✂
chkout	✂	✂	–	✂	✂	✂
price	✂	✂	✂	–	✂	✂
acc	✂	✂	✂	✂	–	✂
office	✂	✂	✂	✂	✂	–

(a) Call graph (b) (In)sensitivities

Fig. 11: Webshop running example after applying Split and Flip (cf. Fig. 10)



(a) Before applying Split (cf. Fig. 1) (b) After applying Split (cf. Fig. 10)

Fig. 12: Proof-of-concept tool applied to the webshop running example

5 Tool

We developed a proof-of-concept tool built on our formalism. Our tools enables architects to draw call graphs as models of architectures. Subsequently, the tool computes and reports stress sets, delay sets, and sensitivities (both textually and visually, with colors). It also proposes a number of candidate refactorings. Currently, we support Flip, Split, and Merge. With a click of a button, architects can select and apply a candidate refactoring, after which the tool recomputes and reports stress sets, delay sets, and sensitivities. Based on this output, architects can make an informed decision about the desirability of the refactoring.

Figure 12 shows two screenshots of the tool, before and after applying refactoring Split in the webshop running example, as also discussed in Sect. 4. The box with the blue text (*chkout*) is the currently selected service; the red/orange/white fill color of a box means that a service is forcibly sensitive/voluntarily sensitive/ insensitive to the currently selected service. The panel on the top right gives a

textual summary of stress sets, delay sets, and sensitivities; the panel on the bottom right lists candidate refactorings.

6 Related Work

Perhaps closest—at least in spirit—to our work on formalizing service-oriented architectures is existing work on formalizing (composition of) service-oriented systems using process calculi (e.g., [2,4,5,8,11,16]). The main difference between those approaches and ours lies in the level of abstraction. Whereas we stay at the higher architectural level, process calculi for service orientation require its users to dive deeper into the local behavior of, and communication between, services. For our current purpose, such details are excessive. From the abstraction level perspective, thus, such calculi are unsuitable for us. Moreover, as such calculi are often rather formal, they are too far from what architects without training in formal methods can comfortably use.

There exists an extensive body of literature on quantitative reasoning about performance in service oriented architecture (e.g., [1,3,6,9,17]). In general, however, these models require load functions, detailed descriptions, or actual implementations for each service. Determining load functions and finding reasonable values for parameters of these models is demanding and might be possible only quite late in the development process. Additionally, calculating the performance of the architecture might not be instant but requires a lengthy simulation. Instead, our work focuses on finding performance dependencies among services, expressed in terms of sensitivity levels, without quantification. The properties can be quickly derived, even manually up to a certain complexity, and tooling can extensively compare alternatives. Our notions of sensitivity, stress, and delay are inspired by work by Lamers and Van Eekelen [10].

7 Conclusion

Summary. We presented a formalism of service-oriented architecture that supports reasoning about refactorings and predicting their performance impact, by non-quantitatively analyzing dependencies among services. The aim of our formalism is to provide a foundation for a tool that aids architecture specialists in their refactoring activities. We also presented such a proof-of-concept tool.

Our formalism consists of syntax to model the structure of architectures (call graphs; Def. 1), semantics to model their behavior in terms of information flows (flow graphs; Def. 2), a composition operation to model refactorings ($\oplus$; Def. 4), and an equivalence relation to prove the safeness of refactorings ($\sim$; Def. 6). We proved that this equivalence relation is, in fact, a congruence relation (Thm. 2), which is of essential importance. We presented three example refactorings (Flip, Split, Merge; Figs. 5 and 6), and proved their safeness (Thms. 3, 4, and 5).

Future work. We see two main directions for future work. The first one is extending our formalism from architectures to *deployments*. Formally, the extension seems straightforward: we can define a deployment as a triple $(M, \gamma, \circledS)$, where M is a set of *machines*, γ a call graph, and $\circledS : M \to 2^S$ a function from the machines in M to sets of services in γ that run on those machines. The challenging part is leveraging such deployment models to automate reasoning about refactorings and performance at the deployment level in a meaningful way.

The second line of future work concerns proving equivalences of architectures. Although Thms. 3, 4, and 5 guarantee the safety of the refactorings in Sect. 3, not all changes that architects may want to apply to an architecture are instances of these refactorings. For such changes, architects still need to manually check preservation of information flows. To assist them in this laborious task, we aim to develop an efficient equivalence checking algorithm and add it to our tool.

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A Proof of Theorem 1

A.1 Sketch

The theorem follows almost immediately by Defs. 4 and 3 of $\oplus$ and $[\![\cdot]\!]$.

A.2 Proof

Proof. Assume:

(A1) $\gamma_1 = (S_1, \blacksquare_1, \square_1, \longrightarrow_1, \longrightarrow\!\!\!\!\prec_1)$

(A2) $\gamma_2 = (S_2, \blacksquare_2, \square_2, \longrightarrow_2, \longrightarrow\!\!\!\!\prec_2)$

Observe:

(Z1) Conclude:

$$\begin{aligned}&(\longrightarrow_1 \uplus \longrightarrow_2)(t) \cup (\longrightarrow\!\!\!\!\prec_1 \uplus \longrightarrow\!\!\!\!\prec_2)(t)^{-1}\\ &= \{\hat{t} \mapsto \xrightarrow{\hat{t}}_1 \cup \xrightarrow{\hat{t}}_2 \mid \hat{t} \in \mathbb{T}\}(t) \cup \{\hat{t} \mapsto \xrightarrow{\hat{t}}\!\!\!\!\prec_1 \cup \xrightarrow{\hat{t}}\!\!\!\!\prec_2 \mid \hat{t} \in \mathbb{T}\}(t)^{-1}\end{aligned}$$

Then, conclude:

$$(\longrightarrow_1 \uplus \longrightarrow_2)(t) \cup (\longrightarrow\!\!\!\!\prec_1 \uplus \longrightarrow\!\!\!\!\prec_2)(t)^{-1} = \xrightarrow{t}_1 \cup \xrightarrow{t}_2 \cup (\xrightarrow{t}\!\!\!\!\prec_1 \cup \xrightarrow{t}\!\!\!\!\prec_2)^{-1}$$

Then, conclude:

$$(\longrightarrow_1 \uplus \longrightarrow_2)(t) \cup (\longrightarrow\!\!\!\!\prec_1 \uplus \longrightarrow\!\!\!\!\prec_2)(t)^{-1} = \xrightarrow{t}_1 \cup \xrightarrow{t}_2 \cup (\xrightarrow{t}\!\!\!\!\prec_1)^{-1} \cup (\xrightarrow{t}\!\!\!\!\prec_2)^{-1}$$

Prove the theorem by the following reduction. By Def. 4 of $\oplus$, conclude:

$$\begin{aligned}&[\![(S_1, \blacksquare_1, \square_1, \longrightarrow_1, \longrightarrow\!\!\!\!\prec_1) \oplus (S_2, \blacksquare_2, \square_2, \longrightarrow_2, \longrightarrow\!\!\!\!\prec_2)]\!]\\ &= [\![(S_1 \cup S_2, \blacksquare_1 \uplus \blacksquare_2, \square_1 \uplus \square_2, \longrightarrow_1 \uplus \longrightarrow_2, \longrightarrow\!\!\!\!\prec_1 \uplus \longrightarrow\!\!\!\!\prec_2)]\!]\end{aligned}$$

Then, by Def. 3 of $[\![\cdot]\!]$, conclude:

$$\begin{aligned}&[\![(S_1, \blacksquare_1, \square_1, \longrightarrow_1, \longrightarrow\!\!\!\!\prec_1) \oplus (S_2, \blacksquare_2, \square_2, \longrightarrow_2, \longrightarrow\!\!\!\!\prec_2)]\!]\\ &= \left(\begin{array}{l}S_1 \cup S_2, \blacksquare_1 \uplus \blacksquare_2, \square_1 \uplus \square_2,\\ \quad \{t \mapsto (\longrightarrow_1 \uplus \longrightarrow_2)(t) \cup (\longrightarrow\!\!\!\!\prec_1 \uplus \longrightarrow\!\!\!\!\prec_2)(t)^{-1} \mid t \in \mathbb{T}\}\end{array}\right)\end{aligned}$$

Then, by (Z1), conclude:

$$\begin{aligned}&[\![(S_1, \blacksquare_1, \square_1, \longrightarrow_1, \longrightarrow\!\!\!\!\prec_1) \oplus (S_2, \blacksquare_2, \square_2, \longrightarrow_2, \longrightarrow\!\!\!\!\prec_2)]\!]\\ &= \left(\begin{array}{l}S_1 \cup S_2, \blacksquare_1 \uplus \blacksquare_2, \square_1 \uplus \square_2,\\ \quad \{t \mapsto \xrightarrow{t}_1 \cup \xrightarrow{t}_2 \cup (\xrightarrow{t}\!\!\!\!\prec_1)^{-1} \cup (\xrightarrow{t}\!\!\!\!\prec_2)^{-1} \mid t \in \mathbb{T}\}\end{array}\right)\end{aligned}$$

Then, conclude:

$$\begin{aligned}&[\![(S_1, \blacksquare_1, \square_1, \longrightarrow_1, \longrightarrow\!\!\!\!\prec_1) \oplus (S_2, \blacksquare_2, \square_2, \longrightarrow_2, \longrightarrow\!\!\!\!\prec_2)]\!]\\ &= \left(\begin{array}{l}S_1 \cup S_2, \blacksquare_1 \uplus \blacksquare_2, \square_1 \uplus \square_2,\\ \quad \{t \mapsto \xrightarrow{t}_1 \cup (\xrightarrow{t}\!\!\!\!\prec_1)^{-1} \mid t \in \mathbb{T}\} \uplus \{t \mapsto \xrightarrow{t}_2 \cup (\xrightarrow{t}\!\!\!\!\prec_2)^{-1} \mid t \in \mathbb{T}\}\end{array}\right)\end{aligned}$$

Then, by Def. 4 of $\oplus$, conclude:

$$\begin{aligned}&[\![(S_1, \blacksquare_1, \square_1, \longrightarrow_1, \longrightarrow\!\!\!\!\prec_1) \oplus (S_2, \blacksquare_2, \square_2, \longrightarrow_2, \longrightarrow\!\!\!\!\prec_2)]\!]\\ =\ &(S_1, \blacksquare_1, \square_1, \{t \mapsto \xrightarrow{t}_1 \cup (\xrightarrow{t}\!\!\!\!\prec_1)^{-1} \mid t \in \mathbb{T}\})\\ &\oplus (S_2, \blacksquare_2, \square_2, \{t \mapsto \xrightarrow{t}_2 \cup (\xrightarrow{t}\!\!\!\!\prec_2)^{-1} \mid t \in \mathbb{T}\})\end{aligned}$$

Then, by Def. 3 of $[\![\cdot]\!]$, conclude:

$$\begin{aligned}&[\![(S_1, \blacksquare_1, \square_1, \longrightarrow_1, \longrightarrow\!\!\!\!\prec_1) \oplus (S_2, \blacksquare_2, \square_2, \longrightarrow_2, \longrightarrow\!\!\!\!\prec_2)]\!]\\ =\ &[\![(S_1, \blacksquare_1, \square_1, \longrightarrow_1, \longrightarrow\!\!\!\!\prec_1)]\!] \oplus [\![(S_2, \blacksquare_2, \square_2, \longrightarrow_2, \longrightarrow\!\!\!\!\prec_2)]\!]\end{aligned}$$

Then, by (A1)(A2), conclude $[\![\varphi_1 \oplus \varphi_2]\!] = [\![\varphi_1]\!] \oplus [\![\varphi_2]\!]$. □

B Proof of Theorem 2

B.1 Sketch

By the antecedent of the theorem and Def. 6 of $\sim$, we derive $\varphi_1 \preceq_{R_1} \varphi_2$ and $\varphi_2 \preceq R_1^{-1}\varphi_2$. Let $R = R_1 \cup \{(s,s) \mid s \in S_2\}$ be a candidate bisimulation relation. The following auxiliary theorem states that R is a simulation relation from $\varphi_1 \oplus \varphi_2$ to $\tilde{\varphi}_1 \oplus \varphi_2$.

Theorem 6.

$$\left[\begin{array}{l} \varphi_1 \preceq_{R_1} \tilde{\varphi}_1 \\ \wedge\ S_B = S_1 \cap S_2 = \tilde{S}_1 \cap S_2 \\ \wedge\ \forall s, \tilde{s}\,.\,\big[\big[s\ R_1\ \tilde{s}\ \wedge\ s \in S_B\big]\ \Rightarrow\ s = \tilde{s}\big] \\ \wedge\ R = R_1 \cup \{(s,s) \mid s \in S_2\} \end{array}\right] \Rightarrow\ \varphi_1 \oplus \varphi_2 \preceq_R \tilde{\varphi}_1 \oplus \varphi_2$$

Proof. See Sect. F. □

We apply Thm. 6 to show also that $R^{-1} = R_1^{-1} \cup \{(s,s) \mid s \in S_2\}$ is a simulation relation from $\tilde{\varphi}_1 \oplus \varphi_2$ to $\varphi \oplus \varphi_2$. The theorem subsequently follows by Def. 6 of $\sim$ (i.e., the candidate bisimulation relation R is, indeed, a bisimulation relation).

B.2 Proof

Proof. Assume:

(A1) $\varphi_1 \sim_{R_1} \tilde{\varphi}_1$

(A2) $S_B = S_1 \cap S_2 = \tilde{S}_1 \cap S_2$

(A3) $\big[\big[s\ R_1\ \tilde{s}$ **and** $s \in S_B\big]$ **implies** $s = \tilde{s}\big]$ **for all** $s, \tilde{s}$

(A4) $\big[\big[s\ R_1\ \tilde{s}$ **and** $\tilde{s} \in S_B\big]$ **implies** $s = \tilde{s}\big]$ **for all** $s, \tilde{s}$

(A5) $R = R_1 \cup \{(s,s) \mid s \in S_2\}$

Observe:

(Z1) Suppose $\varphi_1 \preceq_{R_1} \tilde{\varphi}_1$. Then, by (A2), conclude:

$$\varphi_1 \preceq_{R_1} \tilde{\varphi}_1 \text{ and } S_B = S_1 \cap S_2 = \tilde{S}_1 \cap S_2$$

Then, by (A3), conclude:

$$\begin{array}{l} \varphi_1 \preceq_{R_1} \tilde{\varphi}_1 \\ \text{and } S_B = S_1 \cap S_2 = \tilde{S}_1 \cap S_2 \\ \text{and } \big[\big[\big[s\ R_1\ \tilde{s} \text{ and } s \in S_B\big] \text{ implies } s = \tilde{s}\big] \text{ for all } s, \tilde{s}\big] \end{array}$$

Then, by (A5), conclude:

$$\begin{array}{l} \varphi_1 \preceq_{R_1} \tilde{\varphi}_1 \\ \text{and } S_B = S_1 \cap S_2 = \tilde{S}_1 \cap S_2 \\ \text{and } \big[\big[\big[s\ R_1\ \tilde{s} \text{ and } s \in S_B\big] \text{ implies } s = \tilde{s}\big] \text{ for all } s, \tilde{s}\big] \\ \text{and } R = R_1 \cup \{(s,s) \mid s \in S_2\} \end{array}$$

Then, by Thm. 6, conclude $\varphi_1 \oplus \varphi_2 \preceq_R \tilde{\varphi}_1 \oplus \varphi_2$.

(Z2) By (A4), conclude $\big[\big[\big[s\ R_1\ \tilde{s}$ **and** $\tilde{s} \in S_B\big]$ **implies** $s = \tilde{s}\big]$ **for all** $s, \tilde{s}\big]$. Then, conclude $\big[\big[\big[\tilde{s}\ R_1^{-1}\ s$ **and** $\tilde{s} \in S_B\big]$ **implies** $\tilde{s} = s\big]$ **for all** $s, \tilde{s}\big]$.

(Z3) By (A5), conclude $R = R_1 \cup \{(s, s) \mid s \in S_2\}$. Then, conclude:

$$R^{-1} = R_1 \cup \{(s, s) \mid s \in S_2\}^{-1}$$

Then, conclude $R^{-1} = R_1^{-1} \cup \{(s, s) \mid s \in S_2\}^{-1}$. Then, conclude:

$$R^{-1} = R_1^{-1} \cup \{(s, s) \mid s \in S_2\}$$

(Z4) Suppose $\tilde{\varphi}_1 \preceq_{R_1^{-1}} \varphi_1$. Then, by (A2), conclude:

$$\tilde{\varphi}_1 \preceq_{R_1^{-1}} \varphi_1 \ \textbf{and}\ S_B = S_1 \cap S_2 = \tilde{S}_1 \cap S_2$$

Then, by (Z2), conclude:

$$\begin{array}{ll} & \tilde{\varphi}_1 \preceq_{R_1^{-1}} \varphi_1 \\ \textbf{and} & S_B = S_1 \cap S_2 = \tilde{S}_1 \cap S_2 \\ \textbf{and} & \big[\big[\big[\tilde{s}\ R_1^{-1}\ s\ \textbf{and}\ \tilde{s} \in S_B\big]\ \textbf{implies}\ \tilde{s} = s\big]\ \textbf{for all}\ s, \tilde{s}\big] \end{array}$$

Then, by (Z3), conclude:

$$\begin{array}{ll} & \tilde{\varphi}_1 \preceq_{R_1^{-1}} \varphi_1 \\ \textbf{and} & S_B = S_1 \cap S_2 = \tilde{S}_1 \cap S_2 \\ \textbf{and} & \big[\big[\big[\tilde{s}\ R_1^{-1}\ s\ \textbf{and}\ \tilde{s} \in S_B\big]\ \textbf{implies}\ \tilde{s} = s\big]\ \textbf{for all}\ s, \tilde{s}\big] \\ \textbf{and} & R^{-1} = R_1^{-1} \cup \{(s, s) \mid s \in S_2\} \end{array}$$

Then, by Thm. 6, conclude $\tilde{\varphi}_1 \oplus \varphi_2 \preceq_{R^{-1}} \varphi_1 \oplus \varphi_2$.

Prove the theorem by the following reduction. By (A1), conclude $\varphi_1 \sim_{R_1} \tilde{\varphi}_1$. Then, by Def. 6 of $\sim$, conclude $\big[\varphi_1 \preceq_{R_1} \tilde{\varphi}_1$ **and** $\tilde{\varphi}_1 \preceq_{R_1^{-1}} \varphi_1\big]$. Then, by (Z1), conclude $\big[\varphi_1 \oplus \varphi_2 \preceq_R \tilde{\varphi}_1 \oplus \varphi_2$ **and** $\tilde{\varphi}_1 \preceq_{R_1^{-1}} \varphi_1\big]$. Then, by (Z4), conclude $\big[\varphi_1 \oplus \varphi_2 \preceq_R \tilde{\varphi}_1 \oplus \varphi_2$ **and** $\tilde{\varphi}_1 \oplus \varphi_2 \preceq_{R^{-1}} \varphi_1 \oplus \varphi_2\big]$. Then, by Def. 6 of $\sim$, conclude $\varphi_1 \oplus \varphi_2 \simeq_R \tilde{\varphi}_1 \oplus \varphi_2$. Then, by Def. 6 of $\sim$, conclude $\varphi_1 \oplus \varphi_2 \simeq \tilde{\varphi}_1 \oplus \varphi_2$. □

C Proof of Theorem 3

The scope of auxiliary propositions and auxiliary lemmas in this section is limited to this section.

C.1 Sketch

Let $R = \{(s, s) \mid s \in S_\mathrm{f}\}$ be a candidate bisimulation relation (Prop. 10). First, we prove $[\![\gamma]\!] \preceq_R [\![\mathsf{flip}(\gamma)]\!]$ (Lemma 1). The main step in this proof is that we show that information flows resulting from pushes are equivalent to information flows resulting from corresponding reverse-pulls and vice versa (only the initiative changes, but initiative is not part of the semantics). A proof for $[\![\mathsf{flip}(\gamma)]\!] \preceq_{R^{-1}} [\![\gamma]\!]$ is similar (Lemma 2). The theorem subsequently follows by Def. 6 of $\sim$ (i.e., the candidate bisimulation relation R is, indeed, a bisimulation relation).

C.2 Propositions and Lemmas

Antecedent:

Proposition 1. $\mathsf{CanFlip}(\gamma)$

By Def. 1:

Proposition 2. $\gamma = (S_\mathrm{c}, \blacksquare_\mathrm{c}, \square_\mathrm{c}, \longrightarrow, \longrightarrow\!\!\!\!\prec)$

Proposition 3. $\mathsf{flip}(\gamma) = (\tilde{S}_\mathrm{c}, \tilde{\blacksquare}_\mathrm{c}, \tilde{\square}_\mathrm{c}, \xrightarrow{\sim}, \xrightarrow{\sim}\!\!\!\!\prec)$

By Def. 3:

Proposition 4. $[\![\gamma]\!] = (S_\mathrm{f}, \blacksquare_\mathrm{f}, \square_\mathrm{f}, \longrightarrow\!\!\!\!\!\!\mathrm{I})$

Proposition 5. $[\![\mathsf{flip}(\gamma)]\!] = (\tilde{S}_\mathrm{f}, \tilde{\blacksquare}_\mathrm{f}, \tilde{\square}_\mathrm{f}, \xrightarrow{\sim}\!\!\!\!\!\!\mathrm{I})$

Proposition 6.
$$\begin{aligned} & S_\mathrm{f} = S_\mathrm{c} \\ \textbf{and}\ & \blacksquare_\mathrm{f} = \blacksquare_\mathrm{c} \\ \textbf{and}\ & \square_\mathrm{f} = \square_\mathrm{c} \\ \textbf{and}\ & \longrightarrow\!\!\!\!\!\!\mathrm{I} = \{t \mapsto \xrightarrow{t} \cup (\xrightarrow{t}\!\!\!\!\prec)^{\text{-}1} \mid t \in \mathbb{T}\} \end{aligned}$$

Proposition 7.
$$\begin{aligned} & \tilde{S}_\mathrm{f} = \tilde{S}_\mathrm{c} \\ \textbf{and}\ & \tilde{\blacksquare}_\mathrm{f} = \tilde{\blacksquare}_\mathrm{c} \\ \textbf{and}\ & \tilde{\square}_\mathrm{f} = \tilde{\square}_\mathrm{c} \\ \textbf{and}\ & \xrightarrow{\sim}\!\!\!\!\!\!\mathrm{I} = \{t \mapsto \xrightarrow{\sim t} \cup (\xrightarrow{\sim t}\!\!\!\!\prec)^{\text{-}1} \mid t \in \mathbb{T}\} \end{aligned}$$

Proposition 8.
$$\begin{aligned} & \left[\left[s \in \blacksquare_\mathrm{f}^t\ \textbf{implies}\ s \in S_\mathrm{f}\right]\ \textbf{for all}\ s, t\right] \\ \textbf{and}\ & \left[\left[s \in \square_\mathrm{f}^t\ \textbf{implies}\ s \in S_\mathrm{f}\right]\ \textbf{for all}\ s, t\right] \\ \textbf{and}\ & \left[\left[s \xrightarrow{t}\!\!\!\!\!\!\mathrm{I}\ s'\ \textbf{implies}\ s, s' \in S_\mathrm{f}\right]\ \textbf{for all}\ s, s', t\right] \end{aligned}$$

Proposition 9. $\big[\big[s \in \tilde{\blacksquare}^t_{\mathrm{f}}$ **implies** $s \in \tilde{S}_{\mathrm{f}}\big]$ **for all** $s, t\big]$
and $\big[\big[s \in \tilde{\Box}^t_{\mathrm{f}}$ **implies** $s \in \tilde{S}_{\mathrm{f}}\big]$ **for all** $s, t\big]$
and $\big[\big[s \xrightarrow{\sim t}_{\mathrm{I}} s'$ **implies** $s, s' \in \tilde{S}_{\mathrm{f}}\big]$ **for all** $s, s', t\big]$

Proposition 10. $R = \{(s, s) \mid s \in S_{\mathrm{f}}\}$

By Fig. 6:

Proposition 11. $\tilde{S}_{\mathrm{c}} = S_{\mathrm{c}}$
and $\tilde{\blacksquare}_{\mathrm{c}} = \blacksquare_{\mathrm{c}}$
and $\tilde{\Box}_{\mathrm{c}} = \Box_{\mathrm{c}}$
and $\xrightarrow{\sim} = \{t \mapsto (\xrightarrow{t})^{\text{-}1} \mid t \in \mathbb{T}\}$
and $\xrightarrow{\sim}_{\prec} = \{t \mapsto (\xrightarrow{t}_{\prec})^{\text{-}1} \mid t \in \mathbb{T}\}$

Lemma 1. $[\![\gamma]\!] \preceq_R [\![\mathsf{flip}(\gamma)]\!]$

Proof. Observe:

(Z1) Suppose:

$$s \xrightarrow{t} s' \text{ for some } s, s', t$$

Then, conclude $s' \, (\xrightarrow{t})^{-1} \, s$. Then, conclude $(s', s) \in \{\hat{t} \mapsto (\xrightarrow{\hat{t}})^{-1} \mid \hat{t} \in \mathbb{T}\}(t)$. Then, by Prop. 11, conclude $s' \xrightarrow{\sim t} s$. Then, conclude $s \, (\xrightarrow{\sim t})^{-1} \, s'$.

(Z2) Suppose:

$$s \, (\xrightarrow{t})^{-1} \, s' \text{ for some } s, s', t$$

Then, conclude $(s, s') \in \{\hat{t} \mapsto (\xrightarrow{\hat{t}})^{-1} \mid \hat{t} \in \mathbb{T}\}(t)$. Then, by Prop. 11, conclude $s \xrightarrow{\sim t} s'$.

(Z3) Suppose:

$$s \xrightarrow{t}_{\mathtt{I}} s' \text{ for some } s, s', t$$

Then, by Prop. 6, conclude $(s, s') \in \{\hat{t} \mapsto \xrightarrow{\hat{t}} \cup (\xrightarrow{\hat{t}})^{-1} \mid \hat{t} \in \mathbb{T}\}(t)$. Then, conclude $(s, s') \in \xrightarrow{t} \cup (\xrightarrow{t})^{-1}$. Then, conclude $\left[s \xrightarrow{t} s' \textbf{ or } s \, (\xrightarrow{t})^{-1} \, s'\right]$. Then, by (Z1), conclude $\left[s \, (\xrightarrow{\sim t})^{-1} \, s' \textbf{ or } s \, (\xrightarrow{t})^{-1} \, s'\right]$. Then, by (Z2), conclude $\left[s \, (\xrightarrow{\sim t})^{-1} \, s' \textbf{ or } s \xrightarrow{\sim t} s'\right]$. Then, conclude $(s, s') \in (\xrightarrow{\sim t})^{-1} \cup \xrightarrow{\sim t}$. Then, conclude $(s, s') \in \{\hat{t} \mapsto (\xrightarrow{\hat{t}})^{-1} \cup \xrightarrow{\hat{t}} \mid \hat{t} \in \mathbb{T}\}(t)$. Then, by Prop. 7 conclude $s \xrightarrow{\sim t}_{\mathtt{I}} s'$.

(Z4) Suppose:

$$s \xrightarrow{\sim t}_{\mathtt{I}} s' \text{ for some } s, s', t$$

Then, by Prop. 9, conclude $s, s' \in \tilde{S}_{\mathrm{f}}$. Then, by Prop. 7, conclude $s, s' \in \tilde{S}_{\mathrm{c}}$. Then, by Prop. 11, conclude $s, s' \in S_{\mathrm{c}}$. Then, by Prop. 6, conclude $s, s' \in S_{\mathrm{f}}$. Then, conclude $(s, s), (s, s') \in \{(\hat{s}, \hat{s}) \mid \hat{s} \in S_{\mathrm{f}}\}$. Then, by Prop. 10, conclude $\left[s \mathrel{R} s \textbf{ and } s' \mathrel{R} s'\right]$.

(Z5) Suppose:

$$s_1 \xrightarrow{t}_{\mathtt{I}} \cdots \xrightarrow{t}_{\mathtt{I}} s_n \text{ for some } n, s_1, \ldots, s_n$$

Then, by (Z3), conclude $s_1 \xrightarrow{\sim t}_{\mathtt{I}} \cdots \xrightarrow{\sim t}_{\mathtt{I}} s_n$. Then, by (Z4), conclude:

$$s_1 \xrightarrow{\sim t}_{\mathtt{I}} \cdots \xrightarrow{\sim t}_{\mathtt{I}} s_n \textbf{ and } s_1 \mathrel{R} s_1 \textbf{ and } s_n \mathrel{R} s_n$$

Then, conclude $\left[s_1 \xrightarrow{\sim t}_{\mathtt{I}}{}^{+} s_n \textbf{ and } s_1 \mathrel{R} s_1 \textbf{ and } s_n \mathrel{R} s_n\right]$.

(Z6) Suppose:

$$s \xrightarrow{t}_{\mathtt{I}}{}^{+} s' \text{ for some } s, s', t$$

Then, conclude:

$$\left[s_1 \xrightarrow{t}_{\mathtt{I}} \cdots \xrightarrow{t}_{\mathtt{I}} s_n \textbf{ and } s = s_1 \textbf{ and } s' = s_n\right] \text{ for some } n, s_1, \ldots, s_n$$

Then, by (Z5), conclude:

$$s_1 \xrightarrow{\sim t}_{\mathtt{I}}{}^{+} s_n \textbf{ and } s_1 \mathrel{R} s_1 \textbf{ and } s_n \mathrel{R} s_n \textbf{ and } s = s_1 \textbf{ and } s' = s_n$$

Then, conclude $\left[s_1 \xrightarrow{\sim t}_{\mathtt{I}}{}^{+} s_n \textbf{ and } s \mathrel{R} s_1 \textbf{ and } s' \mathrel{R} s_n\right]$.

Z7 Suppose:

$$s \in \tilde{\blacksquare}^t_{\mathrm{f}} \text{ for some } s, t$$

Then, by Prop. 9, conclude $s \in \tilde{S}_{\mathrm{f}}$. Then, by Prop. 7, conclude $s \in \tilde{S}_{\mathrm{c}}$. Then, by Prop. 11, conclude $s \in S_{\mathrm{c}}$. Then, by Prop. 6, conclude $s \in S_{\mathrm{f}}$. Then, conclude $(s, s) \in \{(\hat{s}, \hat{s}) \mid \hat{s} \in S_{\mathrm{f}}\}$. Then, by Prop. 10, conclude $s \; R \; s$.

Z8 Suppose:

$$s \in \tilde{\Box}^t_{\mathrm{f}} \text{ for some } s, t$$

Then, by a reduction similar to Z7, conclude $s \; R \; s$.

Z9 Suppose:

$$s \in \blacksquare^t_{\mathrm{f}} \text{ for some } s, t$$

Then, by Prop. 6, conclude $s \in \blacksquare^t_{\mathrm{c}}$. Then, by Prop. 11, conclude $s \in \tilde{\blacksquare}^t_{\mathrm{c}}$. Then, by Prop. 7, conclude $s \in \tilde{\blacksquare}^t_{\mathrm{f}}$. Then, by Z7, conclude $\left[s \in \tilde{\blacksquare}^t_{\mathrm{f}} \text{ and } s \; R \; s\right]$. Then, conclude $\left[\left[\tilde{s} \in \tilde{\blacksquare}^t_{\mathrm{f}} \text{ and } s \; R \; \tilde{s}\right] \text{ for some } \tilde{s}\right]$.

Z0 Suppose:

$$s \in \Box^t_{\mathrm{f}} \text{ for some } s, t$$

Then, by a reduction similar to Z9, conclude:

$$\left[\tilde{s} \in \tilde{\Box}^t_{\mathrm{f}} \text{ and } s \; R \; \tilde{s}\right] \text{ for some } \tilde{s}$$

Y1 Suppose:

$$s \; R \; \tilde{s} \text{ for some } s, \tilde{s}$$

Then, by Prop. 10, conclude $(s, \tilde{s}) \in \{(\hat{s}, \hat{s}) \mid \hat{s} \in S_{\mathrm{f}}\}$. Then, conclude $s, \tilde{s} \in S_{\mathrm{f}}$. Then, conclude $(s, \tilde{s}) \in S_{\mathrm{f}} \times S_{\mathrm{f}}$. Then, by Prop. 6, conclude $(s, \tilde{s}) \in S_{\mathrm{f}} \times S_{\mathrm{c}}$. Then, by Prop. 11, conclude $(s, \tilde{s}) \in S_{\mathrm{f}} \times \tilde{S}_{\mathrm{c}}$. Then, by Prop. 7, conclude $(s, \tilde{s}) \in S_{\mathrm{f}} \times \tilde{S}_{\mathrm{f}}$.

Y2 By Y1, conclude $\left[\left[s \; R \; \tilde{s} \text{ implies } (s, \tilde{s}) \in S_{\mathrm{f}} \times \tilde{S}_{\mathrm{f}}\right] \text{ for all } s, \tilde{s}\right]$. Then, conclude $R \subseteq S_{\mathrm{f}} \times \tilde{S}_{\mathrm{f}}$.

Y3 Suppose:

$$s \in S_{\mathrm{f}} \text{ for some } s$$

Then, conclude $s \in \mathrm{Dom}(\{(\hat{s}, \hat{s}) \mid \hat{s} \in S_{\mathrm{f}}\})$. Then, by Prop. 10, conclude $s \in \mathrm{Dom}(R)$.

Y4 By Y3, conclude $\left[\left[s \in S_{\mathrm{f}} \text{ implies } s \in \mathrm{Dom}(R)\right] \text{ for all } s\right]$. Then, conclude $S_{\mathrm{f}} \subseteq \mathrm{Dom}(R)$.

Prove the lemma by the following reduction. By Z6, conclude:

$$\left[s \xrightarrow[1]{t}{}^{+} s' \text{ implies } \left[\begin{array}{c}\left[\tilde{s} \xrightarrow[1]{\sim t}{}^{+} \tilde{s}' \text{ and } s \; R \; \tilde{s} \text{ and } s' \; R \; \tilde{s}'\right] \\ \text{for some } \tilde{s}, s\end{array}\right]\right] \text{ for all } s, s', t$$

Then, by (Z9)(Z0), conclude:

$$\left[\left[s \xrightarrow{t}_{\mathtt{I}}{}^{+} s' \textbf{ implies } \left[\begin{array}{l}\left[\tilde{s} \xrightarrow{\sim t}_{\mathtt{I}}{}^{+} \tilde{s}' \textbf{ and } s\ R\ \tilde{s} \textbf{ and } s'\ R\ \tilde{s}'\right] \\ \textbf{for some } \tilde{s}, s\end{array}\right]\right] \textbf{ for all } \tilde{s}, \tilde{s}'\right]$$

$$\begin{array}{l}\textbf{and } \left[\left[s \in \blacksquare^{t}_{\mathrm{f}} \textbf{ implies } \left[\left[\tilde{s} \in \tilde{\blacksquare}^{t}_{\mathrm{f}} \textbf{ and } s\ R\ \tilde{s}\right] \textbf{ for some } \tilde{s}\right]\right] \textbf{ for all } s, t\right] \\ \textbf{and } \left[\left[s \in \square^{t}_{\mathrm{f}} \textbf{ implies } \left[\left[\tilde{s} \in \tilde{\square}^{t}_{\mathrm{f}} \textbf{ and } s\ R\ \tilde{s}\right] \textbf{ for some } \tilde{s}\right]\right] \textbf{ for all } s, t\right]\end{array}$$

Then, by (Y2), conclude:

$$\left[\left[s \xrightarrow{t}_{\mathtt{I}}{}^{+} s' \textbf{ implies } \left[\begin{array}{l}\left[\tilde{s} \xrightarrow{\sim t}_{\mathtt{I}}{}^{+} \tilde{s}' \textbf{ and } s\ R\ \tilde{s} \textbf{ and } s'\ R\ \tilde{s}'\right] \\ \textbf{for some } \tilde{s}, s\end{array}\right]\right] \textbf{ for all } \tilde{s}, \tilde{s}'\right]$$

$$\begin{array}{l}\textbf{and } \left[\left[s \in \blacksquare^{t}_{\mathrm{f}} \textbf{ implies } \left[\left[\tilde{s} \in \tilde{\blacksquare}^{t}_{\mathrm{f}} \textbf{ and } s\ R\ \tilde{s}\right] \textbf{ for some } \tilde{s}\right]\right] \textbf{ for all } s, t\right] \\ \textbf{and } \left[\left[s \in \square^{t}_{\mathrm{f}} \textbf{ implies } \left[\left[\tilde{s} \in \tilde{\square}^{t}_{\mathrm{f}} \textbf{ and } s\ R\ \tilde{s}\right] \textbf{ for some } \tilde{s}\right]\right] \textbf{ for all } s, t\right] \\ \textbf{and } R \subseteq S_{\mathrm{f}} \times \tilde{S}_{\mathrm{f}}\end{array}$$

Then, by (Y4), conclude:

$$\left[\left[s \xrightarrow{t}_{\mathtt{I}}{}^{+} s' \textbf{ implies } \left[\begin{array}{l}\left[\tilde{s} \xrightarrow{\sim t}_{\mathtt{I}}{}^{+} \tilde{s}' \textbf{ and } s\ R\ \tilde{s} \textbf{ and } s'\ R\ \tilde{s}'\right] \\ \textbf{for some } \tilde{s}, s\end{array}\right]\right] \textbf{ for all } \tilde{s}, \tilde{s}'\right]$$

$$\begin{array}{l}\textbf{and } \left[\left[s \in \blacksquare^{t}_{\mathrm{f}} \textbf{ implies } \left[\left[\tilde{s} \in \tilde{\blacksquare}^{t}_{\mathrm{f}} \textbf{ and } s\ R\ \tilde{s}\right] \textbf{ for some } \tilde{s}\right]\right] \textbf{ for all } s, t\right] \\ \textbf{and } \left[\left[s \in \square^{t}_{\mathrm{f}} \textbf{ implies } \left[\left[\tilde{s} \in \tilde{\square}^{t}_{\mathrm{f}} \textbf{ and } s\ R\ \tilde{s}\right] \textbf{ for some } \tilde{s}\right]\right] \textbf{ for all } s, t\right] \\ \textbf{and } R \subseteq S_{\mathrm{f}} \times \tilde{S}_{\mathrm{f}} \textbf{ and } S_{\mathrm{f}} \subseteq \mathrm{Dom}(R)\end{array}$$

Then, by Def. 5 of $\preceq$, conclude $(S_{\mathrm{f}}, \blacksquare_{\mathrm{f}}, \square_{\mathrm{f}}, \xrightarrow{}_{\mathtt{I}_{\mathrm{f}}}) \preceq_R (\tilde{S}_{\mathrm{f}}, \tilde{\blacksquare}_{\mathrm{f}}, \tilde{\square}_{\mathrm{f}}, \xrightarrow{\sim}_{\mathtt{I}_{\mathrm{f}}})$. Then, by Prop. 4, conclude $[\![\gamma]\!] \preceq_R (\tilde{S}_{\mathrm{f}}, \tilde{\blacksquare}_{\mathrm{f}}, \tilde{\square}_{\mathrm{f}}, \xrightarrow{\sim}_{\mathtt{I}_{\mathrm{f}}})$. Then, by Prop. 5, conclude:

$$[\![\gamma]\!] \preceq_R [\![\mathsf{flip}(\gamma)]\!]$$

□

Lemma 2. $[\![\mathsf{flip}(\gamma)]\!] \preceq_R [\![\gamma]\!]$

Proof. Prove the lemma by a reduction similar to the proof of Lemma 1. □

C.3 Proof

Proof. Observe:

(Z1) Conclude $\{(s,s) \mid s \in S_{\mathrm{f}}\} = \{(s,s) \mid s \in S_{\mathrm{f}}\}^{-1}$. Then, by Prop. 10, conclude $R = R^{-1}$. Then, by Lemma 2, conclude $\big[R = R^{-1}$ **and** $[\![\mathsf{flip}(\gamma)]\!] \preceq_R [\![\gamma]\!]\big]$. Then, conclude $[\![\mathsf{flip}(\gamma)]\!] \preceq_{R^{-1}} [\![\gamma]\!]$.

Prove the theorem by the following reduction. By Lemma 1, conclude:

$$[\![\gamma]\!] \preceq_R [\![\mathsf{flip}(\gamma)]\!]$$

Then, by (Z1), conclude $\big[[\![\gamma]\!] \preceq_R [\![\mathsf{flip}(\gamma)]\!]$ **and** $[\![\mathsf{flip}(\gamma)]\!] \preceq_{R^{-1}} [\![\gamma]\!]\big]$. Then, by Def. 6 of $\sim$, conclude $[\![\gamma]\!] \sim_R [\![\mathsf{flip}(\gamma)]\!]$. Then, by Def. 6 of $\sim$, conclude:

$$[\![\gamma]\!] \sim [\![\mathsf{flip}(\gamma)]\!]$$

□

D Proof of Theorem 4

The scope of auxiliary propositions and auxiliary lemmas in this section is limited to this section.

D.1 Sketch

Let $R = \{(s, s) \mid s \in S_\mathrm{f} \setminus \{s\}\} \cup \{(\overline{s}, \overline{s}_1), (\overline{s}, \overline{s}_2)\}$ be a candidate bisimulation relation (Prop. 21).

First, we prove $[\![\gamma]\!] \preceq_R [\![\mathsf{split}(\gamma; \overline{s}, \overline{s}_1, \overline{s}_2, T_1)]\!]$ (Lemma 7). To do this, we first show that information flows resulting from pushes from a service s to a service s' in γ can be R-mimicked by $\mathsf{split}(\gamma; \overline{s}, \overline{s}_1, \overline{s}_2, T_1)$ (Lemma 3). The main step in this proof is that we show that information flows resulting from pushes in γ that involve $\overline{s}$ (the service to split) can be mimicked with information flows that involve $\overline{s}_1$ or $\overline{s}_2$ in $\mathsf{split}(\gamma; \overline{s}, \overline{s}_1, \overline{s}_2, T_1)$; information flows resulting from pushes in γ that do *not* involve $\overline{s}$ are identical in $\mathsf{split}(\gamma; \overline{s}, \overline{s}_1, \overline{s}_2, T_1)$. Similarly, information flows resulting from pulls in γ can be R-mimicked by $\mathsf{split}(\gamma; \overline{s}, \overline{s}_1, \overline{s}_2, T_1)$ (Lemma 4).

Subsequently, we show that production responsibilities in γ are R-mimicked by production responsibilities in $\mathsf{split}(\gamma; \overline{s}, \overline{s}_1, \overline{s}_2, T_1)$ (Lemma 5). The main step in this proof is that we show that production responsibilities of $\overline{s}$ are mimicked by $\overline{s}_1$ and $\overline{s}_2$; production responsibilities of other services in γ are identical in $\mathsf{split}(\gamma; \overline{s}, \overline{s}_1, \overline{s}_2, T_1)$. Similarly, consumption responsibilities in γ can be R-mimicked by consumption responsibilities in $\mathsf{split}(\gamma; \overline{s}, \overline{s}_1, \overline{s}_2, T_1)$ (Lemma 6).

Using antecedent $\mathsf{CanSplit}(\gamma; \overline{s}, \overline{s}_1, \overline{s}_2, T_1)$, Lemmas 3, 4, 5, and 6, we subsequently prove Lemma 7. A proof for $[\![\mathsf{split}(\gamma; \overline{s}, \overline{s}_1, \overline{s}_2, T_1)]\!] \preceq_{R^{-1}} [\![\gamma]\!]$ is similar (Lemma 8). The theorem subsequently follows by Def. 6 of $\sim$ (i.e., the candidate bisimulation relation R is, indeed, a bisimulation relation).

D.2 Propositions and Lemmas

Antecedent:

Proposition 12. $\mathsf{CanSplit}(\gamma)$

By Def. 1:

Proposition 13. $\gamma = (S_\mathrm{c}, \blacksquare_\mathrm{c}, \square_\mathrm{c}, \longrightarrow, \multimap)$

Proposition 14. $\mathsf{split}(\gamma; \overline{s}, \overline{s}_1, \overline{s}_2, T_1) = (\tilde{S}_\mathrm{c}, \tilde{\blacksquare}_\mathrm{c}, \tilde{\square}_\mathrm{c}, \xrightarrow{\sim}, \overset{\sim}{\multimap})$

Proposition 15.
$$\begin{array}{ll} & \left[\left[s \in \blacksquare_\mathrm{c}^t \text{ implies } s \in S_\mathrm{c}\right] \text{ for all } s, t\right] \\ \text{and} & \left[\left[s \in \square_\mathrm{c}^t \text{ implies } s \in S_\mathrm{c}\right] \text{ for all } s, t\right] \\ \text{and} & \left[\left[s \xrightarrow{t} s' \text{ implies } s, s' \in S_\mathrm{c}\right] \text{ for all } s, s', t\right] \\ \text{and} & \left[\left[s \xrightarrow{t} s' \text{ implies } s, s' \in S_\mathrm{c}\right] \text{ for all } s, s', t\right] \end{array}$$

Proposition 16. $\big[\big[s \in \tilde{\blacksquare}^t_{\mathrm{c}}$ **implies** $s \in \tilde{S}_{\mathrm{c}}\big]$ **for all** $s, t\big]$
and $\big[\big[s \in \tilde{\square}^t_{\mathrm{c}}$ **implies** $s \in \tilde{S}_{\mathrm{c}}\big]$ **for all** $s, t\big]$
and $\big[\big[s \xrightarrow{\sim t} s'$ **implies** $s, s' \in \tilde{S}_{\mathrm{c}}\big]$ **for all** $s, s', t\big]$
and $\big[\big[s \xrightarrow{\sim t}\!\!\!\!\prec\ s'$ **implies** $s, s' \in \tilde{S}_{\mathrm{c}}\big]$ **for all** $s, s', t\big]$

By Def. 3:

Proposition 17. $[\![\gamma]\!] = (S_{\mathrm{f}}, \blacksquare_{\mathrm{f}}, \square_{\mathrm{f}}, \longrightarrow\!\!\!\!\!\!\mathrm{I})$

Proposition 18. $[\![\mathsf{split}(\gamma; \overline{s}, \overline{s}_1, \overline{s}_2, T_1)]\!] = (\tilde{S}_{\mathrm{f}}, \tilde{\blacksquare}_{\mathrm{f}}, \tilde{\square}_{\mathrm{f}}, \xrightarrow{\sim}\!\!\!\!\!\!\mathrm{I})$

Proposition 19. $S_{\mathrm{f}} = S_{\mathrm{c}}$
and $\blacksquare_{\mathrm{f}} = \blacksquare_{\mathrm{c}}$
and $\square_{\mathrm{f}} = \square_{\mathrm{c}}$
and $\longrightarrow\!\!\!\!\!\!\mathrm{I} = \{t \mapsto \xrightarrow{t} \cup (\xrightarrow{t}\!\!\!\!\prec)^{\text{-}1} \mid t \in \mathbb{T}\}$

Proposition 20. $\tilde{S}_{\mathrm{f}} = \tilde{S}_{\mathrm{c}}$
and $\tilde{\blacksquare}_{\mathrm{f}} = \tilde{\blacksquare}_{\mathrm{c}}$
and $\tilde{\square}_{\mathrm{f}} = \tilde{\square}_{\mathrm{c}}$
and $\xrightarrow{\sim}\!\!\!\!\!\!\mathrm{I} = \{t \mapsto \xrightarrow{\sim t} \cup (\xrightarrow{\sim t}\!\!\!\!\prec)^{\text{-}1} \mid t \in \mathbb{T}\}$

Proposition 21. $R = \{(s, s) \mid s \in S_{\mathrm{f}} \setminus \{\overline{s}\}\} \cup \{(\overline{s}, \overline{s}_1), (\overline{s}, \overline{s}_2)\}$

By Fig. 5:

Proposition 22. $\overline{s} \in S_{\mathrm{c}}$ **and** $\overline{s}_1, \overline{s}_2 \notin S_{\mathrm{c}}$

By Fig. 6:

Proposition 23. $\tilde{S}_{\mathrm{c}} = (S_{\mathrm{c}} \setminus \{\overline{s}\}) \cup \{\overline{s}_1, \overline{s}_2\}$
and $\tilde{\blacksquare}_{\mathrm{c}} = \{t \mapsto \blacksquare^t_{\mathrm{c}}[\overline{s}_1/\{\overline{s}\}] \mid t \in T_1\}$
$\cup \{t \mapsto \blacksquare^t_{\mathrm{c}}[\overline{s}_2/\{\overline{s}\}] \mid t \notin T_1\}$
and $\tilde{\square}_{\mathrm{c}} = \{t \mapsto \square^t_{\mathrm{c}}[\overline{s}_1/\{\overline{s}\}] \mid t \in T_1\}$
$\cup \{t \mapsto \square^t_{\mathrm{c}}[\overline{s}_2/\{\overline{s}\}] \mid t \notin T_1\}$
and $\xrightarrow{\sim} = \{t \mapsto \{(\overline{s}_1, \overline{s})\} \circ \xrightarrow{t} \circ \{(\overline{s}, \overline{s}_1)\} \mid t \in T_1\}$
$\cup \{t \mapsto \{(\overline{s}_2, \overline{s})\} \circ \xrightarrow{t} \circ \{(\overline{s}, \overline{s}_2)\} \mid t \notin T_1\}$
and $\xrightarrow{\sim}\!\!\!\!\prec = \{t \mapsto \{(\overline{s}_1, \overline{s})\} \circ \xrightarrow{t}\!\!\!\!\prec \circ \{(\overline{s}, \overline{s}_1)\} \mid t \in T_1\}$
$\cup \{t \mapsto \{(\overline{s}_2, \overline{s})\} \circ \xrightarrow{t}\!\!\!\!\prec \circ \{(\overline{s}, \overline{s}_2)\} \mid t \notin T_1\}$

Lemma 3. $s \xrightarrow{t} s'$ **implies** $\left[\begin{array}{c}[\tilde{s} \xrightarrow{\sim t_1} \tilde{s}' \textbf{ and } s\ R\ \tilde{s} \textbf{ and } s'\ R\ \tilde{s}'] \\ \textbf{for some } \tilde{s}, \tilde{s}'\end{array}\right]$

Proof. Assumptions:

(A1) $s \xrightarrow{t} s'$

Observe:

(Z1) Suppose $[s = \overline{s} \textbf{ and } s' = \overline{s}]$. Then, by (A1), conclude:

$$s = \overline{s} \textbf{ and } s' = \overline{s} \textbf{ and } s \xrightarrow{t} s'$$

Then, conclude $\overline{s} \xrightarrow{t} \overline{s}$. Then, conclude $(\overline{s}_1, \overline{s}_1) \in \{(\overline{s}_1, \overline{s})\} \circ \xrightarrow{t} \circ \{(\overline{s}, \overline{s}_1)\}$.

(Z2) Suppose $[s = \overline{s} \textbf{ and } s' = \overline{s} \textbf{ and } t \in T_1]$. Then, by (Z1), conclude:

$$(\overline{s}_1, \overline{s}_1) \in \{(\overline{s}_1, \overline{s})\} \circ \xrightarrow{t} \circ \{(\overline{s}, \overline{s}_1)\} \textbf{ and } t \in T_1$$

Then, conclude $(\overline{s}_1, \overline{s}_1) \in \{\hat{t} \mapsto \{(\overline{s}_1, \overline{s})\} \circ \xrightarrow{\hat{t}} \circ \{(\overline{s}, \overline{s}_1)\} \mid \hat{t} \in T_1\}(t)$. Then, conclude:

$$(\overline{s}_1, \overline{s}_1) \in \left(\begin{array}{c}\{\hat{t} \mapsto \{(\overline{s}_1, \overline{s})\} \circ \xrightarrow{\hat{t}} \circ \{(\overline{s}, \overline{s}_1)\} \mid \hat{t} \in T_1\} \\ \cup \{\hat{t} \mapsto \{(\overline{s}_2, \overline{s})\} \circ \xrightarrow{\hat{t}} \circ \{(\overline{s}, \overline{s}_2)\} \mid \hat{t} \notin T_1\}\end{array}\right)(t)$$

Then, by Prop. 23, conclude $\overline{s}_1 \xrightarrow{\sim t} \overline{s}_1$.

(Z3) Suppose $[s = \overline{s} \textbf{ and } s' = \overline{s} \textbf{ and } t \notin T_1]$. Then, by a reduction similar to (Z2), conclude $\overline{s}_2 \xrightarrow{\sim t} \overline{s}_2$.

(Z4) Conclude $(\overline{s}, \overline{s}_1) \in \{(\overline{s}, \overline{s}_1), (\overline{s}, \overline{s}_2)\}$. Then, conclude:

$$(\overline{s}, \overline{s}_1) \in \{(s, s) \mid s \in S_{\mathrm{f}} \setminus \{\overline{s}\}\} \cup \{(\overline{s}, \overline{s}_1), (\overline{s}, \overline{s}_2)\}$$

Then, by Prop. 21, conclude $\overline{s}\ R\ \overline{s}_1$.

(Z5) By a reduction similar to (Z4), conclude $\overline{s}\ R\ \overline{s}_2$.

(Z6) Suppose:

$$\hat{s} = \overline{s} \textbf{ for some } \hat{s}$$

Then, by (Z4), conclude $[\hat{s} = \overline{s} \textbf{ and } \overline{s}\ R\ \overline{s}_1]$. Then, conclude $\hat{s}\ R\ \overline{s}_1$.

(Z7) Suppose:

$$\hat{s} = \overline{s} \textbf{ for some } \hat{s}$$

Then, by a reduction similar to (Z6), conclude $\hat{s}\ R\ \overline{s}_2$.

(Z8) Suppose $[s = \overline{s} \textbf{ and } s' = \overline{s} \textbf{ and } t \in T_1]$. Then, by (Z2), conclude:

$$s = \overline{s} \textbf{ and } s' = \overline{s} \textbf{ and } \overline{s}_1 \xrightarrow{\sim t} \overline{s}_1$$

Then, by (Z6), conclude $[\overline{s}_1 \xrightarrow{\sim t} \overline{s}_1 \textbf{ and } s\ R\ \overline{s}_1 \textbf{ and } s'\ R\ \overline{s}_1]$. Then, conclude $[[\tilde{s} \xrightarrow{\sim t} \tilde{s}' \textbf{ and } s\ R\ \tilde{s} \textbf{ and } s'\ R\ \tilde{s}'] \textbf{ for some } \tilde{s}, \tilde{s}']$.

(Z9) Suppose $\big[s = \overline{s}$ **and** $s' = \overline{s}$ **and** $t \notin T_1\big]$. Then, by a reduction similar to (Z8), conclude $\big[\big[\tilde{s} \xrightarrow{\sim t} \tilde{s}'$ **and** $s\ R\ \tilde{s}$ **and** $s'\ R\ \tilde{s}'\big]$ **for some** $\tilde{s}, \tilde{s}'\big]$.

(Z0) Suppose $\big[s = \overline{s}$ **and** $s' \neq \overline{s}\big]$. Then, by (A1), conclude:

$$s = \overline{s} \text{ and } s' \neq \overline{s} \text{ and } s \xrightarrow{t} s'$$

Then, conclude $\big[\overline{s} \xrightarrow{t} s'$ **and** $s' \neq \overline{s}\big]$. Then, conclude:

$$(\overline{s}_1, s') \in \{(\overline{s}_1, \overline{s})\} \circ \xrightarrow{t} \circ \{(\overline{s}, \overline{s}_1)\}$$

(Y1) Suppose $\big[s = \overline{s}$ **and** $s' \neq \overline{s}$ **and** $t \in T_1\big]$. Then, by (Z0), conclude:

$$(\overline{s}_1, s') \in \{(\overline{s}_1, \overline{s})\} \circ \xrightarrow{t} \circ \{(\overline{s}, \overline{s}_1)\} \text{ and } t \in T_1$$

Then, conclude $(\overline{s}_1, s') \in \{\hat{t} \mapsto \{(\overline{s}_1, \overline{s})\} \circ \xrightarrow{\hat{t}} \circ \{(\overline{s}, \overline{s}_1)\} \mid \hat{t} \in T_1\}(t)$. Then, conclude:

$$(\overline{s}_1, s') \in \left(\begin{array}{c} \{\hat{t} \mapsto \{(\overline{s}_1, \overline{s})\} \circ \xrightarrow{\hat{t}} \circ \{(\overline{s}, \overline{s}_1)\} \mid \hat{t} \in T_1\} \\ \cup \{\hat{t} \mapsto \{(\overline{s}_2, \overline{s})\} \circ \xrightarrow{\hat{t}} \circ \{(\overline{s}, \overline{s}_2)\} \mid \hat{t} \notin T_1\} \end{array} \right)(t)$$

Then, by Prop. 23, conclude $\overline{s}_1 \xrightarrow{\sim t} s'$.

(Y2) Suppose $\big[s = \overline{s}$ **and** $s' \neq \overline{s}$ **and** $t \notin T_1\big]$. Then, by a reduction similar to (Y1), conclude $\overline{s}_2 \xrightarrow{\sim t} s'$.

(Y3) Suppose $\big[s \neq \overline{s}$ **and** $s' = \overline{s}$ **and** $t \in T_1\big]$. Then, by a reduction similar to (Y1), conclude $s \xrightarrow{\sim t} \overline{s}_1$.

(Y4) Suppose $\big[s \neq \overline{s}$ **and** $s' = \overline{s}$ **and** $t \notin T_1\big]$. Then, by a reduction similar to (Y1), conclude $s \xrightarrow{\sim t} \overline{s}_2$.

(Y5) From (A1), conclude $s \xrightarrow{t} s'$. Then, by Prop. 15, conclude $s, s' \in S_\mathrm{c}$.

(Y6) Suppose $s' \neq \overline{s}$. Then, by (Y5), conclude $\big[s' \in S_\mathrm{c}$ **and** $s' \neq \overline{s}\big]$. Then, conclude $\big[s' \in S_\mathrm{c}$ **and** $s' \notin \{\overline{s}\}\big]$. Then, conclude $s' \in S_\mathrm{c} \setminus \{\overline{s}\}$. Then, by Prop. 19, conclude $s' \in S_\mathrm{f} \setminus \{\overline{s}\}$. Then, conclude $(s', s') \in \{(\hat{s}, \hat{s}) \mid \hat{s} \in S_\mathrm{f} \setminus \{\overline{s}\}\}$. Then, conclude $(s', s') \in \{(\hat{s}, \hat{s}) \mid \hat{s} \in S_\mathrm{f} \setminus \{\overline{s}\}\} \cup \{(\overline{s}, \overline{s}_1), (\overline{s}, \overline{s}_2)\}$. Then, by Prop. 21, conclude $s'\ R\ s'$.

(Y7) Suppose $s \neq \overline{s}$. Then, by a reduction similar to (Y6), conclude $s\ R\ s$.

(Y8) Suppose $\big[s = \overline{s}$ **and** $s' \neq \overline{s}\big]$. Then, by (Y6), conclude $\big[s = \overline{s}$ **and** $s'\ R\ s'\big]$. Then, by (Z6), conclude $\big[s\ R\ \overline{s}_1$ **and** $s'\ R\ s'\big]$.

(Y9) Suppose $\big[s = \overline{s}$ **and** $s' \neq \overline{s}\big]$. Then, by a reduction similar to (Y8), conclude $\big[s\ R\ \overline{s}_2$ **and** $s'\ R\ s'\big]$.

(Y0) Suppose $\big[s \neq \overline{s}$ **and** $s' = \overline{s}\big]$. Then, by a reduction similar to (Y8), conclude $\big[s\ R\ s$ **and** $s'\ R\ \overline{s}_1\big]$.

(X1) Suppose $\big[s \neq \overline{s}$ **and** $s' = \overline{s}\big]$. Then, by a reduction similar to (Y8), conclude $\big[s\ R\ s$ **and** $s'\ R\ \overline{s}_2\big]$.

(X2) Suppose $\big[s = \overline{s}$ **and** $s' \neq \overline{s}$ **and** $t \in T_1\big]$. Then, by (Y1), conclude:

$$s = \overline{s} \text{ and } s' \neq \overline{s} \text{ and } \overline{s}_1 \xrightarrow{\sim t} s'$$

Then, by (Y8), conclude $\big[\overline{s}_1 \xrightarrow{\sim t} s'$ **and** $s\ R\ \overline{s}_1$ **and** $s'\ R\ s'\big]$. Then, conclude $\big[\big[\tilde{s} \xrightarrow{\sim t} \tilde{s}'$ **and** $s\ R\ \tilde{s}$ **and** $s'\ R\ \tilde{s}'\big]$ **for some** $\tilde{s}, \tilde{s}'\big]$.

(X3) Suppose $\big[s = \overline{s}$ **and** $s' \neq \overline{s}$ **and** $t \notin T_1\big]$. Then, by a reduction similar to (X2), conclude $\big[\big[\tilde{s} \xrightarrow{\sim t} \tilde{s}'$ **and** $s\ R\ \tilde{s}$ **and** $s'\ R\ \tilde{s}'\big]$ **for some** $\tilde{s}, \tilde{s}'\big]$.

(X4) Suppose $\big[s \neq \overline{s}$ **and** $s' = \overline{s}$ **and** $t \in T_1\big]$. Then, by a reduction similar to (X2), conclude $\big[\big[\tilde{s} \xrightarrow{\sim t} \tilde{s}'$ **and** $s\ R\ \tilde{s}$ **and** $s'\ R\ \tilde{s}'\big]$ **for some** $\tilde{s}, \tilde{s}'\big]$.

(X5) Suppose $\big[s \neq \overline{s}$ **and** $s' = \overline{s}$ **and** $t \notin T_1\big]$. Then, by a reduction similar to (X2), conclude $\big[\big[\tilde{s} \xrightarrow{\sim t} \tilde{s}'$ **and** $s\ R\ \tilde{s}$ **and** $s'\ R\ \tilde{s}'\big]$ **for some** $\tilde{s}, \tilde{s}'\big]$.

(X6) Suppose $\big[s \neq \overline{s}$ **and** $s' \neq \overline{s}\big]$. Then, by (A1), conclude:

$$s \neq \overline{s} \text{ and } s' \neq \overline{s} \text{ and } s \xrightarrow{t} s'$$

Then, conclude $(s, s') \in \{(\overline{s}_1, \overline{s})\} \circ \xrightarrow{t} \circ \{(\overline{s}, \overline{s}_1)\}$.

(X7) Suppose $\big[s \neq \overline{s}$ **and** $s' \neq \overline{s}$ **and** $t \in T_1\big]$. Then, by (X6), conclude:

$$(s, s') \in \{(\overline{s}_1, \overline{s})\} \circ \xrightarrow{t} \circ \{(\overline{s}, \overline{s}_1)\} \text{ and } t \in T_1$$

Then, conclude $(s, s') \in \{\hat{t} \mapsto \{(\overline{s}_1, \overline{s})\} \circ \xrightarrow{\hat{t}} \circ \{(\overline{s}, \overline{s}_1)\} \mid \hat{t} \in T_1\}(t)$. Then, conclude:

$$(s, s') \in \left(\begin{array}{c} \{\hat{t} \mapsto \{(\overline{s}_1, \overline{s})\} \circ \xrightarrow{\hat{t}} \circ \{(\overline{s}, \overline{s}_1)\} \mid \hat{t} \in T_1\} \\ \cup \{\hat{t} \mapsto \{(\overline{s}_2, \overline{s})\} \circ \xrightarrow{\hat{t}} \circ \{(\overline{s}, \overline{s}_2)\} \mid \hat{t} \notin T_1\} \end{array} \right)(t)$$

Then, by Prop. 23, conclude $s \xrightarrow{\sim t} s'$.

(X8) Suppose $\big[s \neq \overline{s}$ **and** $s' \neq \overline{s}$ **and** $t \notin T_1\big]$. Then, by a reduction similar to (Y1), conclude $s \xrightarrow{\sim t} s'$.

(X9) Suppose $\big[s \neq \overline{s}$ **and** $s' \neq \overline{s}$ **and** $t \in T_1\big]$. Then, by (X7), conclude:

$$s \xrightarrow{t} s' \text{ and } s \neq \overline{s} \text{ and } s' \neq \overline{s} \text{ and } s \xrightarrow{\sim t} s'$$

Then, by (Y6)(Y7), conclude $\big[s \xrightarrow{\sim t} s'$ **and** $s\ R\ s$ **and** $s'\ R\ s'\big]$. Then, conclude $\big[\big[\tilde{s} \xrightarrow{\sim t} \tilde{s}'$ **and** $s\ R\ \tilde{s}$ **and** $s'\ R\ \tilde{s}'\big]$ **for some** $\tilde{s}, \tilde{s}'\big]$.

(X0) Suppose $\big[s \neq \overline{s}$ **and** $s' \neq \overline{s}$ **and** $t \notin T_1\big]$. Then, by a reduction similar to (X9), conclude $\big[\big[\tilde{s} \xrightarrow{\sim t} \tilde{s}'$ **and** $s\ R\ \tilde{s}$ **and** $s'\ R\ \tilde{s}'\big]$ **for some** $\tilde{s}, \tilde{s}'\big]$.

(W1) Suppose:

$$\tilde{s} \xrightarrow{\sim t} \tilde{s}' \ \ \textbf{for some} \ \ \tilde{s}, \tilde{s}'$$

Then, conclude $(\tilde{s}, \tilde{s}') \in \xrightarrow{\sim t} \cup \, (\overset{\sim t}{\relbar\!\!\prec})^{\text{-}1}$. Then, conclude:

$$(\tilde{s}, \tilde{s}') \in \{\hat{t} \mapsto \xrightarrow{\sim \hat{t}} \cup \, (\overset{\sim \hat{t}}{\relbar\!\!\prec})^{\text{-}1} \mid \hat{t} \in \mathbb{T}\}(t)$$

Then, by Prop. 20, conclude $\tilde{s} \overset{\sim t}{\relbar\!\!\!\mathrel{\text{I}}} \tilde{s}'$.

Proof the lemma by the following reduction. Conclude:

$$\big[s = \overline{s} \ \ \textbf{or} \ \ s \neq \overline{s}\big] \ \ \textbf{and} \ \ \big[s' = \overline{s} \ \ \textbf{or} \ \ s' \neq \overline{s}\big] \ \ \textbf{and} \ \ \big[t \in T_1 \ \ \textbf{or} \ \ t \notin T_1\big]$$

Then, conclude:

$$\begin{array}{ll}
 & \big[s = \overline{s} \ \ \textbf{and} \ \ s' = \overline{s} \ \ \textbf{and} \ \ t \in T_1\big] \\
\textbf{or} & \big[s = \overline{s} \ \ \textbf{and} \ \ s' = \overline{s} \ \ \textbf{and} \ \ t \notin T_1\big] \\
\textbf{or} & \big[s = \overline{s} \ \ \textbf{and} \ \ s' \neq \overline{s} \ \ \textbf{and} \ \ t \in T_1\big] \\
\textbf{or} & \big[s = \overline{s} \ \ \textbf{and} \ \ s' \neq \overline{s} \ \ \textbf{and} \ \ t \notin T_1\big] \\
\textbf{or} & \big[s \neq \overline{s} \ \ \textbf{and} \ \ s' = \overline{s} \ \ \textbf{and} \ \ t \in T_1\big] \\
\textbf{or} & \big[s \neq \overline{s} \ \ \textbf{and} \ \ s' = \overline{s} \ \ \textbf{and} \ \ t \notin T_1\big] \\
\textbf{or} & \big[s \neq \overline{s} \ \ \textbf{and} \ \ s' \neq \overline{s} \ \ \textbf{and} \ \ t \in T_1\big] \\
\textbf{or} & \big[s \neq \overline{s} \ \ \textbf{and} \ \ s' \neq \overline{s} \ \ \textbf{and} \ \ t \notin T_1\big]
\end{array}$$

Then, by (Z8)(Z9), conclude:

$$\begin{array}{ll}
 & \big[\big[\tilde{s} \xrightarrow{\sim t} \tilde{s}' \ \ \textbf{and} \ \ s \mathrel{R} \tilde{s} \ \ \textbf{and} \ \ s' \mathrel{R} \tilde{s}'\big] \ \ \textbf{for some} \ \ \tilde{s}, \tilde{s}'\big] \\
\textbf{or} & \big[\big[\tilde{s} \xrightarrow{\sim t} \tilde{s}' \ \ \textbf{and} \ \ s \mathrel{R} \tilde{s} \ \ \textbf{and} \ \ s' \mathrel{R} \tilde{s}'\big] \ \ \textbf{for some} \ \ \tilde{s}, \tilde{s}'\big] \\
\textbf{or} & \big[s = \overline{s} \ \ \textbf{and} \ \ s' \neq \overline{s} \ \ \textbf{and} \ \ t \in T_1\big] \\
\textbf{or} & \big[s = \overline{s} \ \ \textbf{and} \ \ s' \neq \overline{s} \ \ \textbf{and} \ \ t \notin T_1\big] \\
\textbf{or} & \big[s \neq \overline{s} \ \ \textbf{and} \ \ s' = \overline{s} \ \ \textbf{and} \ \ t \in T_1\big] \\
\textbf{or} & \big[s \neq \overline{s} \ \ \textbf{and} \ \ s' = \overline{s} \ \ \textbf{and} \ \ t \notin T_1\big] \\
\textbf{or} & \big[s \neq \overline{s} \ \ \textbf{and} \ \ s' \neq \overline{s} \ \ \textbf{and} \ \ t \in T_1\big] \\
\textbf{or} & \big[s \neq \overline{s} \ \ \textbf{and} \ \ s' \neq \overline{s} \ \ \textbf{and} \ \ t \notin T_1\big]
\end{array}$$

Then, by (X2)(X3)(X4)(X5), conclude:

$$\begin{array}{ll}
 & \big[\big[\tilde{s} \xrightarrow{\sim t} \tilde{s}' \ \ \textbf{and} \ \ s \mathrel{R} \tilde{s} \ \ \textbf{and} \ \ s' \mathrel{R} \tilde{s}'\big] \ \ \textbf{for some} \ \ \tilde{s}, \tilde{s}'\big] \\
\textbf{or} & \big[\big[\tilde{s} \xrightarrow{\sim t} \tilde{s}' \ \ \textbf{and} \ \ s \mathrel{R} \tilde{s} \ \ \textbf{and} \ \ s' \mathrel{R} \tilde{s}'\big] \ \ \textbf{for some} \ \ \tilde{s}, \tilde{s}'\big] \\
\textbf{or} & \big[\big[\tilde{s} \xrightarrow{\sim t} \tilde{s}' \ \ \textbf{and} \ \ s \mathrel{R} \tilde{s} \ \ \textbf{and} \ \ s' \mathrel{R} \tilde{s}'\big] \ \ \textbf{for some} \ \ \tilde{s}, \tilde{s}'\big] \\
\textbf{or} & \big[\big[\tilde{s} \xrightarrow{\sim t} \tilde{s}' \ \ \textbf{and} \ \ s \mathrel{R} \tilde{s} \ \ \textbf{and} \ \ s' \mathrel{R} \tilde{s}'\big] \ \ \textbf{for some} \ \ \tilde{s}, \tilde{s}'\big] \\
\textbf{or} & \big[\big[\tilde{s} \xrightarrow{\sim t} \tilde{s}' \ \ \textbf{and} \ \ s \mathrel{R} \tilde{s} \ \ \textbf{and} \ \ s' \mathrel{R} \tilde{s}'\big] \ \ \textbf{for some} \ \ \tilde{s}, \tilde{s}'\big] \\
\textbf{or} & \big[\big[\tilde{s} \xrightarrow{\sim t} \tilde{s}' \ \ \textbf{and} \ \ s \mathrel{R} \tilde{s} \ \ \textbf{and} \ \ s' \mathrel{R} \tilde{s}'\big] \ \ \textbf{for some} \ \ \tilde{s}, \tilde{s}'\big] \\
\textbf{or} & \big[s \xrightarrow{t} s' \ \ \textbf{and} \ \ s \neq \overline{s} \ \ \textbf{and} \ \ s' \neq \overline{s} \ \ \textbf{and} \ \ t \in T_1\big] \\
\textbf{or} & \big[s \xrightarrow{t} s' \ \ \textbf{and} \ \ s \neq \overline{s} \ \ \textbf{and} \ \ s' \neq \overline{s} \ \ \textbf{and} \ \ t \notin T_1\big]
\end{array}$$

Then, by (X9)(X0), conclude:

$$\begin{array}{rl}
 & \Big[\big[\tilde{s} \xrightarrow{\sim t} \tilde{s}' \ \textbf{and} \ s\ R\ \tilde{s} \ \textbf{and} \ s'\ R\ \tilde{s}'\big] \ \textbf{for some} \ \tilde{s}, \tilde{s}'\Big] \\
\textbf{or} & \Big[\big[\tilde{s} \xrightarrow{\sim t} \tilde{s}' \ \textbf{and} \ s\ R\ \tilde{s} \ \textbf{and} \ s'\ R\ \tilde{s}'\big] \ \textbf{for some} \ \tilde{s}, \tilde{s}'\Big] \\
\textbf{or} & \Big[\big[\tilde{s} \xrightarrow{\sim t} \tilde{s}' \ \textbf{and} \ s\ R\ \tilde{s} \ \textbf{and} \ s'\ R\ \tilde{s}'\big] \ \textbf{for some} \ \tilde{s}, \tilde{s}'\Big] \\
\textbf{or} & \Big[\big[\tilde{s} \xrightarrow{\sim t} \tilde{s}' \ \textbf{and} \ s\ R\ \tilde{s} \ \textbf{and} \ s'\ R\ \tilde{s}'\big] \ \textbf{for some} \ \tilde{s}, \tilde{s}'\Big] \\
\textbf{or} & \Big[\big[\tilde{s} \xrightarrow{\sim t} \tilde{s}' \ \textbf{and} \ s\ R\ \tilde{s} \ \textbf{and} \ s'\ R\ \tilde{s}'\big] \ \textbf{for some} \ \tilde{s}, \tilde{s}'\Big] \\
\textbf{or} & \Big[\big[\tilde{s} \xrightarrow{\sim t} \tilde{s}' \ \textbf{and} \ s\ R\ \tilde{s} \ \textbf{and} \ s'\ R\ \tilde{s}'\big] \ \textbf{for some} \ \tilde{s}, \tilde{s}'\Big] \\
\textbf{or} & \Big[\big[\tilde{s} \xrightarrow{\sim t} \tilde{s}' \ \textbf{and} \ s\ R\ \tilde{s} \ \textbf{and} \ s'\ R\ \tilde{s}'\big] \ \textbf{for some} \ \tilde{s}, \tilde{s}'\Big] \\
\textbf{or} & \Big[\big[\tilde{s} \xrightarrow{\sim t} \tilde{s}' \ \textbf{and} \ s\ R\ \tilde{s} \ \textbf{and} \ s'\ R\ \tilde{s}'\big] \ \textbf{for some} \ \tilde{s}, \tilde{s}'\Big]
\end{array}$$

Then, conclude $\big[\big[\tilde{s} \xrightarrow{\sim t} \tilde{s}' \ \textbf{and} \ s\ R\ \tilde{s} \ \textbf{and} \ s'\ R\ \tilde{s}'\big] \ \textbf{for some} \ \tilde{s}, \tilde{s}'\big]$. Then, by (W1), conclude $\big[\big[\tilde{s} \overset{\sim t}{\relbar\joinrel\mathrel{\mathrm{I}}} \tilde{s}' \ \textbf{and} \ s\ R\ \tilde{s} \ \textbf{and} \ s'\ R\ \tilde{s}'\big] \ \textbf{for some} \ \tilde{s}, \tilde{s}'\big]$. ☐

Lemma 4. $s\ (\overset{t}{\relbar\joinrel\prec})^{\text{-}1}\ s'$ **implies** $\left[\begin{array}{c}\big[\tilde{s} \overset{\sim t}{\relbar\joinrel\mathrel{\mathrm{I}}} \tilde{s}' \ \textbf{and} \ s\ R\ \tilde{s} \ \textbf{and} \ s'\ R\ \tilde{s}'\big] \\ \textbf{for some} \ \tilde{s}, \tilde{s}'\end{array}\right]$

Proof. Prove the lemma by a reduction similar to the proof of Lemma 3

Lemma 5. $s \in \blacksquare^t_{\mathrm{f}}$ **implies** $\big[\big[\tilde{s} \in \tilde{\blacksquare}^t_{\mathrm{f}}$ **and** $s\ R\ \tilde{s}\big]$ **for some** $\tilde{s}\big]$

Proof. Assume:

(A1) $s \in \blacksquare^t_{\mathrm{f}}$

Observe:

(Z1) Suppose $s = \overline{s}$. Then, by (A1), conclude $\big[s = \overline{s}$ **and** $s \in \blacksquare^t_{\mathrm{f}}\big]$. Then, conclude $\overline{s} \in \blacksquare^t_{\mathrm{f}}$. Then, conclude $\blacksquare^t_{\mathrm{f}} \cap \{\overline{s}\} \neq \emptyset$. Then, conclude:

$$\blacksquare^t_{\mathrm{f}}[\overline{s}_1/\{\overline{s}\}] = (\blacksquare^t_{\mathrm{f}} \setminus \{\overline{s}\}) \cup \{\overline{s}_1\}$$

Then, conclude $\big[\blacksquare^t_{\mathrm{f}}[\overline{s}_1/\{\overline{s}\}] = (\blacksquare^t_{\mathrm{f}} \setminus \{\overline{s}\}) \cup \{\overline{s}_1\}$ **and** $\overline{s}_1 \in (\blacksquare^t_{\mathrm{f}} \setminus \{\overline{s}\}) \cup \{\overline{s}_1\}\big]$. Then, conclude $\overline{s}_1 \in \blacksquare^t_{\mathrm{f}}[\overline{s}_1/\{\overline{s}\}]$. Then, by Prop. 19, conclude $\overline{s}_1 \in \blacksquare^t_{\mathrm{c}}[\overline{s}_1/\{\overline{s}\}]$.

(Z2) Suppose $s = \overline{s}$. Then, by a reduction similar to (Z1), conclude $\overline{s}_2 \in \blacksquare^t_{\mathrm{c}}[\overline{s}_2/\{\overline{s}\}]$.

(Z3) Suppose $\big[s = \overline{s}$ **and** $t \in T_1\big]$. Then, by (Z1), conclude:

$$\overline{s}_1 \in \blacksquare^t_{\mathrm{c}}[\overline{s}_1/\{\overline{s}\}] \text{ and } t \in T_1$$

Then, conclude $\overline{s}_1 \in \{\hat{t} \mapsto \blacksquare^{\hat{t}}_{\mathrm{c}}[\overline{s}_1/\{\overline{s}\}] \mid \hat{t} \in T_1\}(t)$. Then, conclude:

$$\overline{s}_1 \in (\{\hat{t} \mapsto \blacksquare^{\hat{t}}_{\mathrm{c}}[\overline{s}_1/\{\overline{s}\}] \mid \hat{t} \in T_1\} \cup \{\hat{t} \mapsto \blacksquare^{\hat{t}}_{\mathrm{c}}[\overline{s}_2/\{\overline{s}\}] \mid \hat{t} \notin T_1\})(t)$$

Then, by Prop. 23, conclude $\overline{s}_1 \in \tilde{\blacksquare}^t_{\mathrm{c}}$. Then, by Prop. 20, conclude $\overline{s}_1 \in \tilde{\blacksquare}^t_{\mathrm{f}}$.

(Z4) Suppose $\big[s = \overline{s}$ **and** $t \notin T_1\big]$. Then, by a reduction similar to (Z3), conclude $\overline{s}_2 \in \tilde{\blacksquare}^t_{\mathrm{f}}$.

(Z5) Conclude $(\overline{s}, \overline{s}_1) \in \{(\overline{s}, \overline{s}_1), (\overline{s}, \overline{s}_2)\}$. Then, conclude:

$$(\overline{s}, \overline{s}_1) \in \{(s, s) \mid s \in S_{\mathrm{f}} \setminus \{\overline{s}\}\} \cup \{(\overline{s}, \overline{s}_1), (\overline{s}, \overline{s}_2)\}$$

Then, by Prop. 21, conclude $\overline{s}\ R\ \overline{s}_1$.

(Z6) By a reduction similar to (Z5), conclude $\overline{s}\ R\ \overline{s}_2$.

(Z7) Suppose $s = \overline{s}$. Then, by (Z5), conclude $\big[s = \overline{s}$ **and** $\overline{s}\ R\ \overline{s}_1\big]$. Then, conclude $s\ R\ \overline{s}_1$.

(Z8) Suppose $s = \overline{s}$. Then, by a reduction similar to (Z8), conclude $s\ R\ \overline{s}_2$.

(Z9) Suppose $\big[s = \overline{s}$ **and** $t \in T_1\big]$. Then, by (Z3), conclude:

$$s = \overline{s} \text{ and } \overline{s}_1 \in \tilde{\blacksquare}^t_{\mathrm{f}}$$

Then, by (Z7), conclude $\big[\overline{s}_1 \in \tilde{\blacksquare}^t_{\mathrm{f}}$ **and** $s\ R\ \overline{s}_1\big]$. Then, conclude:

$$\big[\tilde{s} \in \tilde{\blacksquare}^t_{\mathrm{f}} \text{ and } s\ R\ \tilde{s}\big] \text{ for some } \tilde{s}$$

Z0 Suppose $\big[s = \overline{s}\ \textbf{and}\ t \notin T_1\big]$. Then, by a reduction similar to Z9, conclude $\big[\big[\tilde{s} \in \tilde{\blacksquare}^t_{\mathrm{f}}\ \textbf{and}\ s\ R\ \tilde{s}\big]\ \textbf{for some}\ \tilde{s}\big]$.

Y1 Suppose $s \neq \overline{s}$. Then, conclude $s \notin \{\overline{s}\}$. Then, by A1, conclude:

$$s \notin \{\overline{s}\}\ \textbf{and}\ s \in \blacksquare^t_{\mathrm{f}}$$

Then, conclude $s \in \blacksquare^t_{\mathrm{f}} \setminus \{\overline{s}\}$. Then, conclude:

$$s \in \blacksquare^t_{\mathrm{f}} \setminus \{\overline{s}\}\ \textbf{and}\ \big[\blacksquare^t_{\mathrm{f}} \cap \{\overline{s}\} = \emptyset\ \textbf{or}\ \blacksquare^t_{\mathrm{f}} \cap \{\overline{s}\} \neq \emptyset\big]$$

Then, conclude:

$$\big[s \in \blacksquare^t_{\mathrm{f}} \setminus \{\overline{s}\}\ \textbf{and}\ \blacksquare^t_{\mathrm{f}} \cap \{\overline{s}\} = \emptyset\big]\ \textbf{or}\ \big[s \in \blacksquare^t_{\mathrm{f}} \setminus \{\overline{s}\}\ \textbf{and}\ \blacksquare^t_{\mathrm{f}} \cap \{\overline{s}\} \neq \emptyset\big]$$

Then, conclude:

$$\begin{matrix}\big[s \in \blacksquare^t_{\mathrm{f}} \setminus \{\overline{s}\}\ \textbf{and}\ \blacksquare^t_{\mathrm{f}}[\overline{s}_1/\{\overline{s}\}] = \blacksquare^t_{\mathrm{f}} \setminus \{\overline{s}\}\big] \\ \textbf{or}\ \big[s \in \blacksquare^t_{\mathrm{f}} \setminus \{\overline{s}\}\ \textbf{and}\ \blacksquare^t_{\mathrm{f}}[\overline{s}_1/\{\overline{s}\}] = (\blacksquare^t_{\mathrm{f}} \setminus \{\overline{s}\}) \cup \{\overline{s}_1\}\big]\end{matrix}$$

Then, conclude:

$$\begin{matrix}\big[s \in \blacksquare^t_{\mathrm{f}} \setminus \{\overline{s}\}\ \textbf{and}\ \blacksquare^t_{\mathrm{f}}[\overline{s}_1/\{\overline{s}\}] = \blacksquare^t_{\mathrm{f}} \setminus \{\overline{s}\}\big] \\ \textbf{or}\ \big[s \in (\blacksquare^t_{\mathrm{f}} \setminus \{\overline{s}\}) \cup \{\overline{s}_1\}\ \textbf{and}\ \blacksquare^t_{\mathrm{f}}[\overline{s}_1/\{\overline{s}\}] = (\blacksquare^t_{\mathrm{f}} \setminus \{\overline{s}\}) \cup \{\overline{s}_1\}\big]\end{matrix}$$

Then, conclude $\big[s \in \blacksquare^t_{\mathrm{f}}[\overline{s}_1/\{\overline{s}\}]\ \textbf{or}\ s \in \blacksquare^t_{\mathrm{f}}[\overline{s}_1/\{\overline{s}\}]\big]$. Then, conclude:

$$s \in \blacksquare^t_{\mathrm{f}}[\overline{s}_1/\{\overline{s}\}]$$

Then, by Prop. 19, conclude $s \in \blacksquare^t_{\mathrm{c}}[\overline{s}_1/\{\overline{s}\}]$.

Y2 Suppose $s \neq \overline{s}$. Then, by a reduction similar to Y1, conclude $s \in \blacksquare^t_{\mathrm{c}}[\overline{s}_2/\{\overline{s}\}]$.

Y3 Suppose $\big[s \neq \overline{s}\ \textbf{and}\ t \in T_1\big]$. Then, by Y1, conclude:

$$s \in \blacksquare^t_{\mathrm{c}}[\overline{s}_1/\{\overline{s}\}]\ \textbf{and}\ t \in T_1$$

Then, conclude $s \in \{\hat{t} \mapsto \blacksquare^{\hat{t}}_{\mathrm{c}}[\overline{s}_1/\{\overline{s}\}] \mid \hat{t} \in T_1\}(t)$. Then, conclude:

$$s \in (\{\hat{t} \mapsto \blacksquare^{\hat{t}}_{\mathrm{c}}[\overline{s}_1/\{\overline{s}\}] \mid \hat{t} \in T_1\} \cup \{\hat{t} \mapsto \blacksquare^{\hat{t}}_{\mathrm{c}}[\overline{s}_2/\{\overline{s}\}] \mid \hat{t} \notin T_1\})(t)$$

Then, by Prop. 23, conclude $s \in \tilde{\blacksquare}^t_{\mathrm{c}}$. Then, by Prop. 20, conclude $s \in \tilde{\blacksquare}^t_{\mathrm{f}}$.

Y4 Suppose $\big[s \neq \overline{s}\ \textbf{and}\ t \notin T_1\big]$. Then, by a reduction similar to Y3, conclude $s \in \tilde{\blacksquare}^t_{\mathrm{f}}$.

Y5 From A1, conclude $s \in \blacksquare^t_{\mathrm{f}}$. Then, by Prop. 19, conclude $s \in \blacksquare^t_{\mathrm{c}}$. Then, by Prop. 15, conclude $s \in S_{\mathrm{c}}$. Then, by Prop. 19, conclude $s \in S_{\mathrm{f}}$.

Y6 Suppose $s \neq \overline{s}$. Then, conclude $s \notin \{\overline{s}\}$. Then, by Y5, conclude:

$$s \notin \{\overline{s}\}\ \textbf{and}\ s \in S_{\mathrm{f}}$$

Then, conclude $s \in S_{\mathrm{f}} \setminus \{\overline{s}\}$. Then, conclude $(s, s) \in \{(\hat{s}, \hat{s}) \mid \hat{s} \in S_{\mathrm{f}} \setminus \{\overline{s}\}\}$. Then, conclude $(s, s) \in \{(\hat{s}, \hat{s}) \mid \hat{s} \in S_{\mathrm{f}} \setminus \{\overline{s}\}\} \cup \{(\overline{s}, \overline{s}_1), (\overline{s}, \overline{s}_2)\}$. Then, by Prop. 21, conclude $s\ R\ s$.

Ⓨ7 Suppose $\big[s \neq \overline{s}$ **and** $t \in T_1\big]$. Then, by Ⓨ3, conclude $\big[s \neq \overline{s}$ **and** $s \in \tilde{\blacksquare}^t_{\mathrm{f}}\big]$. Then, by Ⓨ6, conclude $\big[s \in \tilde{\blacksquare}^t_{\mathrm{f}}$ **and** $s\ R\ s\big]$. Then, conclude:

$$\big[\tilde{s} \in \tilde{\blacksquare}^t_{\mathrm{f}} \textbf{ and } s\ R\ \tilde{s}\big] \textbf{ for some } \tilde{s}$$

Ⓨ8 Suppose $\big[s \neq \overline{s}$ **and** $t \notin T_1\big]$. Then, by a reduction similar to Ⓨ7, conclude $\big[\big[\tilde{s} \in \tilde{\blacksquare}^t_{\mathrm{f}}$ **and** $s\ R\ \tilde{s}\big]$ **for some** $\tilde{s}\big]$.

Prove the lemma by the following reduction. Conclude:

$$\big[s = \overline{s} \textbf{ or } s \neq \overline{s}\big] \textbf{ and } \big[t \in T_1 \textbf{ or } t \notin T_1\big]$$

Then, conclude:

$$\begin{matrix} \big[s = \overline{s} \textbf{ and } t \in T_1\big] \textbf{ or } \big[s = \overline{s} \textbf{ and } t \notin T_1\big] \\ \textbf{or } \big[s \neq \overline{s} \textbf{ and } t \in T_1\big] \textbf{ or } \big[s \neq \overline{s} \textbf{ and } t \notin T_1\big] \end{matrix}$$

Then, by Ⓩ9Ⓩ0, conclude:

$$\begin{matrix} \big[\big[\tilde{s} \in \tilde{\blacksquare}^t_{\mathrm{f}} \textbf{ and } s\ R\ \tilde{s}\big] \textbf{ for some } \tilde{s}\big] \textbf{ or } \big[\big[\tilde{s} \in \tilde{\blacksquare}^t_{\mathrm{f}} \textbf{ and } s\ R\ \tilde{s}\big] \textbf{ for some } \tilde{s}\big] \\ \textbf{or } \big[s \neq \overline{s} \textbf{ and } t \in T_1\big] \textbf{ or } \big[s \neq \overline{s} \textbf{ and } t \notin T_1\big] \end{matrix}$$

Then, by Ⓨ7Ⓨ8, conclude:

$$\begin{matrix} \big[\big[\tilde{s} \in \tilde{\blacksquare}^t_{\mathrm{f}} \textbf{ and } s\ R\ \tilde{s}\big] \textbf{ for some } \tilde{s}\big] \textbf{ or } \big[\big[\tilde{s} \in \tilde{\blacksquare}^t_{\mathrm{f}} \textbf{ and } s\ R\ \tilde{s}\big] \textbf{ for some } \tilde{s}\big] \\ \textbf{or } \big[\big[\tilde{s} \in \tilde{\blacksquare}^t_{\mathrm{f}} \textbf{ and } s\ R\ \tilde{s}\big] \textbf{ for some } \tilde{s}\big] \textbf{ or } \big[\big[\tilde{s} \in \tilde{\blacksquare}^t_{\mathrm{f}} \textbf{ and } s\ R\ \tilde{s}\big] \textbf{ for some } \tilde{s}\big] \end{matrix}$$

Then, conclude $\big[\big[\tilde{s} \in \tilde{\blacksquare}^t_{\mathrm{f}}$ **and** $s\ R\ \tilde{s}\big]$ **for some** $\tilde{s}\big]$. □

Lemma 6. $s \in \square^t_{\mathrm{f}}$ **implies** $\big[\big[\tilde{s} \in \tilde{\square}^t_{\mathrm{f}}$ **and** $s\ R\ \tilde{s}\big]$ **for some** $\tilde{s}\big]$

Proof. Prove the lemma by a reduction similar to the proof of Lemma 5

Lemma 7. $[\![\gamma]\!] \preceq_R [\![\mathsf{split}(\gamma; \overline{s}, \overline{s}_1, \overline{s}_2, T_1)]\!]$

Proof. Observe:

(Z1) Suppose:

$$s \xrightarrow{t}_{\mathbb{I}} s' \text{ for some } s, s', t$$

Then, by Prop. 19, conclude $(s, s') \in \{\hat{t} \mapsto \xrightarrow{\hat{t}} \cup (\xrightarrow{\hat{t}}\!\!\prec)^{-1} \mid \hat{t} \in \mathbb{T}\}(t)$. Then, conclude $(s, s') \in \xrightarrow{t} \cup (\xrightarrow{t}\!\!\prec)^{-1}$. Then, conclude $\left[s \xrightarrow{t} s' \text{ or } s \ (\xrightarrow{t}\!\!\prec)^{-1} \ s'\right]$. Then, by Lemmas 3, 4, conclude:

$$\begin{bmatrix}\left[\tilde{s} \xrightarrow{\sim t}_{\mathbb{I}} \tilde{s}' \text{ and } s \ R \ \tilde{s} \text{ and } s' \ R \ \tilde{s}'\right] \text{ for some } \tilde{s}, \tilde{s}' \\ \text{or} \\ \left[\tilde{s} \xrightarrow{\sim t}_{\mathbb{I}} \tilde{s}' \text{ and } s \ R \ \tilde{s} \text{ and } s' \ R \ \tilde{s}'\right] \text{ for some } \tilde{s}, \tilde{s}'\end{bmatrix}$$

Then, conclude $\left[\left[\tilde{s} \xrightarrow{\sim t}_{\mathbb{I}} \tilde{s}' \text{ and } s \ R \ \tilde{s} \text{ and } s' \ R \ \tilde{s}'\right] \text{ for some } \tilde{s}, \tilde{s}'\right]$.

(Z2) Suppose:

$$\tilde{s} \in S_{\mathrm{f}} \setminus \{\overline{s}\} \text{ for some } \tilde{s}$$

Then, by Prop. 19 conclude $\tilde{s} \in S_{\mathrm{c}} \setminus \{\overline{s}\}$. Then, conclude:

$$\tilde{s} \in (S_{\mathrm{c}} \setminus \{\overline{s}\}) \cup \{\overline{s}_1, \overline{s}_2\}$$

Then, by Prop. 23, conclude $\tilde{s} \in \tilde{S}_{\mathrm{c}}$. Then, by Prop. 20, conclude $\tilde{s} \in \tilde{S}_{\mathrm{f}}$.

(Z3) Suppose:

$$(s, \tilde{s}) \in \{(\hat{s}, \hat{s}) \mid \hat{s} \in S_{\mathrm{f}} \setminus \{\overline{s}\}\} \text{ for some } s, \tilde{s}$$

Then, conclude $s, \tilde{s} \in S_{\mathrm{f}} \setminus \{\overline{s}\}$. Then, conclude $\left[s \in S_{\mathrm{f}} \text{ and } \tilde{s} \in S_{\mathrm{f}} \setminus \{\overline{s}\}\right]$. Then, by (Z2), conclude $\left[s \in S_{\mathrm{f}} \text{ and } \tilde{s} \in \tilde{S}_{\mathrm{f}}\right]$. Then, conclude $(s, \tilde{s}) \in S_{\mathrm{f}} \times \tilde{S}_{\mathrm{f}}$.

(Z4) Suppose:

$$s = \overline{s} \text{ for some } s$$

Then, by Prop. 22, conclude $\left[s = \overline{s} \text{ and } \overline{s} \in S_{\mathrm{c}}\right]$. Then, conclude $s \in S_{\mathrm{c}}$. Then, by Prop. 19, conclude $s \in S_{\mathrm{f}}$.

(Z5) Suppose:

$$\left[\tilde{s} = \overline{s}_1 \text{ or } \tilde{s} = \overline{s}_2\right] \text{ for some } \tilde{s}$$

Then, conclude $\tilde{s} \in \{\overline{s}_1, \overline{s}_2\}$. Then, conclude $\tilde{s} \in (S_{\mathrm{c}} \setminus \{\overline{s}\}) \cup \{\overline{s}_1, \overline{s}_2\}$. Then, by Prop. 23, conclude $\tilde{s} \in \tilde{S}_{\mathrm{c}}$. Then, by Prop. 20, conclude $\tilde{s} \in \tilde{S}_{\mathrm{f}}$.

(Z6) Suppose:

$$(s, \tilde{s}) \in \{(\overline{s}, \overline{s}_1), (\overline{s}, \overline{s}_2)\} \text{ for some } s, \tilde{s}$$

Then, conclude $\left[s = \overline{s} \text{ and } \left[\tilde{s} = \overline{s}_1 \text{ or } \tilde{s} = \overline{s}_2\right]\right]$. Then, by (Z4), conclude $\left[s \in S_{\mathrm{f}} \text{ and } \left[\tilde{s} = \overline{s}_1 \text{ or } \tilde{s} = \overline{s}_2\right]\right]$. Then, by (Z5), conclude:

$$s \in S_{\mathrm{f}} \text{ and } \tilde{s} \in \tilde{S}_{\mathrm{f}}$$

Then, conclude $(s, \tilde{s}) \in S_{\mathrm{f}} \times \tilde{S}_{\mathrm{f}}$.

Z7 Suppose:

$$s \; R \; \tilde{s} \;\; \textbf{for some} \;\; s, \tilde{s}$$

Then, by Prop. 21, conclude $(s, \tilde{s}) \in \{(\hat{s}, \hat{s}) \mid \hat{s} \in S_\mathrm{f} \setminus \{\overline{s}\}\} \cup \{(\overline{s}, \overline{s}_1), (\overline{s}, \overline{s}_2)\}$. Then, conclude $\big[(s, \tilde{s}) \in \{(\hat{s}, \hat{s}) \mid \hat{s} \in S_\mathrm{f} \setminus \{\overline{s}\}\} \;\; \textbf{or} \;\; (s, \tilde{s}) \in \{(\overline{s}, \overline{s}_1), (\overline{s}, \overline{s}_2)\}\big]$. Then, by Z3, conclude $\big[(s, \tilde{s}) \in S_\mathrm{f} \times \tilde{S}_\mathrm{f} \;\; \textbf{or} \;\; (s, \tilde{s}) \in \{(\overline{s}, \overline{s}_1), (\overline{s}, \overline{s}_2)\}\big]$. Then, by Z6, conclude $\big[(s, \tilde{s}) \in S_\mathrm{f} \times \tilde{S}_\mathrm{f} \;\; \textbf{or} \;\; (s, \tilde{s}) \in S_\mathrm{f} \times \tilde{S}_\mathrm{f}\big]$. Then, conclude $(s, \tilde{s}) \in S_\mathrm{f} \times \tilde{S}_\mathrm{f}$.

Z8 By Z7, conclude $\big[\big[s \;\; R \;\; \tilde{s} \;\; \textbf{implies} \;\; (s, \tilde{s}) \in S_\mathrm{f} \times \tilde{S}_\mathrm{f}\big] \;\; \textbf{for all} \;\; s, \tilde{s}\big]$. Then, conclude $R \subseteq S_\mathrm{f} \times \tilde{S}_\mathrm{f}$.

Z9 Suppose:

$$s = \overline{s} \;\; \textbf{for some} \;\; s$$

Then, conclude $\big[s = \overline{s} \;\; \textbf{and} \;\; \overline{s} \in \mathrm{Dom}(\{(\overline{s}, \overline{s}_1), (\overline{s}_1, \overline{s}_2)\})\big]$. Then, conclude $s \in \mathrm{Dom}(\{(\overline{s}, \overline{s}_1), (\overline{s}_1, \overline{s}_2)\})$.

Z0 Suppose:

$$\big[s \in S_\mathrm{f} \;\; \textbf{and} \;\; s \neq \overline{s}\big] \;\; \textbf{for some} \;\; s$$

Then, conclude $\big[s \in S_\mathrm{f} \;\; \textbf{and} \;\; s \notin \{\overline{s}\}\big]$. Then, conclude $s \in S_\mathrm{f} \setminus \{\overline{s}\}$. Then, conclude $s \in \mathrm{Dom}(\{(\hat{s}, \hat{s}) \mid \hat{s} \in S_\mathrm{f} \setminus \{\overline{s}\}\})$.

Y1 Suppose:

$$s \in S_\mathrm{f} \;\; \textbf{for some} \;\; s$$

Then, conclude $\big[s \in S_\mathrm{f} \;\; \textbf{and} \;\; \big[s = \overline{s} \;\; \textbf{or} \;\; s \neq \overline{s}\big]\big]$. Then, conclude:

$$s = \overline{s} \;\; \textbf{or} \;\; \big[s \in S_\mathrm{f} \;\; \textbf{and} \;\; s \neq \overline{s}\big]$$

Then, by Z9, conclude:

$$s \in \mathrm{Dom}(\{(\overline{s}, \overline{s}_1), (\overline{s}_1, \overline{s}_2)\}) \;\; \textbf{or} \;\; \big[s \in S_\mathrm{f} \;\; \textbf{and} \;\; s \neq \overline{s}\big]$$

Then, by Z0, conclude:

$$s \in \mathrm{Dom}(\{(\overline{s}, \overline{s}_1), (\overline{s}_1, \overline{s}_2)\}) \;\; \textbf{or} \;\; s \in \mathrm{Dom}(\{(\hat{s}, \hat{s}) \mid \hat{s} \in S_\mathrm{f} \setminus \{\overline{s}\}\})$$

Then, conclude $s \in \mathrm{Dom}(\{(\overline{s}, \overline{s}_1), (\overline{s}_1, \overline{s}_2)\}) \cup \mathrm{Dom}(\{(\hat{s}, \hat{s}) \mid \hat{s} \in S_\mathrm{f} \setminus \{\overline{s}\}\})$. Then, conclude $s \in \mathrm{Dom}(\{(\overline{s}, \overline{s}_1), (\overline{s}_1, \overline{s}_2)\} \cup \{(\hat{s}, \hat{s}) \mid \hat{s} \in S_\mathrm{f} \setminus \{\overline{s}\}\})$. Then, by Prop. 21, conclude $s \in \mathrm{Dom}(R)$.

Y2 By Y1, conclude $\big[\big[s \in S_\mathrm{f} \;\; \textbf{implies} \;\; s \in \mathrm{Dom}(R)\big] \;\; \textbf{for all} \;\; s\big]$. Then, conclude $S_\mathrm{f} \subseteq \mathrm{Dom}(R)$.

Prove the lemma by the following reduction. By Z1, conclude:

$$\Big[s \xrightarrow{t}{}^{+} s' \;\; \textbf{implies} \;\; \Big[\big[\tilde{s} \xrightarrow{\sim t}{}^{+} \tilde{s}' \;\; \textbf{and} \;\; s \; R \; \tilde{s} \;\; \textbf{and} \;\; s' \; R \; \tilde{s}'\big] \;\; \textbf{for some} \;\; \tilde{s}, s\Big]\Big] \;\; \textbf{for all} \;\; s, s', t$$

Then, by Lemmas 5, 6, conclude:

$$\left[\left[s \xrightarrow{t}_{\mathtt{I}^+} s' \textbf{ implies } \left[\begin{array}{l}\left[\tilde{s} \xrightarrow{\sim t}_{\mathtt{I}^+} \tilde{s}' \textbf{ and } s\ R\ \tilde{s} \textbf{ and } s'\ R\ \tilde{s}'\right] \\ \textbf{for some } \tilde{s}, s\end{array}\right] \right] \textbf{ for all } \tilde{s}, \tilde{s}'\right]$$

$$\begin{array}{l}\textbf{and } \left[\left[s \in \blacksquare^t_{\mathrm{f}} \textbf{ implies } \left[\left[\tilde{s} \in \tilde{\blacksquare}^t_{\mathrm{f}} \textbf{ and } s\ R\ \tilde{s}\right] \textbf{ for some } \tilde{s}\right]\right] \textbf{ for all } s, t\right] \\ \textbf{and } \left[\left[s \in \square^t_{\mathrm{f}} \textbf{ implies } \left[\left[\tilde{s} \in \tilde{\square}^t_{\mathrm{f}} \textbf{ and } s\ R\ \tilde{s}\right] \textbf{ for some } \tilde{s}\right]\right] \textbf{ for all } s, t\right]\end{array}$$

Then, by (Z8), conclude:

$$\left[\left[s \xrightarrow{t}_{\mathtt{I}^+} s' \textbf{ implies } \left[\begin{array}{l}\left[\tilde{s} \xrightarrow{\sim t}_{\mathtt{I}^+} \tilde{s}' \textbf{ and } s\ R\ \tilde{s} \textbf{ and } s'\ R\ \tilde{s}'\right] \\ \textbf{for some } \tilde{s}, s\end{array}\right] \right] \textbf{ for all } \tilde{s}, \tilde{s}'\right]$$

$$\begin{array}{l}\textbf{and } \left[\left[s \in \blacksquare^t_{\mathrm{f}} \textbf{ implies } \left[\left[\tilde{s} \in \tilde{\blacksquare}^t_{\mathrm{f}} \textbf{ and } s\ R\ \tilde{s}\right] \textbf{ for some } \tilde{s}\right]\right] \textbf{ for all } s, t\right] \\ \textbf{and } \left[\left[s \in \square^t_{\mathrm{f}} \textbf{ implies } \left[\left[\tilde{s} \in \tilde{\square}^t_{\mathrm{f}} \textbf{ and } s\ R\ \tilde{s}\right] \textbf{ for some } \tilde{s}\right]\right] \textbf{ for all } s, t\right] \\ \textbf{and } R \subseteq S_{\mathrm{f}} \times \tilde{S}_{\mathrm{f}}\end{array}$$

Then, by (Y2), conclude:

$$\left[\left[s \xrightarrow{t}_{\mathtt{I}^+} s' \textbf{ implies } \left[\begin{array}{l}\left[\tilde{s} \xrightarrow{\sim t}_{\mathtt{I}^+} \tilde{s}' \textbf{ and } s\ R\ \tilde{s} \textbf{ and } s'\ R\ \tilde{s}'\right] \\ \textbf{for some } \tilde{s}, s\end{array}\right] \right] \textbf{ for all } \tilde{s}, \tilde{s}'\right]$$

$$\begin{array}{l}\textbf{and } \left[\left[s \in \blacksquare^t_{\mathrm{f}} \textbf{ implies } \left[\left[\tilde{s} \in \tilde{\blacksquare}^t_{\mathrm{f}} \textbf{ and } s\ R\ \tilde{s}\right] \textbf{ for some } \tilde{s}\right]\right] \textbf{ for all } s, t\right] \\ \textbf{and } \left[\left[s \in \square^t_{\mathrm{f}} \textbf{ implies } \left[\left[\tilde{s} \in \tilde{\square}^t_{\mathrm{f}} \textbf{ and } s\ R\ \tilde{s}\right] \textbf{ for some } \tilde{s}\right]\right] \textbf{ for all } s, t\right] \\ \textbf{and } R \subseteq S_{\mathrm{f}} \times \tilde{S}_{\mathrm{f}} \textbf{ and } S_{\mathrm{f}} \subseteq \mathrm{Dom}(R)\end{array}$$

Then, by Def. 5 of $\preceq$, conclude $(S_{\mathrm{f}}, \blacksquare_{\mathrm{f}}, \square_{\mathrm{f}}, \xrightarrow{}_{\mathtt{I}_{\mathrm{f}}}) \preceq_R (\tilde{S}_{\mathrm{f}}, \tilde{\blacksquare}_{\mathrm{f}}, \tilde{\square}_{\mathrm{f}}, \xrightarrow{\sim}_{\mathtt{I}_{\mathrm{f}}})$. Then, by Prop. 17, conclude $[\![\gamma]\!] \preceq_R (\tilde{S}_{\mathrm{f}}, \tilde{\blacksquare}_{\mathrm{f}}, \tilde{\square}_{\mathrm{f}}, \xrightarrow{\sim}_{\mathtt{I}_{\mathrm{f}}})$. Then, by Prop. 18, conclude:

$$[\![\gamma]\!] \preceq_R [\![\mathsf{split}(\gamma; \overline{s}, \overline{s}_1, \overline{s}_2, T_1)]\!]$$

□

Lemma 8. $[\![\mathsf{split}(\gamma; \overline{s}, \overline{s}_1, \overline{s}_2, T_1)]\!] \preceq_{R^{-1}} [\![\gamma]\!]$

Proof. Prove the lemma by a reduction similar to the proof of Lemma 7. □

D.3 Proof

Proof. Prove the theorem by the following reduction. By Lemma 7, conclude:

$$[\![\gamma]\!] \preceq_R [\![\mathsf{split}(\gamma; \overline{s}, \overline{s}_1, \overline{s}_2, T_1)]\!]$$

Then, by Lemma 8, conclude:

$$[\![\gamma]\!] \preceq_R [\![\mathsf{split}(\gamma; \overline{s}, \overline{s}_1, \overline{s}_2, T_1)]\!] \textbf{ and } [\![\mathsf{split}(\gamma; \overline{s}, \overline{s}_1, \overline{s}_2, T_1)]\!] \preceq_{R^{-1}} [\![\gamma]\!]$$

Then, by Def. 6 of $\sim$, conclude $[\![\gamma]\!] \sim_R [\![\mathsf{split}(\gamma; \overline{s}, \overline{s}_1, \overline{s}_2, T_1)]\!]$. Then, by Def. 6 of $\sim$, conclude $[\![\gamma]\!] \sim [\![\mathsf{split}(\gamma; \overline{s}, \overline{s}_1, \overline{s}_2, T_1)]\!]$. □

E Proof of Theorem 5

The scope of auxiliary propositions and auxiliary lemmas in this section is limited to this section.

E.1 Sketch

Let $R = \{(s, s) \mid s \in S_\mathrm{f} \setminus \overline{S}\} \cup \{(s, \overline{s}) \mid s \in \overline{S}\}$ be a candidate bisimulation relation (Prop. 33).

First, we prove $[\![\gamma]\!] \preceq_R [\![\mathsf{merge}(\gamma; \overline{S}, \overline{s})]\!]$ (Lemma 13). To do this, we first show that information flows resulting from pushes from a service s to a service s' in γ can be R-mimicked by $\mathsf{merge}(\gamma; \overline{S}, \overline{s})$ (Lemma 9). The main step in this proof is that we show that information flows resulting from pushes in γ that involve a service in $\overline{S}$ (the services to merge) can be mimicked with information flows that involve $\overline{s}$ in $\mathsf{merge}(\gamma; \overline{S}, \overline{s})$; information flows resulting from pushes in γ that do *not* involve a service in $\overline{S}$ are identical in $\mathsf{merge}(\gamma; \overline{S}, \overline{s})$. Similarly, information flows resulting from pulls in γ can be R-mimicked by $\mathsf{merge}(\gamma; \overline{S}, \overline{s})$ (Lemma 10).

Subsequently, we show that production responsibilities in γ are R-mimicked by production responsibilities in $\mathsf{merge}(\gamma; \overline{S}, \overline{s})$ (Lemma 11). The main step in this proof is that we show that production responsibilities of services in $\overline{S}$ are mimicked by $\overline{s}$; production responsibilities of other services in γ are identical in $\mathsf{merge}(\gamma; \overline{S}, \overline{s})$. Similarly, consumption responsibilities in γ can be R-mimicked by consumption responsibilities in $\mathsf{merge}(\gamma; \overline{S}, \overline{s})$ (Lemma 12).

Using antecedent $\mathsf{CanMerge}(\gamma; \overline{S}, \overline{s})$, Lemmas 9, 10, 11, and 12, we subsequently prove Lemma 13. A proof for $[\![\mathsf{merge}(\gamma; \overline{S}, \overline{s})]\!] \preceq_{R^{-1}} [\![\gamma]\!]$ is similar (Lemma 14). The theorem subsequently follows by Def. 6 of $\sim$ (i.e., the candidate bisimulation relation R is, indeed, a bisimulation relation).

E.2 Propositions and Lemmas

Antecedent:

Proposition 24. $\mathsf{CanMerge}(\gamma; \overline{S}, \overline{s})$

By Def. 1:

Proposition 25. $\gamma = (S_\mathrm{c}, \blacksquare_\mathrm{c}, \square_\mathrm{c}, \longrightarrow, \multimap)$

Proposition 26. $\mathsf{merge}(\gamma; \overline{S}, \overline{s}) = (\tilde{S}_\mathrm{c}, \tilde{\blacksquare}_\mathrm{c}, \tilde{\square}_\mathrm{c}, \xrightarrow{\sim}, \overset{\sim}{\multimap})$

Proposition 27.
$$\begin{aligned}
&\left[\left[s \in \blacksquare_\mathrm{c}^t \ \textbf{implies}\ s \in S_\mathrm{c}\right] \ \textbf{for all}\ s, t\right] \\
\textbf{and}\ &\left[\left[s \in \square_\mathrm{c}^t \ \textbf{implies}\ s \in S_\mathrm{c}\right] \ \textbf{for all}\ s, t\right] \\
\textbf{and}\ &\left[\left[s \xrightarrow{t} s' \ \textbf{implies}\ s, s' \in S_\mathrm{c}\right] \ \textbf{for all}\ s, s', t\right] \\
\textbf{and}\ &\left[\left[s \multimap^{t} s' \ \textbf{implies}\ s, s' \in S_\mathrm{c}\right] \ \textbf{for all}\ s, s', t\right]
\end{aligned}$$

Proposition 28.
$$\begin{aligned}
&\left[\left[s \in \tilde{\blacksquare}_\mathrm{c}^t \ \textbf{implies}\ s \in \tilde{S}_\mathrm{c}\right] \ \textbf{for all}\ s, t\right] \\
\textbf{and}\ &\left[\left[s \in \tilde{\square}_\mathrm{c}^t \ \textbf{implies}\ s \in \tilde{S}_\mathrm{c}\right] \ \textbf{for all}\ s, t\right] \\
\textbf{and}\ &\left[\left[s \xrightarrow{\sim t} s' \ \textbf{implies}\ s, s' \in \tilde{S}_\mathrm{c}\right] \ \textbf{for all}\ s, s', t\right] \\
\textbf{and}\ &\left[\left[s \overset{\sim t}{\multimap} s' \ \textbf{implies}\ s, s' \in \tilde{S}_\mathrm{c}\right] \ \textbf{for all}\ s, s', t\right]
\end{aligned}$$

By Def. 3:

Proposition 29. $[\![\gamma]\!] = (S_\mathrm{f}, \blacksquare_\mathrm{f}, \square_\mathrm{f}, \relbar\joinrel\mathrel{\mathrm{I}})$

Proposition 30. $[\![\mathsf{merge}(\gamma; \overline{S}, \overline{s})]\!] = (\tilde{S}_\mathrm{f}, \tilde{\blacksquare}_\mathrm{f}, \tilde{\square}_\mathrm{f}, \overset{\sim}{\relbar\joinrel\mathrel{\mathrm{I}}})$

Proposition 31.
$$\begin{aligned} & S_\mathrm{f} = S_\mathrm{c} \\ \textbf{and } & \blacksquare_\mathrm{f} = \blacksquare_\mathrm{c} \\ \textbf{and } & \square_\mathrm{f} = \square_\mathrm{c} \\ \textbf{and } & \relbar\joinrel\mathrel{\mathrm{I}} = \{t \mapsto \xrightarrow{t} \cup (\overset{t}{\relbar\joinrel\prec})^{\text{-}1} \mid t \in \mathbb{T}\} \end{aligned}$$

Proposition 32.
$$\begin{aligned} & \tilde{S}_\mathrm{f} = \tilde{S}_\mathrm{c} \\ \textbf{and } & \tilde{\blacksquare}_\mathrm{f} = \tilde{\blacksquare}_\mathrm{c} \\ \textbf{and } & \tilde{\square}_\mathrm{f} = \tilde{\square}_\mathrm{c} \\ \textbf{and } & \overset{\sim}{\relbar\joinrel\mathrel{\mathrm{I}}} = \{t \mapsto \xrightarrow{\sim t} \cup (\overset{\sim t}{\relbar\joinrel\prec})^{\text{-}1} \mid t \in \mathbb{T}\} \end{aligned}$$

Proposition 33. $R = \{(s, s) \mid s \in S_\mathrm{f} \setminus \overline{S}\} \cup \{(s, \overline{s}) \mid s \in \overline{S}\}$

By Fig. 5:

Proposition 34. $\overline{S} \subseteq S_\mathrm{c}$ **and** $\overline{s} \notin S_\mathrm{c}$

By Fig. 6:

Proposition 35.
$$\begin{aligned} & \tilde{S}_\mathrm{c} = (S_\mathrm{c} \setminus \overline{S}) \cup \{\overline{s}\} \\ \textbf{and } & \tilde{\blacksquare}_\mathrm{c} = \{t \mapsto \blacksquare^t_\mathrm{c}[\overline{s}/\overline{S}] \mid t \in \mathbb{T}\} \\ \textbf{and } & \tilde{\blacksquare}_\mathrm{c} = \{t \mapsto \square^t_\mathrm{c}[\overline{s}/\overline{S}] \mid t \in \mathbb{T}\} \\ \textbf{and } & \overset{\sim}{\longrightarrow} = \{t \mapsto (\{\overline{s}\} \times \overline{S}) \circ \xrightarrow{t} \circ (\overline{S} \times \{\overline{s}\}) \mid t \in \mathbb{T}\} \\ \textbf{and } & \overset{\sim}{\relbar\joinrel\prec} = \{t \mapsto (\{\overline{s}\} \times \overline{S}) \circ \xrightarrow{t} \circ (\overline{S} \times \{\overline{s}\}) \mid t \in \mathbb{T}\} \end{aligned}$$

Lemma 9. $s \xrightarrow{t} s'$ **implies** $\left[\begin{array}{c}[\tilde{s} \xrightarrow{\sim t_{\mathrm{I}}} \tilde{s}' \textbf{ and } s\ R\ \tilde{s} \textbf{ and } s'\ R\ \tilde{s}'] \\ \textbf{for some } \tilde{s}, \tilde{s}'\end{array}\right]$

Proof. Assumptions:

(A1) $s \xrightarrow{t} s'$

Observe:

(Z1) Suppose $s, s' \in \overline{S}$. Then, conclude $\big[(\overline{s}, s) \in \{\overline{s}\} \times \overline{S}$ **and** $(s', \overline{s}) \in \overline{S} \times \{\overline{s}\}\big]$. Then, by (A1), conclude:

$$(\overline{s}, s) \in \{\overline{s}\} \times \overline{S} \textbf{ and } (s', \overline{s}) \in \overline{S} \times \{\overline{s}\} \textbf{ and } s \xrightarrow{t} s'$$

Then, conclude $(\overline{s}, \overline{s}) \in (\{\overline{s}\} \times \overline{S}) \circ \xrightarrow{t} \circ (\overline{S} \times \{\overline{s}\})$. Then, conclude:

$$(\overline{s}, \overline{s}) \in \{\hat{t} \mapsto (\{\overline{s}\} \times \overline{S}) \circ \xrightarrow{\hat{t}} \circ (\overline{S} \times \{\overline{s}\}) \mid \hat{t} \in \mathbb{T}\}$$

Then, by Prop. 35, conclude $\overline{s} \xrightarrow{\sim t} \overline{s}$.

(Z2) Suppose:

$$\hat{s} \in \overline{S} \textbf{ for some } \hat{s}$$

Then, conclude $(\hat{s}, \overline{s}) \in \{(\check{s}, \overline{s}) \mid \check{s} \in \overline{S}\}$. Then, conclude:

$$(\hat{s}, \overline{s}) \in \{(\check{s}, \check{s}) \mid \check{s} \in S_{\mathrm{f}} \setminus \overline{S}\}\{(\check{s}, \overline{s}) \mid \check{s} \in \overline{S}\}$$

Then, by Prop. 33, conclude $\hat{s}\ R\ \overline{s}$.

(Z3) Suppose $s, s' \in \overline{S}$. Then, by (Z1), conclude $\big[s, s' \in \overline{S}$ **and** $\overline{s} \xrightarrow{\sim t} \overline{s}\big]$. Then, by (Z5), conclude $\big[\overline{s} \xrightarrow{\sim t} \overline{s}$ **and** $s\ R\ \overline{s}$ **and** $s'\ R\ \overline{s}\big]$. Then, conclude:

$$\big[\tilde{s} \xrightarrow{\sim t} \tilde{s}' \textbf{ and } s\ R\ \tilde{s} \textbf{ and } s'\ R\ \tilde{s}'\big] \textbf{ for some } \tilde{s}, \tilde{s}'$$

(Z4) Suppose $\big[s \in \overline{S}$ **and** $s' \notin \overline{S}\big]$. Then, conclude $\big[(\overline{s}, s) \in \{\overline{s}\} \times \overline{S}$ **and** $s' \notin \overline{S}\big]$. Then, by (A1), conclude $\big[(\overline{s}, s) \in \{\overline{s}\} \times \overline{S}$ **and** $s' \notin \overline{S}$ **and** $s \xrightarrow{t} s'\big]$. Then, conclude $(\overline{s}, s') \in (\{\overline{s}\} \times \overline{S}) \circ \xrightarrow{t} \circ (\overline{S} \times \{\overline{s}\})$. Then, conclude:

$$(\overline{s}, s') \in \{\hat{t} \mapsto (\{\overline{s}\} \times \overline{S}) \circ \xrightarrow{\hat{t}} \circ (\overline{S} \times \{\overline{s}\}) \mid \hat{t} \in \mathbb{T}\}$$

Then, by Prop. 35, conclude $\overline{s} \xrightarrow{\sim t} s'$.

(Z5) Suppose $\big[s \notin \overline{S}$ **and** $s' \in \overline{S}\big]$. Then, by a reduction similar to (Y1), conclude $s \xrightarrow{\sim t} \overline{s}$.

(Z6) From (A1), conclude $s \xrightarrow{t} s'$. Then, by Prop. 27, conclude $s, s' \in S_{\mathrm{c}}$.

(Z7) Suppose $s' \notin \overline{S}$. Then, by (Z6), conclude $\big[s' \in S_{\mathrm{c}}$ **and** $s' \notin \overline{S}\big]$. Then, conclude $s' \in S_{\mathrm{c}} \setminus \overline{S}$. Then, by Prop. 31, conclude $s' \in S_{\mathrm{f}} \setminus \overline{S}$. Then, conclude $(s', s') \in \{(\hat{s}, \hat{s}) \mid \hat{s} \in S_{\mathrm{f}} \setminus \overline{S}\}$. Then, conclude:

$$(s', s') \in \{(\hat{s}, \hat{s}) \mid \hat{s} \in S_{\mathrm{f}} \setminus \overline{S}\} \cup \{(\hat{s}, \overline{s}) \mid \hat{s} \in \overline{S}\}$$

Then, by Prop. 33, conclude $s'\ R\ s'$.

Z8 Suppose $s \notin \overline{S}$. Then, by a reduction similar to Z7, conclude $s \; R \; s$.

Z9 Suppose $\left[s \in \overline{S} \textbf{ and } s' \notin \overline{S}\right]$. Then, by Z7, conclude $\left[s = \overline{s} \textbf{ and } s' \; R \; s'\right]$. Then, by Z2, conclude $\left[s \; R \; \overline{s} \textbf{ and } s' \; R \; s'\right]$.

Z0 Suppose $\left[s \notin \overline{S} \textbf{ and } s' \in \overline{S}\right]$. Then, by a reduction similar to Z9, conclude $\left[s \; R \; s \textbf{ and } s' \; R \; \overline{s}\right]$.

Y1 Suppose $\left[s \in \overline{S} \textbf{ and } s' \notin \overline{S}\right]$. Then, by Z4, conclude:

$$s \in \overline{S} \textbf{ and } s' \notin \overline{S} \textbf{ and } \overline{s} \xrightarrow{\sim t} s'$$

Then, by Z9, conclude $\left[\overline{s}_1 \xrightarrow{\sim t} s' \textbf{ and } s \; R \; \overline{s} \textbf{ and } s' \; R \; s'\right]$. Then, conclude $\left[\left[\tilde{s} \xrightarrow{\sim t} \tilde{s}' \textbf{ and } s \; R \; \tilde{s} \textbf{ and } s' \; R \; \tilde{s}'\right] \textbf{ for some } \tilde{s}, \tilde{s}'\right]$.

Y2 Suppose $\left[s \notin \overline{S} \textbf{ and } s' \in \overline{S}\right]$. Then, by a reduction similar to Y1, conclude $\left[\left[\tilde{s} \xrightarrow{\sim t} \tilde{s}' \textbf{ and } s \; R \; \tilde{s} \textbf{ and } s' \; R \; \tilde{s}'\right] \textbf{ for some } \tilde{s}, \tilde{s}'\right]$.

Y3 Suppose $s, s' \notin \overline{S}$. Then, by A1, conclude $\left[s, s' \notin \overline{S} \textbf{ and } s \xrightarrow{t} s'\right]$. Then, conclude $(s, s') \in (\{\overline{s}\} \times \overline{S}) \circ \overset{\hat{t}}{\longrightarrow} \circ (\overline{S} \times \{\overline{s}\})$. Then, conclude:

$$(s, s') \in \{\hat{t} \mapsto (\{\overline{s}\} \times \overline{S}) \circ \overset{\hat{t}}{\longrightarrow} \circ (\overline{S} \times \{\overline{s}\}) \mid \hat{t} \in \mathbb{T}\}$$

Then, by Prop. 35, conclude $s \xrightarrow{\sim t} s'$.

Y4 Suppose $s, s' \notin \overline{S}$. Then, by Y3, conclude $\left[s, s' \notin \overline{S} \textbf{ and } s \xrightarrow{\sim t} s'\right]$. Then, by Z7 Z8, conclude $\left[s \xrightarrow{\sim t} s' \textbf{ and } s \; R \; s \textbf{ and } s' \; R \; s'\right]$. Then, conclude $\left[\left[\tilde{s} \xrightarrow{\sim t} \tilde{s}' \textbf{ and } s \; R \; \tilde{s} \textbf{ and } s' \; R \; \tilde{s}'\right] \textbf{ for some } \tilde{s}, \tilde{s}'\right]$.

Y5 Suppose:

$$\tilde{s} \xrightarrow{\sim t} \tilde{s}' \textbf{ for some } \tilde{s}, \tilde{s}'$$

Then, conclude $(\tilde{s}, \tilde{s}') \in \xrightarrow{\sim t} \cup (\xrightarrow{\sim t})^{-1}$. Then, conclude:

$$(\tilde{s}, \tilde{s}') \in \{\hat{t} \mapsto \xrightarrow{\sim \hat{t}} \cup (\xrightarrow{\sim \hat{t}})^{-1} \mid \hat{t} \in \mathbb{T}\}(t)$$

Then, by Prop. 32, conclude $\tilde{s} \xrightarrow{\sim t}_{\mathrm{I}} \tilde{s}'$.

Proof the lemma by the following reduction. Conclude:

$$\left[s \in \overline{S} \textbf{ or } s \notin \overline{S}\right] \textbf{ and } \left[s' \in \overline{S} \textbf{ or } s' \notin \overline{S}\right]$$

Then, conclude:

$$s, s' \in \overline{S} \textbf{ or } \left[s \in \overline{S} \textbf{ and } s' \notin \overline{S}\right] \textbf{ or } \left[s \notin \overline{S} \textbf{ and } s' \in \overline{S}\right] \textbf{ or } s, s' \notin \overline{S}$$

Then, by Z3, conclude:

$$\begin{gathered}\left[\left[\tilde{s} \xrightarrow{\sim t} \tilde{s}' \textbf{ and } s \; R \; \tilde{s} \textbf{ and } s' \; R \; \tilde{s}'\right] \textbf{ for some } \tilde{s}, \tilde{s}'\right] \\ \textbf{or } \left[s \in \overline{S} \textbf{ and } s' \notin \overline{S}\right] \textbf{ or } \left[s \notin \overline{S} \textbf{ and } s' \in \overline{S}\right] \textbf{ or } s, s' \notin \overline{S}\end{gathered}$$

Then, by (Y1)(Y2), conclude:

$$\begin{array}{ll}
 & \Big[\big[\tilde{s} \xrightarrow{\sim t} \tilde{s}' \ \textbf{and}\ s\ R\ \tilde{s}\ \textbf{and}\ s'\ R\ \tilde{s}'\big]\ \textbf{for some}\ \tilde{s}, \tilde{s}'\Big] \\
\textbf{or} & \Big[\big[\tilde{s} \xrightarrow{\sim t} \tilde{s}' \ \textbf{and}\ s\ R\ \tilde{s}\ \textbf{and}\ s'\ R\ \tilde{s}'\big]\ \textbf{for some}\ \tilde{s}, \tilde{s}'\Big] \\
\textbf{or} & \Big[\big[\tilde{s} \xrightarrow{\sim t} \tilde{s}' \ \textbf{and}\ s\ R\ \tilde{s}\ \textbf{and}\ s'\ R\ \tilde{s}'\big]\ \textbf{for some}\ \tilde{s}, \tilde{s}'\Big] \\
\textbf{or} & s, s' \notin \overline{S}
\end{array}$$

Then, by (Y4), conclude:

$$\begin{array}{ll}
 & \Big[\big[\tilde{s} \xrightarrow{\sim t} \tilde{s}' \ \textbf{and}\ s\ R\ \tilde{s}\ \textbf{and}\ s'\ R\ \tilde{s}'\big]\ \textbf{for some}\ \tilde{s}, \tilde{s}'\Big] \\
\textbf{or} & \Big[\big[\tilde{s} \xrightarrow{\sim t} \tilde{s}' \ \textbf{and}\ s\ R\ \tilde{s}\ \textbf{and}\ s'\ R\ \tilde{s}'\big]\ \textbf{for some}\ \tilde{s}, \tilde{s}'\Big] \\
\textbf{or} & \Big[\big[\tilde{s} \xrightarrow{\sim t} \tilde{s}' \ \textbf{and}\ s\ R\ \tilde{s}\ \textbf{and}\ s'\ R\ \tilde{s}'\big]\ \textbf{for some}\ \tilde{s}, \tilde{s}'\Big] \\
\textbf{or} & \Big[\big[\tilde{s} \xrightarrow{\sim t} \tilde{s}' \ \textbf{and}\ s\ R\ \tilde{s}\ \textbf{and}\ s'\ R\ \tilde{s}'\big]\ \textbf{for some}\ \tilde{s}, \tilde{s}'\Big]
\end{array}$$

Then, conclude $\Big[\big[\tilde{s} \xrightarrow{\sim t} \tilde{s}' \ \textbf{and}\ s\ R\ \tilde{s}\ \textbf{and}\ s'\ R\ \tilde{s}'\big]\ \textbf{for some}\ \tilde{s}, \tilde{s}'\Big]$. Then, by (Y5), conclude $\Big[\big[\tilde{s} \overset{\sim t}{\relbar\!\!\relbar\!\!\text{I}} \tilde{s}' \ \textbf{and}\ s\ R\ \tilde{s}\ \textbf{and}\ s'\ R\ \tilde{s}'\big]\ \textbf{for some}\ \tilde{s}, \tilde{s}'\Big]$. ☐

Lemma 10. $s\ (\overset{t}{\relbar\!\!\relbar\!\!\prec})^{\text{-}1}\ s'$ **implies** $\left[\begin{array}{c}\big[\tilde{s} \overset{\sim t}{\relbar\!\!\relbar\!\!\text{I}} \tilde{s}' \ \textbf{and}\ s\ R\ \tilde{s}\ \textbf{and}\ s'\ R\ \tilde{s}'\big] \\ \textbf{for some}\ \tilde{s}, \tilde{s}'\end{array}\right]$

Proof. Prove the lemma by a reduction similar to the proof of Lemma 9

Lemma 11. $s \in \blacksquare^t_{\mathrm{f}}$ **implies** $\big[\big[\tilde{s} \in \tilde{\blacksquare}^t_{\mathrm{f}}$ **and** $s\ R\ \tilde{s}\big]$ **for some** $\tilde{s}\big]$

Proof. Assume:

(A1) $s \in \blacksquare^t_{\mathrm{f}}$

Observe:

(Z1) Suppose $s \in \overline{S}$. Then, by (A1), conclude $\big[s \in \overline{S}$ **and** $s \in \blacksquare^t_{\mathrm{f}}\big]$. Then, conclude $\overline{S} \cap \blacksquare^t_{\mathrm{f}} \neq \emptyset$. Then, conclude $\blacksquare^t_{\mathrm{f}}[\overline{s}/\overline{S}] = (\blacksquare^t_{\mathrm{f}} \setminus \overline{S}) \cup \{\overline{s}\}$. Then, conclude $\big[\blacksquare^t_{\mathrm{f}}[\overline{s}/\overline{S}] = (\blacksquare^t_{\mathrm{f}} \setminus \overline{S}) \cup \{\overline{s}\}$ **and** $\overline{s} \in (\blacksquare^t_{\mathrm{f}} \setminus \overline{S}) \cup \{\overline{s}\}\big]$. Then, conclude:

$$\overline{s} \in \blacksquare^t_{\mathrm{f}}[\overline{s}/\overline{S}]$$

Then, by Prop. 31, conclude $\overline{s} \in \blacksquare^t_{\mathrm{c}}[\overline{s}/\overline{S}]$. Then, conclude:

$$\overline{s} \in \{\hat{t} \mapsto \blacksquare^{\hat{t}}_{\mathrm{c}}[\overline{s}/\overline{S}] \mid \hat{t} \in \mathbb{T}\}(t)$$

Then, by Prop. 35, conclude $\overline{s} \in \tilde{\blacksquare}^t_{\mathrm{c}}$. Then, by Prop. 32, conclude $\overline{s} \in \tilde{\blacksquare}^t_{\mathrm{f}}$.

(Z2) Suppose:

$$\hat{s} \in \overline{S} \textbf{ for some } \hat{s}$$

Then, conclude $(\hat{s}, \overline{s}) \in \{(\check{s}, \overline{s}) \mid \check{s} \in \overline{S}\}$. Then, conclude:

$$(\hat{s}, \overline{s}) \in \{(\check{s}, \check{s}) \mid \check{s} \in S_{\mathrm{f}} \setminus \overline{S}\}\{(\check{s}, \overline{s}) \mid \check{s} \in \overline{S}\}$$

Then, by Prop. 33, conclude $\hat{s}\ R\ \overline{s}$.

(Z3) Suppose $s \in \overline{S}$. Then, by (Z1), conclude $\big[s \in \overline{S}$ **and** $\overline{s} \in \tilde{\blacksquare}^t_{\mathrm{f}}\big]$. Then, by (Z2), conclude $\big[\overline{s} \in \tilde{\blacksquare}^t_{\mathrm{f}}$ **and** $s\ R\ \overline{s}\big]$. Then, conclude:

$$\big[\tilde{s} \in \tilde{\blacksquare}^t_{\mathrm{f}} \textbf{ and } s\ R\ \tilde{s}\big] \textbf{ for some } \tilde{s}$$

(Z4) Suppose $s \notin \overline{S}$. Then, conclude $s \notin \overline{S}$. Then, by (A1), conclude:

$$s \notin \overline{S} \textbf{ and } s \in \blacksquare^t_{\mathrm{f}}$$

Then, conclude $s \in \blacksquare^t_{\mathrm{f}} \setminus \overline{S}$. Then, conclude:

$$s \in \blacksquare^t_{\mathrm{f}} \setminus \overline{S} \textbf{ and } \big[\blacksquare^t_{\mathrm{f}} \cap \overline{S} = \emptyset \textbf{ or } \blacksquare^t_{\mathrm{f}} \cap \overline{S} \neq \emptyset\big]$$

Then, conclude:

$$\big[s \in \blacksquare^t_{\mathrm{f}} \setminus \overline{S} \textbf{ and } \blacksquare^t_{\mathrm{f}} \cap \overline{S} = \emptyset\big] \textbf{ or } \big[s \in \blacksquare^t_{\mathrm{f}} \setminus \overline{S} \textbf{ and } \blacksquare^t_{\mathrm{f}} \cap \overline{S} \neq \emptyset\big]$$

Then, conclude:

$$\begin{array}{c}\big[s \in \blacksquare^t_{\mathrm{f}} \setminus \overline{S} \textbf{ and } \blacksquare^t_{\mathrm{f}}[\overline{s}/\overline{S}] = \blacksquare^t_{\mathrm{f}} \setminus \overline{S}\big] \\ \textbf{or } \big[s \in \blacksquare^t_{\mathrm{f}} \setminus \overline{S} \textbf{ and } \blacksquare^t_{\mathrm{f}}[\overline{s}/\overline{S}] = (\blacksquare^t_{\mathrm{f}} \setminus \overline{S}) \cup \{\overline{s}\}\big]\end{array}$$

Then, conclude:

$$\begin{array}{c} \big[s \in \blacksquare^t_{\mathrm{f}} \setminus \overline{S} \ \textbf{and} \ \blacksquare^t_{\mathrm{f}}[\overline{s}/\overline{S}] = \blacksquare^t_{\mathrm{f}} \setminus \overline{S}\big] \\ \textbf{or} \ \big[s \in (\blacksquare^t_{\mathrm{f}} \setminus \overline{S}) \cup \{\overline{s}\} \ \textbf{and} \ \blacksquare^t_{\mathrm{f}}[\overline{s}/\overline{S}] = (\blacksquare^t_{\mathrm{f}} \setminus \overline{S}) \cup \{\overline{s}\}\big] \end{array}$$

Then, conclude $\big[s \in \blacksquare^t_{\mathrm{f}}[\overline{s}/\overline{S}] \ \textbf{or} \ s \in \blacksquare^t_{\mathrm{f}}[\overline{s}/\overline{S}]\big]$. Then, conclude $s \in \blacksquare^t_{\mathrm{f}}[\overline{s}/\overline{S}]$. Then, by Prop. 31, conclude $s \in \blacksquare^t_{\mathrm{c}}[\overline{s}/\overline{S}]$. Then, conclude:

$$s \in \{\hat{t} \mapsto \blacksquare^{\hat{t}}_{\mathrm{c}}[\overline{s}_1/\overline{S}] \mid \hat{t} \in \mathbb{T}\}(t)$$

Then, by Prop. 35, conclude $s \in \tilde{\blacksquare}^t_{\mathrm{c}}$. Then, by Prop. 32, conclude $s \in \tilde{\blacksquare}^t_{\mathrm{f}}$.

(Z5) From (A1), conclude $s \in \blacksquare^t_{\mathrm{f}}$. Then, by Prop. 31, conclude $s \in \blacksquare^t_{\mathrm{c}}$. Then, by Prop. 27, conclude $s \in S_{\mathrm{c}}$.

(Z6) Suppose $s \notin \overline{S}$. Then, by (Z5), conclude $\big[s \in S_{\mathrm{c}} \ \textbf{and} \ s' \notin \overline{S}\big]$. Then, conclude $s \in S_{\mathrm{c}} \setminus \overline{S}$. Then, by Prop. 31, conclude $s \in S_{\mathrm{f}} \setminus \overline{S}$. Then, conclude:

$$(s, s) \in \{(\hat{s}, \hat{s}) \mid \hat{s} \in S_{\mathrm{f}} \setminus \overline{S}\}$$

Then, conclude $(s, s) \in \{(\hat{s}, \hat{s}) \mid \hat{s} \in S_{\mathrm{f}} \setminus \overline{S}\} \cup \{(\hat{s}, \overline{s}) \mid \hat{s} \in \overline{S}\}$. Then, by Prop. 33, conclude $s \ R \ s$.

(Z7) Suppose $s \notin \overline{S}$. Then, by (Z4), conclude $\big[s \notin \overline{S} \ \textbf{and} \ s \in \tilde{\blacksquare}^t_{\mathrm{f}}\big]$. Then, by (Z6), conclude $\big[s \in \tilde{\blacksquare}^t_{\mathrm{f}} \ \textbf{and} \ s \ R \ s\big]$. Then, conclude:

$$\big[\tilde{s} \in \tilde{\blacksquare}^t_{\mathrm{f}} \ \textbf{and} \ s \ R \ \tilde{s}\big] \ \textbf{for some} \ \tilde{s}$$

Prove the lemma by the following reduction. Conclude $\big[s \in \overline{S} \ \textbf{or} \ s \notin \overline{S}\big]$. Then, by (Z3), conclude $\big[\big[\tilde{s} \in \tilde{\blacksquare}^t_{\mathrm{f}} \ \textbf{and} \ s \ R \ \tilde{s}\big] \ \textbf{for some} \ \tilde{s}\big] \ \textbf{or} \ s \notin \overline{S}\big]$. Then, by (Z7), conclude:

$$\big[\big[\tilde{s} \in \tilde{\blacksquare}^t_{\mathrm{f}} \ \textbf{and} \ s \ R \ \tilde{s}\big] \ \textbf{for some} \ \tilde{s}\big] \ \textbf{or} \ \big[\big[\tilde{s} \in \tilde{\blacksquare}^t_{\mathrm{f}} \ \textbf{and} \ s \ R \ \tilde{s}\big] \ \textbf{for some} \ \tilde{s}\big]$$

Then, conclude $\big[\big[\tilde{s} \in \tilde{\blacksquare}^t_{\mathrm{f}} \ \textbf{and} \ s \ R \ \tilde{s}\big] \ \textbf{for some} \ \tilde{s}\big]$. □

Lemma 12. $s \in \square^t_{\mathrm{f}}$ **implies** $\big[\big[\tilde{s} \in \tilde{\square}^t_{\mathrm{f}} \ \textbf{and} \ s \ R \ \tilde{s}\big] \ \textbf{for some} \ \tilde{s}\big]$

Proof. Prove the lemma by a reduction similar to the proof of Lemma 11

Lemma 13. $[\![\gamma]\!] \preceq_R [\![\mathsf{merge}(\gamma; \overline{S}, \overline{s})]\!]$

Proof. Observe:

(Z1) Suppose:

$$s \xrightarrow{t}_{\mathbb{I}} s' \text{ for some } s, s', t$$

Then, by Prop. 31, conclude $(s, s') \in \{\hat{t} \mapsto \xrightarrow{\hat{t}} \cup (\xrightarrow{\hat{t}})^{-1} \mid \hat{t} \in \mathbb{T}\}(t)$. Then, conclude $(s, s') \in \xrightarrow{t} \cup (\xrightarrow{t})^{-1}$. Then, conclude $\left[s \xrightarrow{t} s' \text{ or } s \ (\xrightarrow{t})^{-1} \ s'\right]$. Then, by Lemmas 9, 10, conclude:

$$\begin{array}{r} \left[\left[\tilde{s} \xrightarrow{\sim t}_{\mathbb{I}} \tilde{s}' \text{ and } s \ R \ \tilde{s} \text{ and } s' \ R \ \tilde{s}'\right] \text{ for some } \tilde{s}, \tilde{s}'\right] \\ \text{or } \left[\left[\tilde{s} \xrightarrow{\sim t}_{\mathbb{I}} \tilde{s}' \text{ and } s \ R \ \tilde{s} \text{ and } s' \ R \ \tilde{s}'\right] \text{ for some } \tilde{s}, \tilde{s}'\right] \end{array}$$

Then, conclude $\left[\left[\tilde{s} \xrightarrow{\sim t}_{\mathbb{I}} \tilde{s}' \text{ and } s \ R \ \tilde{s} \text{ and } s' \ R \ \tilde{s}'\right] \text{ for some } \tilde{s}, \tilde{s}'\right]$.

(Z2) Suppose:

$$\tilde{s} \in S_{\mathrm{f}} \setminus \overline{S} \text{ for some } \tilde{s}$$

Then, by Prop. 31 conclude $\tilde{s} \in S_{\mathrm{c}} \setminus \overline{S}$. Then, conclude $\tilde{s} \in (S_{\mathrm{c}} \setminus \overline{S}) \cup \{\overline{s}\}$. Then, by Prop. 35, conclude $\tilde{s} \in \tilde{S}_{\mathrm{c}}$. Then, by Prop. 32, conclude $\tilde{s} \in \tilde{S}_{\mathrm{f}}$.

(Z3) Suppose:

$$(s, \tilde{s}) \in \{(\hat{s}, \hat{s}) \mid \hat{s} \in S_{\mathrm{f}} \setminus \overline{S}\} \text{ for some } s, \tilde{s}$$

Then, conclude $s, \tilde{s} \in S_{\mathrm{f}} \setminus \overline{S}$. Then, conclude $\left[s \in S_{\mathrm{f}} \text{ and } \tilde{s} \in S_{\mathrm{f}} \setminus \overline{S}\right]$. Then, by (Z2), conclude $\left[s \in S_{\mathrm{f}} \text{ and } \tilde{s} \in \tilde{S}_{\mathrm{f}}\right]$. Then, conclude $(s, \tilde{s}) \in S_{\mathrm{f}} \times \tilde{S}_{\mathrm{f}}$.

(Z4) Suppose:

$$s \in \overline{S} \text{ for some } s$$

Then, by Prop. 34, conclude $\left[s \in \overline{S} \text{ and } \overline{S} \subseteq S_{\mathrm{c}}\right]$. Then, conclude $s \in S_{\mathrm{c}}$. Then, by Prop. 31, conclude $s \in S_{\mathrm{f}}$.

(Z5) Suppose:

$$\tilde{s} = \overline{s} \text{ for some } \tilde{s}$$

Then, conclude $\tilde{s} \in \{\overline{s}\}$. Then, conclude $\tilde{s} \in (S_{\mathrm{c}} \setminus \overline{S}) \cup \{\overline{s}\}$. Then, by Prop. 35, conclude $\tilde{s} \in \tilde{S}_{\mathrm{c}}$. Then, by Prop. 32, conclude $\tilde{s} \in \tilde{S}_{\mathrm{f}}$.

(Z6) Suppose:

$$(s, \tilde{s}) \in \{(\hat{s}, \overline{s}) \mid \hat{s} \in \overline{S}\} \text{ for some } s, \tilde{s}$$

Then, conclude $\left[s \in \overline{S} \text{ and } \tilde{s} = \overline{s}\right]$. Then, by (Z4), conclude:

$$s \in S_{\mathrm{f}} \text{ and } \tilde{s} = \overline{s}$$

Then, by (Z5), conclude $\left[s \in S_{\mathrm{f}} \text{ and } \tilde{s} \in \tilde{S}_{\mathrm{f}}\right]$. Then, conclude $(s, \tilde{s}) \in S_{\mathrm{f}} \times \tilde{S}_{\mathrm{f}}$.

(Z7) Suppose:

$$s\ R\ \tilde{s}\ \textbf{ for some }\ s, \tilde{s}$$

Then, by Prop. 33, conclude $(s, \tilde{s}) \in \{(\hat{s}, \hat{s}) \mid \hat{s} \in S_{\mathrm{f}} \setminus \overline{S}\} \cup \{(\hat{s}, \overline{s}) \mid \hat{s} \in \overline{S}\}$. Then, conclude $\big[(s, \tilde{s}) \in \{(\hat{s}, \hat{s}) \mid \hat{s} \in S_{\mathrm{f}} \setminus \overline{S}\}$ **or** $(s, \tilde{s}) \in \{(\hat{s}, \overline{s}) \mid \hat{s} \in \overline{S}\}\big]$. Then, by (Z3), conclude $\big[(s, \tilde{s}) \in S_{\mathrm{f}} \times \tilde{S}_{\mathrm{f}}$ **or** $(s, \tilde{s}) \in \{(\hat{s}, \overline{s}) \mid \hat{s} \in \overline{S}\}\big]$. Then, by (Z6), conclude $\big[(s, \tilde{s}) \in S_{\mathrm{f}} \times \tilde{S}_{\mathrm{f}}$ **or** $(s, \tilde{s}) \in S_{\mathrm{f}} \times \tilde{S}_{\mathrm{f}}\big]$. Then, conclude $(s, \tilde{s}) \in S_{\mathrm{f}} \times \tilde{S}_{\mathrm{f}}$.

(Z8) By (Z7), conclude $\big[\big[s\ R\ \tilde{s}$ **implies** $(s, \tilde{s}) \in S_{\mathrm{f}} \times \tilde{S}_{\mathrm{f}}\big]$ **for all** $s, \tilde{s}\big]$. Then, conclude $R \subseteq S_{\mathrm{f}} \times \tilde{S}_{\mathrm{f}}$.

(Z9) Suppose:

$$\big[s \in S_{\mathrm{f}}\ \textbf{ and }\ s \notin \overline{S}\big]\ \textbf{ for some }\ s$$

Then, conclude $s \in S_{\mathrm{f}} \setminus \overline{S}$. Then, conclude $s \in \mathrm{Dom}(\{(\hat{s}, \hat{s}) \mid \hat{s} \in S_{\mathrm{f}} \setminus \overline{S}\})$.

(Z0) Suppose:

$$s \in S_{\mathrm{f}}\ \textbf{ for some }\ s$$

Then, conclude $\big[s \in S_{\mathrm{f}}$ **and** $\big[s \in \overline{S}$ **or** $s \notin \overline{S}\big]\big]$. Then, conclude:

$$s \in \overline{S}\ \textbf{ or }\ \big[s \in S_{\mathrm{f}}\ \textbf{ and }\ s \notin \overline{S}\big]$$

Then, conclude $\big[s \in \mathrm{Dom}(\{(\hat{s}, \overline{s}) \mid \hat{s} \in \overline{S}\})$ **or** $\big[s \in S_{\mathrm{f}}$ **and** $s \notin \overline{S}\big]\big]$. Then, by (Z9), conclude:

$$s \in \mathrm{Dom}(\{(\hat{s}, \overline{s}) \mid \hat{s} \in \overline{S}\})\ \textbf{ or }\ s \in \mathrm{Dom}(\{(\hat{s}, \hat{s}) \mid \hat{s} \in S_{\mathrm{f}} \setminus \overline{S}\})$$

Then, conclude $s \in \mathrm{Dom}(\{(\hat{s}, \overline{s}) \mid \hat{s} \in \overline{S}\}) \cup \mathrm{Dom}(\{(\hat{s}, \hat{s}) \mid \hat{s} \in S_{\mathrm{f}} \setminus \overline{S}\})$. Then, conclude $s \in \mathrm{Dom}(\{(\hat{s}, \overline{s}) \mid \hat{s} \in \overline{S}\} \cup \{(\hat{s}, \hat{s}) \mid \hat{s} \in S_{\mathrm{f}} \setminus \overline{S}\})$. Then, by Prop. 33, conclude $s \in \mathrm{Dom}(R)$.

(Y1) By (Y1), conclude $\big[\big[s \in S_{\mathrm{f}}$ **implies** $s \in \mathrm{Dom}(R)\big]$ **for all** $s\big]$. Then, conclude $S_{\mathrm{f}} \subseteq \mathrm{Dom}(R)$.

Prove the lemma by the following reduction. By (Z1), conclude:

$$\Big[s \xrightarrow{t}{}^{+}_{\mathtt{I}}\ s'\ \textbf{ implies }\ \Big[\big[\tilde{s} \xrightarrow{\sim t}{}^{+}_{\mathtt{I}}\ \tilde{s}'\ \textbf{ and }\ s\ R\ \tilde{s}\ \textbf{ and }\ s'\ R\ \tilde{s}'\big]\ \textbf{ for some }\ \tilde{s}, s\Big]\Big]\ \textbf{ for all }\ s, s', t$$

Then, by Lemmas 11, 12, conclude:

$$\Big[\big[s \xrightarrow{t}{}^{+}_{\mathtt{I}}\ s'\ \textbf{ implies }\ \Big[\big[\tilde{s} \xrightarrow{\sim t}{}^{+}_{\mathtt{I}}\ \tilde{s}'\ \textbf{ and }\ s\ R\ \tilde{s}\ \textbf{ and }\ s'\ R\ \tilde{s}'\big]\ \textbf{ for some }\ \tilde{s}, s\Big]\big]\ \textbf{ for all }\ \tilde{s}, \tilde{s}'\Big]$$

$$\textbf{and }\ \big[\big[s \in \blacksquare^{t}_{\mathrm{f}}\ \textbf{ implies }\ \big[\big[\tilde{s} \in \tilde{\blacksquare}^{t}_{\mathrm{f}}\ \textbf{ and }\ s\ R\ \tilde{s}\big]\ \textbf{ for some }\ \tilde{s}\big]\big]\ \textbf{ for all }\ s, t\big]$$
$$\textbf{and }\ \big[\big[s \in \square^{t}_{\mathrm{f}}\ \textbf{ implies }\ \big[\big[\tilde{s} \in \tilde{\square}^{t}_{\mathrm{f}}\ \textbf{ and }\ s\ R\ \tilde{s}\big]\ \textbf{ for some }\ \tilde{s}\big]\big]\ \textbf{ for all }\ s, t\big]$$

Then, by (Z8), conclude:

$$\left[\left[s \xrightarrow{t}_{\mathtt{I}}{}^{+} s' \textbf{ implies } \left[\begin{array}{l}\left[\tilde{s} \xrightarrow{\sim t}_{\mathtt{I}}{}^{+} \tilde{s}' \textbf{ and } s\ R\ \tilde{s} \textbf{ and } s'\ R\ \tilde{s}'\right] \\ \textbf{for some } \tilde{s}, s\end{array}\right]\right] \textbf{ for all } \tilde{s}, \tilde{s}'\right]$$

$$\begin{array}{l}\textbf{and } \left[\left[s \in \blacksquare^{t}_{\mathrm{f}} \textbf{ implies } \left[\left[\tilde{s} \in \tilde{\blacksquare}^{t}_{\mathrm{f}} \textbf{ and } s\ R\ \tilde{s}\right] \textbf{ for some } \tilde{s}\right]\right] \textbf{ for all } s, t\right] \\ \textbf{and } \left[\left[s \in \square^{t}_{\mathrm{f}} \textbf{ implies } \left[\left[\tilde{s} \in \tilde{\square}^{t}_{\mathrm{f}} \textbf{ and } s\ R\ \tilde{s}\right] \textbf{ for some } \tilde{s}\right]\right] \textbf{ for all } s, t\right] \\ \textbf{and } R \subseteq S_{\mathrm{f}} \times \tilde{S}_{\mathrm{f}}\end{array}$$

Then, by (Y1), conclude:

$$\left[\left[s \xrightarrow{t}_{\mathtt{I}}{}^{+} s' \textbf{ implies } \left[\begin{array}{l}\left[\tilde{s} \xrightarrow{\sim t}_{\mathtt{I}}{}^{+} \tilde{s}' \textbf{ and } s\ R\ \tilde{s} \textbf{ and } s'\ R\ \tilde{s}'\right] \\ \textbf{for some } \tilde{s}, s\end{array}\right]\right] \textbf{ for all } \tilde{s}, \tilde{s}'\right]$$

$$\begin{array}{l}\textbf{and } \left[\left[s \in \blacksquare^{t}_{\mathrm{f}} \textbf{ implies } \left[\left[\tilde{s} \in \tilde{\blacksquare}^{t}_{\mathrm{f}} \textbf{ and } s\ R\ \tilde{s}\right] \textbf{ for some } \tilde{s}\right]\right] \textbf{ for all } s, t\right] \\ \textbf{and } \left[\left[s \in \square^{t}_{\mathrm{f}} \textbf{ implies } \left[\left[\tilde{s} \in \tilde{\square}^{t}_{\mathrm{f}} \textbf{ and } s\ R\ \tilde{s}\right] \textbf{ for some } \tilde{s}\right]\right] \textbf{ for all } s, t\right] \\ \textbf{and } R \subseteq S_{\mathrm{f}} \times \tilde{S}_{\mathrm{f}} \textbf{ and } S_{\mathrm{f}} \subseteq \mathrm{Dom}(R)\end{array}$$

Then, by Def. 5 of $\preceq$, conclude $(S_{\mathrm{f}}, \blacksquare_{\mathrm{f}}, \square_{\mathrm{f}}, \xrightarrow{}_{\mathtt{I}_{\mathrm{f}}}) \preceq_R (\tilde{S}_{\mathrm{f}}, \tilde{\blacksquare}_{\mathrm{f}}, \tilde{\square}_{\mathrm{f}}, \xrightarrow{\sim}_{\mathtt{I}_{\mathrm{f}}})$. Then, by Prop. 29, conclude $[\![\gamma]\!] \preceq_R (\tilde{S}_{\mathrm{f}}, \tilde{\blacksquare}_{\mathrm{f}}, \tilde{\square}_{\mathrm{f}}, \xrightarrow{\sim}_{\mathtt{I}_{\mathrm{f}}})$. Then, by Prop. 30, conclude:

$$[\![\gamma]\!] \preceq_R [\![\mathsf{split}(\gamma; \overline{s}, \overline{s}_1, \overline{s}_2, T_1)]\!]$$

□

Lemma 14. $[\![\mathsf{merge}(\gamma; \overline{S}, \overline{s})]\!] \preceq_{R^{-1}} [\![\gamma]\!]$

Proof. Prove the lemma by a reduction similar to the proof of Lemma 13. □

E.3 Proof

Proof. Prove the theorem by the following reduction. By Lemma 13, conclude:

$$[\![\gamma]\!] \preceq_R [\![\mathsf{merge}(\gamma; \overline{S}, \overline{s})]\!]$$

Then, by Lemma 14, conclude:

$$[\![\gamma]\!] \preceq_R [\![\mathsf{merge}(\gamma; \overline{S}, \overline{s})]\!] \textbf{ and } [\![\mathsf{merge}(\gamma; \overline{S}, \overline{s})]\!] \preceq_{R^{-1}} [\![\gamma]\!]$$

Then, by Def. 6 of $\sim$, conclude $[\![\gamma]\!] \sim_R [\![\mathsf{merge}(\gamma; \overline{S}, \overline{s})]\!]$. Then, by Def. 6 of $\sim$, conclude $[\![\gamma]\!] \sim [\![\mathsf{merge}(\gamma; \overline{S}, \overline{s})]\!]$. □

F Proof of Theorem 6

The scope of auxiliary propositions and auxiliary lemmas in this section is limited to this section.

F.1 Sketch

We first prove that every sequence of information flows from a service s_1 to a service s_n in $\varphi_1 \oplus \varphi_2'$ either has a prefix in φ_1 or in φ_2 that ends on a boundary service or is completely in φ_1 or in φ_2 (Lemma 15). This proof is by induction on the length of the sequence.

Subsequently, let $R = R_1 \cup \{(s, s) \mid s \in S_2\}$ be a candidate bisimulation relation (Prop. 39). Using Lemma 15, we prove that for every sequence of information flows from a service s_1 to a service s_n in $\varphi_1 \oplus \varphi_2'$, there exists a sequence of information flows from a service $\tilde{s}$ to a service $\tilde{s}$ in $\varphi_1' \oplus \varphi_2$ such that $\tilde{s}, \tilde{s}'$ can R-mimic s_1, s_n (Lemma 16). The proof works by showing, by induction, that the sequence of information flows between s_1 and s_n can be split into a number of subsequences to which Lemma 15 applies.

The theorem subsequently follows by Lemma 16 and Def. 5 of $\preceq$.

F.2 Propositions and Lemmas

Antecedent:

Proposition 36. $\varphi_1 \preceq_{R_1} \tilde{\varphi}_1$

Proposition 37. $S_B = S_1 \cap S_2 = \tilde{S}_1 \cap S_2$

Proposition 38. $\big[\big[s\ R_1\ \tilde{s}$ **and** $s \in S_B\big]$ **implies** $s = \tilde{s}\big]$ **for all** $s, \tilde{s}$

Proposition 39. $R = R_1 \cup \{(s, s) \mid s \in S_2\}$

By Def. 2:

Proposition 40. $\varphi_1 = (S_1, \blacksquare_1, \square_1, \longrightarrow\!\!\!\!\!\!\mathtt{I}_1)$

Proposition 41. $\tilde{\varphi}_1 = (\tilde{S}_1, \tilde{\blacksquare}_1, \tilde{\square}_1, \overset{\sim}{\longrightarrow}\!\!\!\!\!\!\mathtt{I}_1)$

Proposition 42. $\varphi_2 = (S_2, \blacksquare_2, \square_2, \longrightarrow\!\!\!\!\!\!\mathtt{I}_2)$

By Def. 4:

Proposition 43. $\varphi_1 \oplus \varphi_2 = (S, \blacksquare, \square, \longrightarrow\!\!\!\!\!\!\mathtt{I})$

Proposition 44. $\tilde{\varphi}_1 \oplus \varphi_2 = (\tilde{S}, \tilde{\blacksquare}, \tilde{\square}, \overset{\sim}{\longrightarrow}\!\!\!\!\!\!\mathtt{I})$

Proposition 45.

$S = S_1 \cup S_2$ **and** $\blacksquare = \blacksquare_1 \uplus \blacksquare_2$ **and** $\square = \square_1 \uplus \square_2$ **and** $\longrightarrow\!\!\!\!\!\!\mathtt{I} = \longrightarrow\!\!\!\!\!\!\mathtt{I}_1 \uplus \longrightarrow\!\!\!\!\!\!\mathtt{I}_2$

Proposition 46.

$\tilde{S} = \tilde{S}_1 \cup S_2$ **and** $\tilde{\blacksquare} = \tilde{\blacksquare}_1 \uplus \blacksquare_2$ **and** $\tilde{\square} = \tilde{\square}_1 \uplus \square_2$ **and** $\overset{\sim}{\longrightarrow}\!\!\!\!\!\!\mathtt{I} = \overset{\sim}{\longrightarrow}\!\!\!\!\!\!\mathtt{I}_1 \uplus \longrightarrow\!\!\!\!\!\!\mathtt{I}_2$

Lemma 15.

$$s_1 \xrightarrow{t}_{\mathtt{I}} \cdots \xrightarrow{t}_{\mathtt{I}} s_n \ \textbf{implies}\ \left[\begin{bmatrix}\left[s_1 \xrightarrow{t}_{\mathtt{I}_1^+} s_j \ \textbf{or}\ s_1 \xrightarrow{t}_{\mathtt{I}_2^+} s_j\right]\\ \textbf{and}\ \left[s_j \in S_B \ \textbf{or}\ j = n\right]\end{bmatrix}\right] \ \textbf{for some}\ j\Big]$$

Proof. Assume:

(A1) $s_1 \xrightarrow{t}_{\mathtt{I}} \cdots \xrightarrow{t}_{\mathtt{I}} s_n$

Prove the lemma by the following induction on $n \geq 2$.

- Base $n = 2$
  Prove the base by the following reduction. By (A1), conclude $s_1 \xrightarrow{t}_{\mathtt{I}} \cdots \xrightarrow{t}_{\mathtt{I}} s_n$. Then, by Base, conclude $s_1 \xrightarrow{t}_{\mathtt{I}} \cdots \xrightarrow{t}_{\mathtt{I}} s_2$. Then, conclude $s_1 \xrightarrow{t}_{\mathtt{I}} s_2$. Then, by Prop. 45, conclude $(s_1, s_2) \in (\longrightarrow_{\mathtt{I}_1} \uplus \longrightarrow_{\mathtt{I}_2})(t)$. Then, conclude:

  $$(s_1, s_2) \in \{\hat{t} \mapsto \xrightarrow{\hat{t}}_{\mathtt{I}_1} \cup \xrightarrow{\hat{t}}_{\mathtt{I}_2} \mid \hat{t} \in T\}(t)$$

  Then, conclude $(s_1, s_2) \in \xrightarrow{t}_{\mathtt{I}_1} \cup \xrightarrow{t}_{\mathtt{I}_2}$. Then, conclude:

  $$s_1 \xrightarrow{t}_{\mathtt{I}_1} s_2 \ \textbf{or}\ s_1 \xrightarrow{t}_{\mathtt{I}_2} s_2$$

  Then, conclude $\left[s_1 \xrightarrow{t}_{\mathtt{I}_1^+} s_2 \ \textbf{or}\ s_1 \xrightarrow{t}_{\mathtt{I}_2^+} s_2\right]$. Then, conclude:

  $$\left[\left[s_1 \xrightarrow{t}_{\mathtt{I}_1^+} s_j \ \textbf{or}\ s_1 \xrightarrow{t}_{\mathtt{I}_2^+} s_j\right] \ \textbf{and}\ j = 2\right] \ \textbf{for some}\ j$$

  Then, by Base, conclude $\left[\left[s_1 \xrightarrow{t}_{\mathtt{I}_1^+} s_j \ \textbf{or}\ s_1 \xrightarrow{t}_{\mathtt{I}_2^+} s_j\right] \ \textbf{and}\ j = n\right]$. Then conclude $\left[\left[s_1 \xrightarrow{t}_{\mathtt{I}_1^+} s_j \ \textbf{or}\ s_1 \xrightarrow{t}_{\mathtt{I}_2^+} s_j\right] \ \textbf{and}\ \left[s_j \in S_B \ \textbf{or}\ j = n\right]\right]$.
- IH

  $$\left[\begin{bmatrix}\hat{s}_1 \xrightarrow{\hat{t}}_{\mathtt{I}} \cdots \xrightarrow{\hat{t}}_{\mathtt{I}} \hat{s}_{\hat{n}}\\ \textbf{and}\ 2 \leq \hat{n} < n\end{bmatrix}\right] \ \textbf{implies}\ \left[\begin{bmatrix}\left[\hat{s}_1 \xrightarrow{\hat{t}}_{\mathtt{I}_1^+} \hat{s}_j \ \textbf{or}\ \hat{s}_1 \xrightarrow{\hat{t}}_{\mathtt{I}_2^+} \hat{s}_j\right]\\ \textbf{and}\ \left[\hat{s}_j \in S_B \ \textbf{or}\ j = \hat{n}\right]\end{bmatrix} \\ \textbf{for some}\ j\end{bmatrix}\right]$$

  $$\textbf{for all}\ \hat{n}, \hat{s}_1, \ldots, \hat{s}_{\hat{n}}, \hat{t}$$

- Step $n > 2$
  Observe:

  (Z1) By Step, conclude $n > 2$. Then, conclude $2 \leq n - 1 < n$.

  (Z2) By (Z1), conclude $2 \leq n - 1 < n$. Then, by (A1), conclude $s_{n-1} \xrightarrow{t}_{\mathtt{I}} s_n$. Then, by Prop. 45, conclude $(s_{n-1}, s_n) \in (\longrightarrow_{\mathtt{I}_1} \uplus \longrightarrow_{\mathtt{I}_2})(t)$. Then, conclude:

  $$(s_{n-1}, s_n) \in \{\hat{t} \mapsto \xrightarrow{\hat{t}}_{\mathtt{I}_1} \cup \xrightarrow{\hat{t}}_{\mathtt{I}_2} \mid \hat{t} \in T\}(t)$$

  Then, conclude $(s_{n-1}, s_n) \in \xrightarrow{t}_{\mathtt{I}_1} \cup \xrightarrow{t}_{\mathtt{I}_2}$. Then, conclude:

  $$s_{n-1} \xrightarrow{t}_{\mathtt{I}_1} s_n \ \textbf{or}\ s_{n-1} \xrightarrow{t}_{\mathtt{I}_2} s_n$$

(Z3) By Prop. 40, conclude $\varphi_1 = (S_1, \blacksquare_1, \square_1, \multimap\!\!\!\!\!\!\mathrm{I}_1)$. Then, by Def. 2 of φ, conclude $\big[\varphi_1 = (S_1, \blacksquare_1, \square_1, \multimap\!\!\!\!\!\!\mathrm{I}_1)$ **and** $\varphi \in \mathbb{F}\mathrm{low}\big]$. Then, conclude:

$$(S_1, \blacksquare_1, \square_1, \multimap\!\!\!\!\!\!\mathrm{I}_1) \in \mathbb{F}\mathrm{low}$$

Then, by Def. 2 of $\mathbb{F}\mathrm{low}$, conclude $\multimap\!\!\!\!\!\!\mathrm{I}_1 : \mathbb{T} \to 2^{S_1 \times S_1}$. Then, conclude:

$$\overset{\hat{t}}{\multimap\!\!\!\!\!\!\mathrm{I}}_1 \subseteq S_1 \times S_1 \textbf{ for all } \hat{t}$$

Then, conclude $\big[\big[\hat{s} \overset{\hat{t}}{\multimap\!\!\!\!\!\!\mathrm{I}}_1 \hat{s}'$ **implies** $(\hat{s}, \hat{s}') \in S_1 \times S_1\big]$ **for all** $\hat{s}, \hat{s}', \hat{t}\big]$. Then, conclude $\big[\big[\hat{s} \overset{\hat{t}}{\multimap\!\!\!\!\!\!\mathrm{I}}_1 \hat{s}'$ **implies** $\hat{s}, \hat{s}' \in S_1\big]$ **for all** $\hat{s}, \hat{s}', \hat{t}\big]$.

(Z4) By a reduction similar to (Z3), conclude:

$$\big[\hat{s} \overset{\hat{t}}{\multimap\!\!\!\!\!\!\mathrm{I}}_2 \hat{s}' \textbf{ implies } \hat{s}, \hat{s}' \in S_2\big] \textbf{ for all } \hat{s}, \hat{s}', \hat{t}$$

(Z5) Suppose $s_1 \overset{t}{\multimap\!\!\!\!\!\!\mathrm{I}}_1^+ s_{n-1} \overset{t}{\multimap\!\!\!\!\!\!\mathrm{I}}_2 s_n$. Then, by (Z3)(Z4), conclude:

$$s_{n-1} \in S_1 \textbf{ and } s_{n-1} \in S_2$$

Then, conclude $s_{n-1} \in S_1 \cap S_2$. Then, by Prop. 37, conclude $s_{n-1} \in S_B$.

(Z6) Suppose $s_1 \overset{t}{\multimap\!\!\!\!\!\!\mathrm{I}}_2^+ s_{n-1} \overset{t}{\multimap\!\!\!\!\!\!\mathrm{I}}_1 s_n$. By a reduction similar to (Z5), conclude:

$$s_{n-1} \in S_B$$

(Z7) Suppose $\big[s_1 \overset{t}{\multimap\!\!\!\!\!\!\mathrm{I}}_1^+ s_{n-1} \overset{t}{\multimap\!\!\!\!\!\!\mathrm{I}}_2 s_n$ **and** $s_1 \overset{t}{\multimap\!\!\!\!\!\!\mathrm{I}}_2^+ s_{n-1} \overset{t}{\multimap\!\!\!\!\!\!\mathrm{I}}_1 s_n\big]$. Then, by (Z5)(Z6), conclude $\big[s_{n-1} \in S_B$ **or** $s_{n-1} \in S_B\big]$. Then, conclude $s_{n-1} \in S_B$.

(Z8) Suppose:

$$\big[\big[s_1 \overset{t}{\multimap\!\!\!\!\!\!\mathrm{I}}_1^+ s_j \textbf{ or } s_1 \overset{t}{\multimap\!\!\!\!\!\!\mathrm{I}}_2^+ s_j\big] \textbf{ and } j = n-1\big] \textbf{ for some } j$$

Then, conclude $\big[s_1 \overset{t}{\multimap\!\!\!\!\!\!\mathrm{I}}_1^+ s_{n-1}$ **or** $s_1 \overset{t}{\multimap\!\!\!\!\!\!\mathrm{I}}_2^+ s_{n-1}\big]$. Then, by (Z2), conclude:

$$\big[s_1 \overset{t}{\multimap\!\!\!\!\!\!\mathrm{I}}_1^+ s_{n-1} \textbf{ or } s_1 \overset{t}{\multimap\!\!\!\!\!\!\mathrm{I}}_2^+ s_{n-1}\big] \textbf{ and } \big[s_{n-1} \overset{t}{\multimap\!\!\!\!\!\!\mathrm{I}}_1 s_n \textbf{ or } s_{n-1} \overset{t}{\multimap\!\!\!\!\!\!\mathrm{I}}_2 s_n\big]$$

Then, conclude:

$$\begin{aligned} &s_1 \overset{t}{\multimap\!\!\!\!\!\!\mathrm{I}}_1^+ s_{n-1} \overset{t}{\multimap\!\!\!\!\!\!\mathrm{I}}_1 s_n \textbf{ or } s_1 \overset{t}{\multimap\!\!\!\!\!\!\mathrm{I}}_1^+ s_{n-1} \overset{t}{\multimap\!\!\!\!\!\!\mathrm{I}}_2 s_n \\ \textbf{or } &s_1 \overset{t}{\multimap\!\!\!\!\!\!\mathrm{I}}_2^+ s_{n-1} \overset{t}{\multimap\!\!\!\!\!\!\mathrm{I}}_1 s_n \textbf{ or } s_1 \overset{t}{\multimap\!\!\!\!\!\!\mathrm{I}}_2^+ s_{n-1} \overset{t}{\multimap\!\!\!\!\!\!\mathrm{I}}_2 s_n \end{aligned}$$

Then, by (Z7), conclude:

$$s_1 \overset{t}{\multimap\!\!\!\!\!\!\mathrm{I}}_1^+ s_{n-1} \overset{t}{\multimap\!\!\!\!\!\!\mathrm{I}}_1 s_n \textbf{ or } s_{n-1} \in S_B \textbf{ or } s_1 \overset{t}{\multimap\!\!\!\!\!\!\mathrm{I}}_2^+ s_{n-1} \overset{t}{\multimap\!\!\!\!\!\!\mathrm{I}}_2 s_n$$

Then, conclude $\big[s_1 \overset{t}{\multimap\!\!\!\!\!\!\mathrm{I}}_1^+ s_n$ **or** $s_{n-1} \in S_B$ **or** $s_1 \overset{t}{\multimap\!\!\!\!\!\!\mathrm{I}}_2^+ s_n\big]$.

(Z9) Suppose:

$$\left[\left[s_1 \xrightarrow{t} \mathrm{I}_1^+ s_j \textbf{ or } s_1 \xrightarrow{t} \mathrm{I}_2^+ s_j\right] \textbf{ and } j = n-1\right] \textbf{ for some } j$$

Then, by (Z8), conclude:

$$\begin{array}{l}\left[s_1 \xrightarrow{t} \mathrm{I}_1^+ s_j \textbf{ or } s_1 \xrightarrow{t} \mathrm{I}_2^+ s_j\right] \textbf{ and } j = n-1 \\ \textbf{and } \left[s_1 \xrightarrow{t} \mathrm{I}_1^+ s_n \textbf{ or } s_{n-1} \in S_B \textbf{ or } s_1 \xrightarrow{t} \mathrm{I}_2^+ s_n\right]\end{array}$$

Then, conclude:

$$\begin{array}{l}\left[s_1 \xrightarrow{t} \mathrm{I}_1^+ s_j \textbf{ or } s_1 \xrightarrow{t} \mathrm{I}_2^+ s_j\right] \\ \textbf{and } \left[s_1 \xrightarrow{t} \mathrm{I}_1^+ s_n \textbf{ or } s_j \in S_B \textbf{ or } s_1 \xrightarrow{t} \mathrm{I}_2^+ s_n\right]\end{array}$$

Then, conclude:

$$\begin{array}{l}\left[\left[s_1 \xrightarrow{t} \mathrm{I}_1^+ s_j \textbf{ or } s_1 \xrightarrow{t} \mathrm{I}_2^+ s_j\right] \textbf{ and } s_j \in S_B\right] \\ \textbf{or } \left[s_1 \xrightarrow{t} \mathrm{I}_1^+ s_n \textbf{ or } s_1 \xrightarrow{t} \mathrm{I}_2^+ s_n\right]\end{array}$$

Then, conclude:

$$\begin{array}{l}\left[\left[s_1 \xrightarrow{t} \mathrm{I}_1^+ s_j \textbf{ or } s_1 \xrightarrow{t} \mathrm{I}_2^+ s_j\right] \textbf{ and } s_j \in S_B\right] \\ \textbf{or } \left[\left[\left[s_1 \xrightarrow{t} \mathrm{I}_1^+ s_{j'} \textbf{ or } s_1 \xrightarrow{t} \mathrm{I}_2^+ s_{j'}\right] \textbf{ and } j' = n\right] \textbf{ for some } j'\right]\end{array}$$

(Z0) Suppose:

$$\left[\begin{array}{l}\left[\left[s_1 \xrightarrow{t} \mathrm{I}_1^+ s_j \textbf{ or } s_1 \xrightarrow{t} \mathrm{I}_2^+ s_j\right] \textbf{ and } s_j \in S_B\right] \\ \textbf{or } \left[\left[s_1 \xrightarrow{t} \mathrm{I}_1^+ s_j \textbf{ or } s_1 \xrightarrow{t} \mathrm{I}_2^+ s_j\right] \textbf{ and } s_j \in S_B\right]\end{array}\right] \textbf{ for some } j$$

Then, conclude $\left[\left[\left[s_1 \xrightarrow{t} \mathrm{I}_1^+ s_j \textbf{ or } s_1 \xrightarrow{t} \mathrm{I}_2^+ s_j\right] \textbf{ and } s_j \in S_B\right]\right]$. Then, conclude $\left[\left[\left[s_1 \xrightarrow{t} \mathrm{I}_1^+ s_{j'} \textbf{ or } s_1 \xrightarrow{t} \mathrm{I}_2^+ s_{j'}\right] \textbf{ and } s_{j'} \in S_B\right] \textbf{ for some } j'\right]$.

Prove the step by the following reduction. By (Z1), conclude $2 \leq n-1 < n$. Then, by (A1), conclude $\left[2 \leq n-1 < n \textbf{ and } s_1 \xrightarrow{t} \mathrm{I} \cdots \xrightarrow{t} \mathrm{I} s_{n-1}\right]$. Then, by IH, conclude:

$$\left[\left[s_1 \xrightarrow{t} \mathrm{I}_1^+ s_j \textbf{ or } s_1 \xrightarrow{t} \mathrm{I}_2^+ s_j\right] \textbf{ and } \left[s_j \in S_B \textbf{ or } j = n-1\right]\right] \textbf{ for some } j$$

Then, conclude:

$$\begin{array}{r}\left[\left[s_1 \xrightarrow{t} \mathrm{I}_1^+ s_j \textbf{ or } s_1 \xrightarrow{t} \mathrm{I}_2^+ s_j\right] \textbf{ and } s_j \in S_B\right] \\ \textbf{or } \left[\left[s_1 \xrightarrow{t} \mathrm{I}_1^+ s_j \textbf{ or } s_1 \xrightarrow{t} \mathrm{I}_2^+ s_j\right] \textbf{ and } j = n-1\right]\end{array}$$

Then, by (Z9), conclude:

$$\begin{array}{l}\left[\left[s_1 \xrightarrow{t} \mathrm{I}_1^+ s_j \textbf{ or } s_1 \xrightarrow{t} \mathrm{I}_2^+ s_j\right] \textbf{ and } s_j \in S_B\right] \\ \textbf{or } \left[\left[s_1 \xrightarrow{t} \mathrm{I}_1^+ s_j \textbf{ or } s_1 \xrightarrow{t} \mathrm{I}_2^+ s_j\right] \textbf{ and } s_j \in S_B\right] \\ \textbf{or } \left[\left[\left[s_1 \xrightarrow{t} \mathrm{I}_1^+ s_{j'} \textbf{ or } s_1 \xrightarrow{t} \mathrm{I}_2^+ s_{j'}\right] \textbf{ and } j' = n\right] \textbf{ for some } j'\right]\end{array}$$

Then, by (Z0), conclude:

$$\begin{array}{r}\Big[\Big[\big[s_1 \xrightarrow{t} \mathrm{I}_1^+ \; s_{j'} \;\; \textbf{or} \;\; s_1 \xrightarrow{t} \mathrm{I}_2^+ \; s_{j'}\big] \;\; \textbf{and} \;\; s_{j'} \in S_B\big] \;\; \textbf{for some} \;\; j'\Big] \\ \textbf{or} \;\; \Big[\big[s_1 \xrightarrow{t} \mathrm{I}_1^+ \; s_{j'} \;\; \textbf{or} \;\; s_1 \xrightarrow{t} \mathrm{I}_2^+ \; s_{j'}\big] \;\; \textbf{and} \;\; j' = n\big] \;\; \textbf{for some} \;\; j'\Big]\end{array}$$

Then, conclude:

$$\left[\begin{array}{l}\Big[\big[s_1 \xrightarrow{t} \mathrm{I}_1^+ \; s_{j'} \;\; \textbf{or} \;\; s_1 \xrightarrow{t} \mathrm{I}_2^+ \; s_{j'}\big] \;\; \textbf{and} \;\; s_{j'} \in S_B\big] \\ \textbf{or} \;\; \Big[\big[s_1 \xrightarrow{t} \mathrm{I}_1^+ \; s_{j'} \;\; \textbf{or} \;\; s_1 \xrightarrow{t} \mathrm{I}_2^+ \; s_{j'}\big] \;\; \textbf{and} \;\; j' = n\big]\end{array}\right] \;\; \textbf{for some} \;\; j'$$

Then, conclude:

$$\big[\big[s_1 \xrightarrow{t} \mathrm{I}_1^+ \; s_{j'} \;\; \textbf{or} \;\; s_1 \xrightarrow{t} \mathrm{I}_2^+ \; s_{j'}\big] \;\; \textbf{and} \;\; \big[s_{j'} \in S_B \;\; \textbf{or} \;\; j' = n\big]\big] \;\; \textbf{for some} \;\; j'$$

□

Lemma 16.

$$s_1 \overset{t}{\text{—I}} \cdots \overset{t}{\text{—I}} s_n \textbf{ implies } \left[\begin{array}{c}\left[\tilde{s} \overset{\sim t}{\text{—I}}{}^{+} \tilde{s}' \textbf{ and } s_1 \; R \; \tilde{s} \textbf{ and } s_n \; R \; \tilde{s}'\right] \\ \textbf{for some } \tilde{s}, \tilde{s}'\end{array}\right]$$

Proof. Assume:

(A1) $s_1 \overset{t}{\text{—I}} \cdots \overset{t}{\text{—I}} s_n$

Observe:

(Z1) Suppose:

$$\tilde{s}_1 \overset{t}{\text{—I}}_1 \cdots \overset{t}{\text{—I}}_1 \tilde{s}_m \textbf{ for some } m, \tilde{s}_1, \ldots, \tilde{s}_m$$

Then, conclude $(\tilde{s}_1, \tilde{s}_2), \ldots, (\tilde{s}_{m-1}, \tilde{s}_m) \in \overset{t}{\text{—I}}_1 \cup \overset{t}{\text{—I}}_2$. Then, by Prop. 46, conclude $\tilde{s}_1 \overset{t}{\text{—I}} \cdots \overset{t}{\text{—I}} \tilde{s}_m$. Then, conclude $\tilde{s}_1 \overset{t}{\text{—I}}{}^{+} \tilde{s}_m$.

(Z2) Suppose:

$$\tilde{s}_1 \overset{t}{\text{—I}}_2 \cdots \overset{t}{\text{—I}}_2 \tilde{s}_m \textbf{ for some } m, \tilde{s}_1, \ldots, \tilde{s}_m$$

Then, by a reduction similar to (Z1), conclude $\tilde{s}_1 \overset{t}{\text{—I}}{}^{+} \tilde{s}_m$.

(Z3) Suppose:

$$\tilde{s}_1 \overset{\sim t}{\text{—I}}_1 \cdots \overset{\sim t}{\text{—I}}_1 \tilde{s}_m \textbf{ for some } m, \tilde{s}_1, \ldots, \tilde{s}_m$$

Then, by a reduction similar to (Z1), conclude $\tilde{s}_1 \overset{\sim t}{\text{—I}}{}^{+} \tilde{s}_m$.

(Z4) Suppose:

$$\tilde{s}_1 \overset{t}{\text{—I}}_2 \cdots \overset{t}{\text{—I}}_2 \tilde{s}_m \textbf{ for some } m, \tilde{s}_1, \ldots, \tilde{s}_m$$

Then, by a reduction similar to (Z1), conclude $\tilde{s}_1 \overset{\sim t}{\text{—I}}{}^{+} \tilde{s}_m$.

(Z5) Suppose:

$$\tilde{s} \overset{t}{\text{—I}}{}^{+}_1 \tilde{s}' \textbf{ for some } \tilde{s}, \tilde{s}'$$

Then, conclude:

$$\left[\begin{array}{l}\tilde{s}_1 \overset{t}{\text{—I}}_1 \cdots \overset{t}{\text{—I}}_1 \tilde{s}_m \\ \textbf{and } \tilde{s} = \tilde{s}_1 \textbf{ and } \tilde{s}' = \tilde{s}_m\end{array}\right] \textbf{ for some } m, \tilde{s}_1, \ldots, \tilde{s}_m$$

Then, by (Z3), conclude $\left[\tilde{s}_1 \overset{t}{\text{—I}}{}^{+} \tilde{s}_m \textbf{ and } \tilde{s} = \tilde{s}_1 \textbf{ and } \tilde{s}' = \tilde{s}_m\right]$. Then, conclude $\tilde{s} \overset{t}{\text{—I}}{}^{+} \tilde{s}'$.

(Z6) Suppose:

$$\tilde{s} \overset{t}{\text{—I}}{}^{+}_2 \tilde{s}' \textbf{ for some } \tilde{s}, \tilde{s}'$$

Then, by a reduction similar to (Z5), conclude $\tilde{s} \overset{t}{\text{—I}}{}^{+} \tilde{s}'$.

(Z7) Suppose:

$$\tilde{s} \overset{\sim t}{\text{—I}}{}^{+}_1 \tilde{s}' \textbf{ for some } \tilde{s}, \tilde{s}'$$

Then, by a reduction similar to (Z5), conclude $\tilde{s} \overset{\sim t}{\text{—I}}{}^{+} \tilde{s}'$.

(Z8) Suppose:

$$\tilde{s} \xrightarrow{t}_{\mathrm{I}_2^+} \tilde{s}' \text{ for some } \tilde{s}, \tilde{s}'$$

Then, by a reduction similar to (Z5), conclude $\tilde{s} \xrightarrow{\sim t}_{\mathrm{I}^+} \tilde{s}'$.

(Z9) Suppose:

$$s \; R_1 \; \tilde{s} \text{ for some } s, \tilde{s}$$

Then, conclude $(s, \tilde{s}) \in R_1 \cup \{(\hat{s}, \hat{s}) \mid \hat{s} \in S_2\}$. Then, by Prop. 39, conclude $s \; R \; \tilde{s}$.

(Z0) By Prop. 36, conclude $\varphi_1 \preceq_{R_1} \tilde{\varphi}_1$. Then, by Prop. 40, conclude:

$$(S_1, \blacksquare_1, \square_1, \xrightarrow{}_{\mathrm{I}_1}) \preceq_{R_1} \tilde{\varphi}_1$$

Then, by Prop. 41, conclude $(S_1, \blacksquare_1, \square_1, \xrightarrow{}_{\mathrm{I}_1}) \preceq_{R_1} (\tilde{S}_1, \tilde{\blacksquare}_1, \tilde{\square}_1, \xrightarrow{\sim}_{\mathrm{I}_1})$. Then, by Def. 5 of $\preceq$, conclude:

$$\Big[s \xrightarrow{t}_{\mathrm{I}_1^+} s' \text{ implies } \Big[\big[\tilde{s} \xrightarrow{\sim t}_{\mathrm{I}_1^+} \tilde{s}' \text{ and } s \; R_1 \; \tilde{s} \text{ and } s' \; R_1 \; \tilde{s}'\big] \text{ for some } \tilde{s}, \tilde{s}'\Big]\Big] \text{ for all } s, s', t$$

(Y1) Suppose:

$$s_1 \xrightarrow{t}_{\mathrm{I}_1^+} s_j \text{ for some } j$$

Then, by (Z0), conclude:

$$\big[\tilde{s} \xrightarrow{\sim t}_{\mathrm{I}_1^+} \tilde{s}' \text{ and } s_1 \; R_1 \; \tilde{s} \text{ and } s_j \; R_1 \; \tilde{s}'\big] \text{ for some } \tilde{s}, \tilde{s}'$$

Then, by (Z9), conclude $\big[\tilde{s} \xrightarrow{\sim t}_{\mathrm{I}_1^+} \tilde{s}' \text{ and } s_1 \; R \; \tilde{s} \text{ and } s_j \; R \; \tilde{s}'\big]$. Then, by (Z7), conclude $\big[\tilde{s} \xrightarrow{\sim t}_{\mathrm{I}^+} \tilde{s}' \text{ and } s_1 \; R \; \tilde{s} \text{ and } s_j \; R \; \tilde{s}'\big]$.

(Y2) By Prop. 40, conclude $\varphi_1 = (S_1, \blacksquare_1, \square_1, \xrightarrow{}_{\mathrm{I}_1})$. Then, by Def. 2 of φ, conclude $\big[\varphi_1 = (S_1, \blacksquare_1, \square_1, \xrightarrow{}_{\mathrm{I}_1}) \text{ and } \varphi \in \mathbb{F}\mathrm{low}\big]$. Then, conclude:

$$(S_1, \blacksquare_1, \square_1, \xrightarrow{}_{\mathrm{I}_1}) \in \mathbb{F}\mathrm{low}$$

Then, by Def. 2 of $\mathbb{F}\mathrm{low}$, conclude $\xrightarrow{}_{\mathrm{I}_1} : \mathbb{T} \to 2^{S_1 \times S_1}$. Then, conclude:

$$\xrightarrow{\hat{t}}_{\mathrm{I}_1} \subseteq S_1 \times S_1 \text{ for all } \hat{t}$$

Then, conclude $\big[\big[\hat{s} \xrightarrow{\hat{t}}_{\mathrm{I}_1} \hat{s}' \text{ implies } (\hat{s}, \hat{s}') \in S_1 \times S_1\big] \text{ for all } \hat{s}, \hat{s}', \hat{t}\big]$. Then, conclude $\big[\big[\hat{s} \xrightarrow{\hat{t}}_{\mathrm{I}_1} \hat{s}' \text{ implies } \hat{s}, \hat{s}' \in S_1\big] \text{ for all } \hat{s}, \hat{s}', \hat{t}\big]$.

(Y3) By a reduction similar to (Y2), conclude:

$$\big[\hat{s} \xrightarrow{\hat{t}}_{\mathrm{I}_2} \hat{s}' \text{ implies } \hat{s}, \hat{s}' \in S_2\big] \text{ for all } \hat{s}, \hat{s}', \hat{t}$$

(Y4) Suppose:

$$\tilde{s} \xrightarrow{t}_{\mathtt{I}_2^+} \tilde{s}' \textbf{ for some } \tilde{s}, \tilde{s}'$$

Then, by (Y3), conclude $\tilde{s}, \tilde{s}' \in S_2$. Then, conclude:

$$(\tilde{s}, \tilde{s}), (\tilde{s}', \tilde{s}') \in \{(s, s) \mid s \in S_2\}$$

Then, conclude $(\tilde{s}, \tilde{s}), (\tilde{s}', \tilde{s}') \in R_1 \cup \{(s, s) \mid s \in S_2\}$. Then, by Prop. 39, conclude $\left[\tilde{s}\ R\ \tilde{s} \textbf{ and } \tilde{s}'\ R\ \tilde{s}'\right]$.

(Y5) Suppose:

$$\left[\tilde{s} \xrightarrow{t}_{\mathtt{I}_2^+} \tilde{s}' \textbf{ and } s_1 = \tilde{s} \textbf{ and } s_2 = \tilde{s}'\right] \textbf{ for some } \tilde{s}, \tilde{s}'$$

Then, by (Y4), conclude $\left[\tilde{s}\ R\ \tilde{s} \textbf{ and } \tilde{s}'\ R\ \tilde{s}' \textbf{ and } s_1 = \tilde{s} \textbf{ and } s_2 = \tilde{s}'\right]$. Then, conclude $\left[s_1\ R\ \tilde{s} \textbf{ and } s_2\ R\ \tilde{s}'\right]$.

(Y6) Suppose:

$$s_1 \xrightarrow{t}_{\mathtt{I}_2^+} s_j \textbf{ for some } j$$

Then, conclude:

$$\left[\tilde{s} \xrightarrow{t}_{\mathtt{I}_2^+} \tilde{s}' \textbf{ and } s_1 = \tilde{s} \textbf{ and } s_j = \tilde{s}'\right] \textbf{ for some } \tilde{s}, \tilde{s}'$$

Then, by (Y5), conclude $\left[\tilde{s} \xrightarrow{t}_{\mathtt{I}_2^+} \tilde{s}' \textbf{ and } s_1\ R\ \tilde{s} \textbf{ and } s_j\ R\ \tilde{s}'\right]$. Then, by (Z6), conclude $\left[\tilde{s} \xrightarrow{\sim t}_{\mathtt{I}^+} \tilde{s}' \textbf{ and } s_1\ R\ \tilde{s} \textbf{ and } s_j\ R\ \tilde{s}'\right]$.

Prove the lemma by the following induction on $n \geq 2$.

– Base $n = 2$
Prove the base by the following reduction. By (A1), conclude $s_1 \xrightarrow{t}_{\mathtt{I}} \cdots \xrightarrow{t}_{\mathtt{I}} s_n$. Then, by Base, conclude $s_1 \xrightarrow{t}_{\mathtt{I}} \cdots \xrightarrow{t}_{\mathtt{I}} s_2$. Then, conclude $s_1 \xrightarrow{t}_{\mathtt{I}} s_2$. Then, by Prop. 45, conclude $(s_1, s_2) \in (\longrightarrow_{\mathtt{I}_1} \uplus \longrightarrow_{\mathtt{I}_2})(t)$. Then, conclude:

$$(s_1, s_2) \in \{\hat{t} \mapsto \xrightarrow{\hat{t}}_{\mathtt{I}_1} \cup \xrightarrow{\hat{t}}_{\mathtt{I}_2} \mid \hat{t} \in T\}(t)$$

Then, conclude $(s_1, s_2) \in \xrightarrow{t}_{\mathtt{I}_1} \cup \xrightarrow{t}_{\mathtt{I}_2}$. Then, conclude:

$$s_1 \xrightarrow{t}_{\mathtt{I}_1} s_2 \textbf{ or } s_1 \xrightarrow{t}_{\mathtt{I}_2} s_2$$

Then, conclude $\left[s_1 \xrightarrow{t}_{\mathtt{I}_1^+} s_2 \textbf{ or } s_1 \xrightarrow{t}_{\mathtt{I}_2^+} s_2\right]$. Then, by (Y1), conclude:

$$\left[\left[\tilde{s} \xrightarrow{\sim t}_{\mathtt{I}^+} \tilde{s}' \textbf{ and } s_1\ R\ \tilde{s} \textbf{ and } s_2\ R\ \tilde{s}'\right] \textbf{ for some } \tilde{s}, \tilde{s}'\right] \textbf{ or } s_1 \xrightarrow{t}_{\mathtt{I}_2^+} s_2$$

Then, by (Y6), conclude:

$$\begin{array}{c}\left[\left[\tilde{s} \xrightarrow{\sim t}_{\mathtt{I}^+} \tilde{s}' \textbf{ and } s_1\ R\ \tilde{s} \textbf{ and } s_2\ R\ \tilde{s}'\right] \textbf{ for some } \tilde{s}, \tilde{s}'\right] \\ \textbf{or} \\ \left[\left[\tilde{s} \xrightarrow{\sim t}_{\mathtt{I}^+} \tilde{s}' \textbf{ and } s_1\ R\ \tilde{s} \textbf{ and } s_2\ R\ \tilde{s}'\right] \textbf{ for some } \tilde{s}, \tilde{s}'\right]\end{array}$$

Then, conclude $\left[\left[\tilde{s} \xrightarrow{\sim t}_{\mathtt{I}^+} \tilde{s}' \textbf{ and } s_1\ R\ \tilde{s} \textbf{ and } s_2\ R\ \tilde{s}'\right] \textbf{ for some } \tilde{s}, \tilde{s}'\right]$. Then, by applying Base, conclude:

$$\left[\tilde{s} \xrightarrow{\sim t}_{\mathtt{I}^+} \tilde{s}' \textbf{ and } s_1\ R\ \tilde{s} \textbf{ and } s_n\ R\ \tilde{s}'\right] \textbf{ for some } \tilde{s}, \tilde{s}'$$

– IH

$$\left[\hat{s}_1 \xrightarrow{\hat{t}}_{\mathtt{I}} \cdots \xrightarrow{\hat{t}}_{\mathtt{I}} \hat{s}_{\hat{n}} \;\textbf{ and }\; 2 \leq \hat{n} < n\right] \;\textbf{ implies }\; \left[\begin{array}{c}\left[\tilde{s} \overset{\sim t}{\longrightarrow}_{\mathtt{I}}^{+} \tilde{s}' \;\textbf{ and }\; \hat{s}_1 \; R \; \tilde{s} \;\textbf{ and }\; \hat{s}_{\hat{n}} \; R \; \tilde{s}'\right] \\ \textbf{for some } \tilde{s}, \tilde{s}'\end{array}\right]$$

$$\textbf{for all } \hat{n}, \hat{s}_1, \ldots, \hat{s}_{\hat{n}}, \hat{t}$$

– Step $n > 2$
Observe:

(X1) Suppose:

$$s_1 \xrightarrow{t}_{\mathtt{I}_1}^{+} s_j \;\textbf{ for some }\; j$$

Then, by (A1), conclude $s_1 \xrightarrow{t}_{\mathtt{I}_1}^{+} s_j \xrightarrow{t}_{\mathtt{I}_1} \cdots \xrightarrow{t}_{\mathtt{I}_1} s_n$. Then, conclude $\left[2 \leq j < n \;\textbf{ or }\; j = n\right]$.

(X2) Suppose:

$$s_1 \xrightarrow{t}_{\mathtt{I}_2}^{+} s_j \;\textbf{ for some }\; j$$

Then, by a reduction similar to (X1), conclude $\left[2 \leq j < n \;\textbf{ or }\; j = n\right]$.

(X3) Suppose:

$$s_1 \xrightarrow{t}_{\mathtt{I}_1}^{+} s_j \;\textbf{ for some }\; j$$

Then, by (X1), conclude $\left[s_1 \xrightarrow{t}_{\mathtt{I}_1}^{+} s_j \;\textbf{ and }\; \left[2 \leq j < n \;\textbf{ or }\; j = n\right]\right]$. Then, conclude $\left[\left[s_1 \xrightarrow{t}_{\mathtt{I}_1}^{+} s_j \;\textbf{ and }\; 2 \leq j < n\right] \;\textbf{ or }\; \left[s_1 \xrightarrow{t}_{\mathtt{I}_1}^{+} s_j \;\textbf{ and }\; j = n\right]\right]$.

(X4) Suppose:

$$s_1 \xrightarrow{t}_{\mathtt{I}_2}^{+} s_j \;\textbf{ for some }\; j$$

Then, by a reduction similar to (X3), conclude:

$$\left[s_1 \xrightarrow{t}_{\mathtt{I}_2}^{+} s_j \;\textbf{ and }\; 2 \leq j < n\right] \;\textbf{ or }\; \left[s_1 \xrightarrow{t}_{\mathtt{I}_2}^{+} s_j \;\textbf{ and }\; j = n\right]$$

(X5) Suppose:

$$\left[s_1 \xrightarrow{t}_{\mathtt{I}_1}^{+} s_j \;\textbf{ and }\; s_j \in S_B\right] \;\textbf{ for some }\; j$$

Then, by applying (X3), conclude:

$$\left[\begin{array}{l}\textbf{or} \; \left[s_1 \xrightarrow{t}_{\mathtt{I}_1}^{+} s_j \;\textbf{ and }\; 2 \leq j < n\right] \\ \phantom{\textbf{or}} \; \left[s_1 \xrightarrow{t}_{\mathtt{I}_1}^{+} s_j \;\textbf{ and }\; j = n\right]\end{array}\right] \;\textbf{ and }\; s_j \in S_B$$

Then, conclude:

$$\begin{array}{l}\left[s_1 \xrightarrow{t}_{\mathtt{I}_1}^{+} s_j \;\textbf{ and }\; 2 \leq j < n \;\textbf{ and }\; s_j \in S_B\right] \\ \textbf{or} \; \left[s_1 \xrightarrow{t}_{\mathtt{I}_1}^{+} s_j \;\textbf{ and }\; j = n\right]\end{array}$$

(X6) Suppose:

$$\left[s_1 \xrightarrow{t}_{\mathtt{I}_2}^{+} s_j \;\textbf{ and }\; s_j \in S_B\right] \;\textbf{ for some }\; j$$

Then, by a reduction similar to (X5), conclude:

$$\begin{array}{l}\left[s_1 \xrightarrow{t}_{\mathtt{I}_2}^{+} s_j \;\textbf{ and }\; 2 \leq j < n \;\textbf{ and }\; s_j \in S_B\right] \\ \textbf{or} \; \left[s_1 \xrightarrow{t}_{\mathtt{I}_2}^{+} s_j \;\textbf{ and }\; j = n\right]\end{array}$$

(X7) Suppose:

$$s_1 \xrightarrow{t}{}_{\mathtt{I}_1^+} s_j \quad \textbf{for some } j$$

Then, by (A1), conclude $s_j \xrightarrow{t}{}_{\mathtt{I}_1} \cdots \xrightarrow{t}{}_{\mathtt{I}_1} s_n$. Then, conclude:

$$(s_j, s_{j+1}), \ldots, (s_{n-1}, s_n) \in \xrightarrow{t}{}_{\mathtt{I}_1} \cup \xrightarrow{t}{}_{\mathtt{I}_2}$$

Then, by Prop. 46, conclude $s_j \xrightarrow{t}{}_{\mathtt{I}} \cdots \xrightarrow{t}{}_{\mathtt{I}} s_n$.

(X8) Suppose:

$$s_1 \xrightarrow{t}{}_{\mathtt{I}_2^+} s_j \quad \textbf{for some } j$$

Then, by a reduction similar to (X7), conclude $s_j \xrightarrow{t}{}_{\mathtt{I}} \cdots \xrightarrow{t}{}_{\mathtt{I}} s_n$.

(X9) Suppose:

$$\left[s_1 \xrightarrow{t}{}_{\mathtt{I}_1^+} s_j \ \textbf{and}\ 2 \leq j < n\right] \quad \textbf{for some } j$$

Then, by (X7), conclude $\left[s_j \xrightarrow{t}{}_{\mathtt{I}} \cdots \xrightarrow{t}{}_{\mathtt{I}} s_n \ \textbf{and}\ 2 \leq j < n\right]$. Then, conclude:

$$\begin{bmatrix} \hat{s}_1 \xrightarrow{t}{}_{\mathtt{I}} \cdots \xrightarrow{t}{}_{\mathtt{I}} \hat{s}_{\hat{n}} \\ \textbf{and}\ s_j = \hat{s}_1 \ \textbf{and}\ \cdots \ \textbf{and}\ s_n = \hat{s}_{\hat{n}} \\ \textbf{and}\ 2 \leq \hat{n} < n \end{bmatrix} \quad \textbf{for some } \hat{n}, \hat{s}_1, \ldots, \hat{s}_{\hat{n}}$$

Then, by IH, conclude:

$$\begin{gathered} s_j = \hat{s}_1 \ \textbf{and}\ \cdots \ \textbf{and}\ s_n = \hat{s}_{\hat{n}} \\ \textbf{and}\ \left[\left[\tilde{s} \xrightarrow{\sim t}{}_{\mathtt{I}^+} \tilde{s}' \ \textbf{and}\ \hat{s}_1\ R\ \tilde{s} \ \textbf{and}\ \hat{s}_{\hat{n}}\ R\ \tilde{s}'\right] \ \textbf{for some}\ \tilde{s}, \tilde{s}'\right] \end{gathered}$$

Then, conclude $\left[\left[\tilde{s} \xrightarrow{\sim t}{}_{\mathtt{I}^+} \tilde{s}' \ \textbf{and}\ s_j\ R\ \tilde{s} \ \textbf{and}\ s_n\ R\ \tilde{s}'\right] \ \textbf{for some}\ \tilde{s}, \tilde{s}'\right]$.

(X0) Suppose:

$$\left[s_1 \xrightarrow{t}{}_{\mathtt{I}_2^+} s_j \ \textbf{and}\ 2 \leq j < n\right] \quad \textbf{for some } j$$

Then, by a reduction similar to (X9), conclude:

$$\left[\tilde{s} \xrightarrow{\sim t}{}_{\mathtt{I}^+} \tilde{s}' \ \textbf{and}\ s_j\ R\ \tilde{s} \ \textbf{and}\ s_n\ R\ \tilde{s}'\right] \ \textbf{for some}\ \tilde{s}, \tilde{s}'$$

(W1) Suppose:

$$\left[s_1 \xrightarrow{t}{}_{\mathtt{I}_1^+} s_j \ \textbf{and}\ 2 \leq j < n\right] \quad \textbf{for some } j$$

Then, by (X9), conclude:

$$\begin{bmatrix} s_1 \xrightarrow{t}{}_{\mathtt{I}_1^+} s_j \\ \textbf{and}\ \tilde{s}^\ddagger \xrightarrow{\sim t}{}_{\mathtt{I}^+} \tilde{s}' \ \textbf{and}\ s_j\ R\ \tilde{s}^\ddagger \ \textbf{and}\ s_n\ R\ \tilde{s}' \end{bmatrix} \quad \textbf{for some } \tilde{s}^\ddagger, \tilde{s}'$$

Then, by (Y1), conclude:

$$\begin{bmatrix} \tilde{s} \xrightarrow{\sim t}{}_{\mathtt{I}^+} \tilde{s}^\dagger \ \textbf{and}\ s_1\ R\ \tilde{s} \ \textbf{and}\ s_j\ R\ \tilde{s}^\dagger \\ \textbf{and}\ \tilde{s}^\ddagger \xrightarrow{\sim t}{}_{\mathtt{I}^+} \tilde{s}' \ \textbf{and}\ s_j\ R\ \tilde{s}^\ddagger \ \textbf{and}\ s_n\ R\ \tilde{s}' \end{bmatrix} \quad \textbf{for some } \tilde{s}, \tilde{s}^\dagger$$

(W2) Suppose:

$$s \; R \; \tilde{s} \;\; \textbf{for some} \;\; s, \tilde{s}$$

Then, by Prop. 39, conclude $(s, \tilde{s}) \in R_1 \cup \{(\hat{s}, \hat{s}) \mid \hat{s} \in S_2\}$. Then, conclude $\big[s \; R_1 \; \tilde{s} \;\; \textbf{or} \;\; (s, \tilde{s}) \in \{(\hat{s}, \hat{s}) \mid \hat{s} \in S_2\}\big]$. Then, conclude $\big[s \; R_1 \; \tilde{s} \;\; \textbf{or} \;\; s = \tilde{s}\big]$.

(W3) Suppose:

$$\big[s \; R \; \tilde{s} \;\; \textbf{and} \;\; s \in S_B\big] \;\; \textbf{for some} \;\; s, \tilde{s}$$

Then, by (W2), conclude $\big[\big[s \; R_1 \; \tilde{s} \;\; \textbf{or} \;\; s = \tilde{s}\big] \;\; \textbf{and} \;\; s \in S_B\big]$. Then, conclude $\big[\big[s \; R_1 \; \tilde{s} \;\; \textbf{and} \;\; s \in S_B\big] \;\; \textbf{or} \;\; s = \tilde{s}\big]$. Then, by Prop. 38, conclude $\big[s = \tilde{s} \;\; \textbf{or} \;\; s = \tilde{s}\big]$. Then, conclude $s = \tilde{s}$.

(W4) Suppose:

$$\left[\begin{array}{c} \tilde{s} \xrightarrow{\sim t}{}_{\mathtt{I}}{}^{+} \; \tilde{s}^{\dagger} \;\; \textbf{and} \;\; \tilde{s}^{\ddagger} \xrightarrow{\sim t}{}_{\mathtt{I}}{}^{+} \; \tilde{s}' \\ \textbf{and} \;\; s_j \; R \; \tilde{s}^{\dagger} \;\; \textbf{and} \;\; s_j \; R \; \tilde{s}^{\ddagger} \;\; \textbf{and} \;\; s_j \in S_B \end{array}\right] \;\; \textbf{for some} \;\; j, \tilde{s}, \tilde{s}^{\dagger}, \tilde{s}^{\ddagger}, \tilde{s}'$$

Then, by (W3), conclude:

$$\tilde{s} \xrightarrow{\sim t}{}_{\mathtt{I}}{}^{+} \; \tilde{s}^{\dagger} \;\; \textbf{and} \;\; \tilde{s}^{\ddagger} \xrightarrow{\sim t}{}_{\mathtt{I}}{}^{+} \; \tilde{s}' \;\; \textbf{and} \;\; s_j = \tilde{s}^{\dagger} \;\; \textbf{and} \;\; s_j = \tilde{s}^{\ddagger}$$

Then, conclude $\tilde{s} \xrightarrow{\sim t}{}_{\mathtt{I}}{}^{+} \; s_j \xrightarrow{\sim t}{}_{\mathtt{I}}{}^{+} \; \tilde{s}'$. Then, conclude $\tilde{s} \xrightarrow{\sim t}{}_{\mathtt{I}}{}^{+} \; \tilde{s}'$.

(W5) Suppose:

$$\big[s_1 \xrightarrow{t}{}_{\mathtt{I}_1}{}^{+} \; s_j \;\; \textbf{and} \;\; 2 \leq j < n \;\; \textbf{and} \;\; s_j \in S_B\big] \;\; \textbf{for some} \;\; j$$

Then, by (W1), conclude:

$$\left[\begin{array}{c} \tilde{s} \xrightarrow{\sim t}{}_{\mathtt{I}}{}^{+} \; \tilde{s}^{\dagger} \;\; \textbf{and} \;\; s_1 \; R \; \tilde{s} \;\; \textbf{and} \;\; s_j \; R \; \tilde{s}^{\dagger} \\ \textbf{and} \;\; \tilde{s}^{\ddagger} \xrightarrow{\sim t}{}_{\mathtt{I}}{}^{+} \; \tilde{s}' \;\; \textbf{and} \;\; s_j \; R \; \tilde{s}^{\ddagger} \;\; \textbf{and} \;\; s_n \; R \; \tilde{s}' \\ \textbf{and} \;\; s_j \in S_B \end{array}\right] \;\; \textbf{for some} \;\; \tilde{s}, \tilde{s}^{\dagger}, \tilde{s}^{\ddagger}, \tilde{s}'$$

Then, by (W4), conclude $\big[s_n \; R \; \tilde{s}' \;\; \textbf{and} \;\; s_1 \; R \; \tilde{s} \;\; \textbf{and} \;\; \tilde{s} \xrightarrow{\sim t}{}_{\mathtt{I}}{}^{+} \; \tilde{s}'\big]$.

(W6) Suppose:

$$\big[s_1 \xrightarrow{t}{}_{\mathtt{I}_2}{}^{+} \; s_j \;\; \textbf{and} \;\; 2 \leq j < n\big] \;\; \textbf{for some} \;\; j$$

Then, by (X0), conclude:

$$\left[\begin{array}{c} s_1 \xrightarrow{t}{}_{\mathtt{I}_2}{}^{+} \; s_j \\ \textbf{and} \;\; \tilde{s}^{\ddagger} \xrightarrow{\sim t}{}_{\mathtt{I}}{}^{+} \; \tilde{s}' \;\; \textbf{and} \;\; s_j \; R \; \tilde{s}^{\ddagger} \;\; \textbf{and} \;\; s_n \; R \; \tilde{s}' \end{array}\right] \;\; \textbf{for some} \;\; \tilde{s}^{\ddagger}, \tilde{s}'$$

Then, by (Y6), conclude:

$$\left[\begin{array}{c} \tilde{s} \xrightarrow{\sim t}{}_{\mathtt{I}}{}^{+} \; \tilde{s}^{\dagger} \;\; \textbf{and} \;\; s_1 \; R \; \tilde{s} \;\; \textbf{and} \;\; s_j \; R \; \tilde{s}^{\dagger} \\ \textbf{and} \;\; \tilde{s}^{\ddagger} \xrightarrow{\sim t}{}_{\mathtt{I}}{}^{+} \; \tilde{s}' \;\; \textbf{and} \;\; s_j \; R \; \tilde{s}^{\ddagger} \;\; \textbf{and} \;\; s_n \; R \; \tilde{s}' \end{array}\right] \;\; \textbf{for some} \;\; \tilde{s}, \tilde{s}^{\dagger}$$

(W7) Suppose:

$$\left[s_1 \xrightarrow{t}_{\mathtt{I}2}^{+} s_j \text{ and } 2 \leq j < n \text{ and } s_j \in S_B\right] \text{ for some } j$$

Then, by (W6), conclude:

$$\left[\begin{array}{l} \tilde{s} \xrightarrow{\sim t}_{\mathtt{I}1}^{+} \tilde{s}^{\dagger} \text{ and } s_1 \, R \, \tilde{s} \text{ and } s_j \, R \, \tilde{s}^{\dagger} \\ \text{and } \tilde{s}^{\ddagger} \xrightarrow{\sim t}_{\mathtt{I}1}^{+} \tilde{s}' \text{ and } s_j \, R \, \tilde{s}^{\ddagger} \text{ and } s_n \, R \, \tilde{s}' \\ \text{and } s_j \in S_B \end{array}\right] \text{ for some } \tilde{s}, \tilde{s}^{\dagger}, \tilde{s}^{\ddagger}, \tilde{s}'$$

Then, by (W4), conclude $\left[s_n \, R \, \tilde{s}' \text{ and } s_1 \, R \, \tilde{s} \text{ and } \tilde{s} \xrightarrow{\sim t}_{\mathtt{I}1}^{+} \tilde{s}'\right]$.

(W8) Suppose:

$$\left[\begin{array}{l} \left[s_1 \xrightarrow{t}_{\mathtt{I}1}^{+} s_j \text{ and } 2 \leq j < n \text{ and } s_j \in S_B\right] \\ \text{or } \left[s_1 \xrightarrow{t}_{\mathtt{I}2}^{+} s_j \text{ and } 2 \leq j < n \text{ and } s_j \in S_B\right] \end{array}\right] \text{ for some } j$$

Then, by (W5), conclude:

$$\begin{array}{l} \left[\left[s_n \, R \, \tilde{s}' \text{ and } s_1 \, R \, \tilde{s} \text{ and } \tilde{s} \xrightarrow{\sim t}_{\mathtt{I}1}^{+} \tilde{s}'\right] \text{ for some } \tilde{s}, \tilde{s}'\right] \\ \text{or } \left[s_1 \xrightarrow{t}_{\mathtt{I}2}^{+} s_j \text{ and } 2 \leq j < n \text{ and } s_j \in S_B\right] \end{array}$$

Then, by (W7), conclude:

$$\begin{array}{l} \left[\left[s_n \, R \, \tilde{s}' \text{ and } s_1 \, R \, \tilde{s} \text{ and } \tilde{s} \xrightarrow{\sim t}_{\mathtt{I}1}^{+} \tilde{s}'\right] \text{ for some } \tilde{s}, \tilde{s}'\right] \\ \text{or } \left[\left[s_n \, R \, \tilde{s}' \text{ and } s_1 \, R \, \tilde{s} \text{ and } \tilde{s} \xrightarrow{\sim t}_{\mathtt{I}1}^{+} \tilde{s}'\right] \text{ for some } \tilde{s}, \tilde{s}'\right] \end{array}$$

Then, conclude $\left[\left[s_n \, R \, \tilde{s}' \text{ and } s_1 \, R \, \tilde{s} \text{ and } \tilde{s} \xrightarrow{\sim t}_{\mathtt{I}1}^{+} \tilde{s}'\right] \text{ for some } \tilde{s}, \tilde{s}'\right]$.

(W9) Suppose:

$$\left[s_1 \xrightarrow{t}_{\mathtt{I}1}^{+} s_j \text{ and } j = n\right] \text{ for some } j$$

Then, by (Y1), conclude:

$$\left[\left[\tilde{s} \xrightarrow{\sim t}_{\mathtt{I}1}^{+} \tilde{s}' \text{ and } s_1 \, R \, \tilde{s} \text{ and } s_j \, R \, \tilde{s}'\right] \text{ for some } \tilde{s}, \tilde{s}'\right] \text{ and } j = n$$

Then, conclude $\left[\left[\tilde{s} \xrightarrow{\sim t}_{\mathtt{I}1}^{+} \tilde{s}' \text{ and } s_1 \, R \, \tilde{s} \text{ and } s_n \, R \, \tilde{s}'\right] \text{ for some } \tilde{s}, \tilde{s}'\right]$.

(W0) Suppose:

$$\left[s_1 \xrightarrow{t}_{\mathtt{I}2}^{+} s_j \text{ and } j = n\right] \text{ for some } j$$

Then, by (Y6), conclude:

$$\left[\left[\tilde{s} \xrightarrow{\sim t}_{\mathtt{I}1}^{+} \tilde{s}' \text{ and } s_1 \, R \, \tilde{s} \text{ and } s_j \, R \, \tilde{s}'\right] \text{ for some } \tilde{s}, \tilde{s}'\right] \text{ and } j = n$$

Then, conclude $\left[\left[\tilde{s} \xrightarrow{\sim t}_{\mathtt{I}1}^{+} \tilde{s}' \text{ and } s_1 \, R \, \tilde{s} \text{ and } s_n \, R \, \tilde{s}'\right] \text{ for some } \tilde{s}, \tilde{s}'\right]$.

(V1) Suppose:

$$\left[\begin{array}{l} \left[s_1 \xrightarrow{t}_{\mathtt{I}1}^{+} s_j \text{ and } j = n\right] \\ \text{or } \left[s_1 \xrightarrow{t}_{\mathtt{I}2}^{+} s_j \text{ and } j = n\right] \\ \text{or } \left[s_1 \xrightarrow{t}_{\mathtt{I}1}^{+} s_j \text{ and } j = n\right] \\ \text{or } \left[s_1 \xrightarrow{t}_{\mathtt{I}2}^{+} s_j \text{ and } j = n\right] \end{array}\right] \text{ for some } j$$

Then, by $\textcircled{\texttt{W9}}$, conclude:

$$\begin{array}{rl}
& \big[\big[\tilde{s} \overset{\sim t}{\relbar\!\!\mathtt{I}}^{+} \tilde{s}' \ \textbf{and}\ s_1 \mathrel{R} \tilde{s} \ \textbf{and}\ s_n \mathrel{R} \tilde{s}'\big] \ \textbf{for some}\ \tilde{s}, \tilde{s}'\big] \\
\textbf{or} & \big[s_1 \overset{t}{\relbar\!\!\mathtt{I}}^{+}_{2} s_j \ \textbf{and}\ j = n\big] \\
\textbf{or} & \big[\big[\tilde{s} \overset{\sim t}{\relbar\!\!\mathtt{I}}^{+} \tilde{s}' \ \textbf{and}\ s_1 \mathrel{R} \tilde{s} \ \textbf{and}\ s_n \mathrel{R} \tilde{s}'\big] \ \textbf{for some}\ \tilde{s}, \tilde{s}'\big] \\
\textbf{or} & \big[s_1 \overset{t}{\relbar\!\!\mathtt{I}}^{+}_{2} s_j \ \textbf{and}\ j = n\big]
\end{array}$$

Then, by $\textcircled{\texttt{W0}}$, conclude:

$$\begin{array}{rl}
& \big[\big[\tilde{s} \overset{\sim t}{\relbar\!\!\mathtt{I}}^{+} \tilde{s}' \ \textbf{and}\ s_1 \mathrel{R} \tilde{s} \ \textbf{and}\ s_n \mathrel{R} \tilde{s}'\big] \ \textbf{for some}\ \tilde{s}, \tilde{s}'\big] \\
\textbf{or} & \big[\big[\tilde{s} \overset{\sim t}{\relbar\!\!\mathtt{I}}^{+} \tilde{s}' \ \textbf{and}\ s_1 \mathrel{R} \tilde{s} \ \textbf{and}\ s_n \mathrel{R} \tilde{s}'\big] \ \textbf{for some}\ \tilde{s}, \tilde{s}'\big] \\
\textbf{or} & \big[\big[\tilde{s} \overset{\sim t}{\relbar\!\!\mathtt{I}}^{+} \tilde{s}' \ \textbf{and}\ s_1 \mathrel{R} \tilde{s} \ \textbf{and}\ s_n \mathrel{R} \tilde{s}'\big] \ \textbf{for some}\ \tilde{s}, \tilde{s}'\big] \\
\textbf{or} & \big[\big[\tilde{s} \overset{\sim t}{\relbar\!\!\mathtt{I}}^{+} \tilde{s}' \ \textbf{and}\ s_1 \mathrel{R} \tilde{s} \ \textbf{and}\ s_n \mathrel{R} \tilde{s}'\big] \ \textbf{for some}\ \tilde{s}, \tilde{s}'\big]
\end{array}$$

Then, conclude $\big[\big[\tilde{s} \overset{\sim t}{\relbar\!\!\mathtt{I}}^{+} \tilde{s}' \ \textbf{and}\ s_1 \mathrel{R} \tilde{s} \ \textbf{and}\ s_n \mathrel{R} \tilde{s}'\big] \ \textbf{for some}\ \tilde{s}, \tilde{s}'\big]$.

Prove the step by the following reduction. By $\textcircled{\texttt{A1}}$, conclude $s_1 \overset{t}{\relbar\!\!\mathtt{I}} \cdots \overset{t}{\relbar\!\!\mathtt{I}} s_n$. Then, by Lemma 15, conclude:

$$\big[\big[s_1 \overset{t}{\relbar\!\!\mathtt{I}}^{+}_{1} s_j \ \textbf{or}\ s_1 \overset{t}{\relbar\!\!\mathtt{I}}^{+}_{2} s_j\big] \ \textbf{and}\ \big[s_j \in S_B \ \textbf{or}\ j = n\big]\big] \ \textbf{for some}\ j$$

Then, conclude:

$$\begin{array}{rl}
& \big[\big[s_1 \overset{t}{\relbar\!\!\mathtt{I}}^{+}_{1} s_j \ \textbf{or}\ s_1 \overset{t}{\relbar\!\!\mathtt{I}}^{+}_{2} s_j\big] \ \textbf{and}\ s_j \in S_B\big] \\
\textbf{or} & \big[\big[s_1 \overset{t}{\relbar\!\!\mathtt{I}}^{+}_{1} s_j \ \textbf{or}\ s_1 \overset{t}{\relbar\!\!\mathtt{I}}^{+}_{2} s_j\big] \ \textbf{and}\ j = n\big]
\end{array}$$

Then, conclude:

$$\begin{array}{rl}
& \big[s_1 \overset{t}{\relbar\!\!\mathtt{I}}^{+}_{1} s_j \ \textbf{and}\ s_j \in S_B\big] \ \textbf{or}\ \big[s_1 \overset{t}{\relbar\!\!\mathtt{I}}^{+}_{2} s_j \ \textbf{and}\ s_j \in S_B\big] \\
\textbf{or} & \big[s_1 \overset{t}{\relbar\!\!\mathtt{I}}^{+}_{1} s_j \ \textbf{and}\ j = n\big] \ \textbf{or}\ \big[s_1 \overset{t}{\relbar\!\!\mathtt{I}}^{+}_{2} s_j \ \textbf{and}\ j = n\big]
\end{array}$$

Then, by $\textcircled{\texttt{X5}}\textcircled{\texttt{X6}}$, conclude:

$$\begin{array}{rl}
& \big[s_1 \overset{t}{\relbar\!\!\mathtt{I}}^{+}_{1} s_j \ \textbf{and}\ 2 \le j < n \ \textbf{and}\ s_j \in S_B\big] \\
\textbf{or} & \big[s_1 \overset{t}{\relbar\!\!\mathtt{I}}^{+}_{1} s_j \ \textbf{and}\ j = n\big] \\
\textbf{or} & \big[s_1 \overset{t}{\relbar\!\!\mathtt{I}}^{+}_{2} s_j \ \textbf{and}\ 2 \le j < n \ \textbf{and}\ s_j \in S_B\big] \\
\textbf{or} & \big[s_1 \overset{t}{\relbar\!\!\mathtt{I}}^{+}_{2} s_j \ \textbf{and}\ j = n\big] \\
\textbf{or} & \big[s_1 \overset{t}{\relbar\!\!\mathtt{I}}^{+}_{1} s_j \ \textbf{and}\ j = n\big] \ \textbf{or}\ \big[s_1 \overset{t}{\relbar\!\!\mathtt{I}}^{+}_{2} s_j \ \textbf{and}\ j = n\big]
\end{array}$$

Then, by $\textcircled{\texttt{W8}}$, conclude:

$$\begin{array}{rl}
& \big[s_1 \overset{t}{\relbar\!\!\mathtt{I}}^{+}_{1} s_j \ \textbf{and}\ j = n\big] \ \textbf{or}\ \big[s_1 \overset{t}{\relbar\!\!\mathtt{I}}^{+}_{2} s_j \ \textbf{and}\ j = n\big] \\
\textbf{or} & \big[s_1 \overset{t}{\relbar\!\!\mathtt{I}}^{+}_{1} s_j \ \textbf{and}\ j = n\big] \ \textbf{or}\ \big[s_1 \overset{t}{\relbar\!\!\mathtt{I}}^{+}_{2} s_j \ \textbf{and}\ j = n\big] \\
\textbf{or} & \big[\big[\tilde{s} \overset{\sim t}{\relbar\!\!\mathtt{I}}^{+} \tilde{s}' \ \textbf{and}\ s_1 \mathrel{R} \tilde{s} \ \textbf{and}\ s_n \mathrel{R} \tilde{s}'\big] \ \textbf{for some}\ \tilde{s}, \tilde{s}'\big]
\end{array}$$

Then, by $\textcircled{\texttt{V1}}$, conclude:

$$\begin{array}{rl}
& \big[\big[\tilde{s} \overset{\sim t}{\relbar\!\!\mathtt{I}}^{+} \tilde{s}' \ \textbf{and}\ s_1 \mathrel{R} \tilde{s} \ \textbf{and}\ s_n \mathrel{R} \tilde{s}'\big] \ \textbf{for some}\ \tilde{s}, \tilde{s}'\big] \\
\textbf{or} & \big[\big[\tilde{s} \overset{\sim t}{\relbar\!\!\mathtt{I}}^{+} \tilde{s}' \ \textbf{and}\ s_1 \mathrel{R} \tilde{s} \ \textbf{and}\ s_n \mathrel{R} \tilde{s}'\big] \ \textbf{for some}\ \tilde{s}, \tilde{s}'\big]
\end{array}$$

Then, conclude $\big[\big[\tilde{s} \overset{\sim t}{\relbar\!\!\mathtt{I}}^{+} \tilde{s}' \ \textbf{and}\ s_1 \mathrel{R} \tilde{s} \ \textbf{and}\ s_n \mathrel{R} \tilde{s}'\big] \ \textbf{for some}\ \tilde{s}, \tilde{s}'\big]$.

□

F.3 Proof

Proof. Observe:

(Z1) Suppose:

$$s \xrightarrow{t}_{\mathtt{I}}{}^{+} s' \text{ for some } s, s', t$$

Then, conclude:

$$\left[s_1 \xrightarrow{t}_{\mathtt{I}} \cdots \xrightarrow{t}_{\mathtt{I}} s_n \text{ and } s = s_1 \text{ and } s' = s_n\right] \text{ for some } n, s_1, \ldots, s_n$$

Then, by Lemma 16, conclude:

$$\left[\begin{array}{c}\left[\tilde{s} \xrightarrow{\sim t}_{\mathtt{I}}{}^{+} \tilde{s}' \text{ and } s_1 \mathrel{R} \tilde{s} \text{ and } s_n \mathrel{R} \tilde{s}'\right] \\ \text{for some } \tilde{s}, \tilde{s}'\end{array}\right] \text{ and } s = s_1 \text{ and } s' = s_n$$

Then, conclude $\left[\left[\tilde{s} \xrightarrow{\sim t}_{\mathtt{I}}{}^{+} \tilde{s}' \text{ and } s \mathrel{R} \tilde{s} \text{ and } s' \mathrel{R} \tilde{s}'\right] \text{ for some } \tilde{s}, \tilde{s}'\right]$.

(Z2) Suppose:

$$\tilde{s} \in \tilde{\blacksquare}_1^t \text{ for some } \tilde{s}, t$$

Then, conclude $\tilde{s} \in \tilde{\blacksquare}_1^t \cup \blacksquare_2^t$. Then, conclude $\tilde{s} \in \{\hat{t} \mapsto \tilde{\blacksquare}_1^{\hat{t}} \cup \blacksquare_2^{\hat{t}} \mid \hat{t} \in \mathbb{T}\}(t)$. Then, conclude $\tilde{s} \in (\tilde{\blacksquare}_1 \uplus \blacksquare_2)(t)$. Then, by Prop. 46, conclude $\tilde{s} \in \tilde{\blacksquare}^t$.

(Z3) Suppose:

$$\tilde{s} \in \blacksquare_2^t \text{ for some } \tilde{s}, t$$

Then, by a reduction similar to (Z2), conclude $\tilde{s} \in \tilde{\blacksquare}^t$.

(Z4) Suppose:

$$\tilde{s} \in \tilde{\square}_1^t \text{ for some } \tilde{s}, t$$

Then, by a reduction similar to (Z2), conclude $\tilde{s} \in \tilde{\square}^t$.

(Z5) Suppose:

$$\tilde{s} \in \square_2^t \text{ for some } \tilde{s}, t$$

Then, by a reduction similar to (Z2), conclude $\tilde{s} \in \tilde{\square}^t$.

(Z6) Suppose:

$$s \mathrel{R_1} \tilde{s} \text{ for some } s, \tilde{s}$$

Then, conclude $(s, \tilde{s}) \in R_1 \cup \{(\hat{s}, \hat{s}) \mid \hat{s} \in S_2\}$. Then, by Prop. 39, conclude $s \mathrel{R} \tilde{s}$.

(Z7) By Prop. 36, conclude $\varphi_1 \preceq_{R_1} \tilde{\varphi}_1$. Then, by Prop. 40, conclude:

$$(S_1, \blacksquare_1, \square_1, \xrightarrow{}_{\mathtt{I}1}) \preceq_{R_1} \tilde{\varphi}_1$$

Then, by Prop. 41, conclude $(S_1, \blacksquare_1, \square_1, \xrightarrow{}_{\mathtt{I}1}) \preceq_{R_1} (\tilde{S}_1, \tilde{\blacksquare}_1, \tilde{\square}_1, \xrightarrow{\sim}_{\mathtt{I}1})$. Then, by Def. 5 of $\preceq$, conclude:

$$\left[s \in \blacksquare_1^t \text{ implies } \left[\begin{array}{c}\left[\tilde{s} \in \tilde{\blacksquare}_1^t \text{ and } s \mathrel{R_1} \tilde{s}\right] \\ \text{for some } \tilde{s}\end{array}\right]\right] \text{ for all } s, t$$

(Z8) By a reduction similar to (Z7), conclude:

$$\big[s \in \square_1^t \ \textbf{implies} \ \Big[\big[\tilde{s} \in \tilde{\square}_1^t \ \textbf{and} \ s \ R_1 \ \tilde{s}\big] \ \textbf{for some} \ \tilde{s}\Big]\big] \ \textbf{for all} \ s, t$$

(Z9) Suppose:

$$s \in \blacksquare_1^t \ \textbf{for some} \ s, t$$

Then, by (Z7), conclude:

$$\big[\tilde{s} \in \tilde{\blacksquare}_1^t \ \textbf{and} \ s \ R_1 \ \tilde{s}\big] \ \textbf{for some} \ \tilde{s}$$

Then, by (Z2), conclude $\big[\tilde{s} \in \tilde{\blacksquare}^t \ \textbf{and} \ s \ R_1 \ \tilde{s}\big]$. Then, by (Z6), conclude:

$$\tilde{s} \in \tilde{\blacksquare}^t \ \textbf{and} \ s \ R \ \tilde{s}$$

(Z0) Suppose:

$$s \in \square_1^t \ \textbf{for some} \ s, t$$

Then, by a reduction similar to (Z9), conclude $\big[\tilde{s} \in \tilde{\square}^t \ \textbf{and} \ s \ R \ \tilde{s}\big]$.

(Y1) By Prop. 42, conclude $\varphi_2 = (S_2, \blacksquare_2, \square_2, \longrightarrow\!\!\!\!\!\!\shortmid_2)$. Then, by Def. 2 of φ, conclude $\big[\varphi_2 = (S_2, \blacksquare_2, \square_2, \longrightarrow\!\!\!\!\!\!\shortmid_2) \ \textbf{and} \ \varphi \in \mathbb{F}\text{low}\big]$. Then, conclude:

$$(S_2, \blacksquare_2, \square_2, \longrightarrow\!\!\!\!\!\!\shortmid_2) \in \mathbb{F}\text{low}$$

Then, by Def. 2 of $\mathbb{F}$low, conclude $\blacksquare_2 : \mathbb{T} \to 2^{S_2}$. Then, conclude:

$$\blacksquare_2^{\hat{t}} \subseteq S_2 \ \textbf{for all} \ \hat{t}$$

Then, conclude $\big[\big[\hat{s} \in \blacksquare_2^{\hat{t}} \ \textbf{implies} \ \hat{s} \in S_2\big] \ \textbf{for all} \ \hat{s}, \hat{t}\big]$.

(Y2) By a reduction similar to (Y1), conclude:

$$\big[\hat{s} \in \square_2^{\hat{t}} \ \textbf{implies} \ \hat{s} \in S_2\big] \ \textbf{for all} \ \hat{s}, \hat{t}$$

(Y3) Suppose:

$$s \in \blacksquare_2^t \ \textbf{for some} \ s, t$$

Then, by (Y1), conclude $s \in S_2$. Then, conclude $(s, s) \in \{(\hat{s}, \hat{s}) \mid \hat{s} \in S_2\}$. Then, conclude $(s, s) \in R_1 \cup \{(\hat{s}, \hat{s}) \mid \hat{s} \in S_2\}$. Then, by Prop. 39, conclude $s \ R \ s$.

(Y4) Suppose:

$$s \in \square_2^t \ \textbf{for some} \ s, t$$

Then, by a reduction similar to (Y3), conclude $s \ R \ s$.

(Y5) Suppose:

$$s \in \blacksquare_2^t \textbf{ for some } s, t$$

Then, by (Z3), conclude $\left[s \in \blacksquare_2^t \textbf{ and } s \in \tilde{\blacksquare}^t\right]$. Then, by (Y3), conclude:

$$s \in \tilde{\blacksquare}^t \textbf{ and } s \mathrel{R} s$$

Then, conclude $\left[\left[\tilde{s} \in \tilde{\blacksquare}^t \textbf{ and } s \mathrel{R} \tilde{s}\right] \textbf{ for some } \tilde{s}\right]$.

(Y6) Suppose:

$$s \in \square_2^t \textbf{ for some } s, t$$

Then, by a reduction similar to (Y5), conclude:

$$\left[\tilde{s} \in \tilde{\square}^t \textbf{ and } s \mathrel{R} \tilde{s}\right] \textbf{ for some } \tilde{s}$$

(Y7) Suppose:

$$s \in \blacksquare^t \textbf{ for some } s, t$$

Then, by Prop. 45, conclude $s \in (\blacksquare_1 \uplus \blacksquare_2)(t)$. Then, conclude:

$$s \in \{\hat{t} \mapsto \blacksquare_1^{\hat{t}} \cup \blacksquare_2^{\hat{t}} \mid \hat{t} \in \mathbb{T}\}(t)$$

Then, conclude $s \in \blacksquare_1^t \cup \blacksquare_2^t$. Then, conclude $\left[s \in \blacksquare_1^t \textbf{ or } s \in \blacksquare_2^t\right]$. Then, by (Z9), conclude $\left[\left[\left[\tilde{s} \in \tilde{\blacksquare}^t \textbf{ and } s \mathrel{R} \tilde{s}\right] \textbf{ for some } \tilde{s}\right] \textbf{ or } s \in \blacksquare_2^t\right]$. Then, by (Y5), conclude:

$$\left[\left[\tilde{s} \in \tilde{\blacksquare}^t \textbf{ and } s \mathrel{R} \tilde{s}\right] \textbf{ for some } \tilde{s}\right] \textbf{ or } \left[\left[\tilde{s} \in \tilde{\blacksquare}^t \textbf{ and } s \mathrel{R} \tilde{s}\right] \textbf{ for some } \tilde{s}\right]$$

Then, conclude $\left[\left[\tilde{s} \in \tilde{\blacksquare}^t \textbf{ and } s \mathrel{R} \tilde{s}\right] \textbf{ for some } \tilde{s}\right]$.

(Y8) Suppose:

$$s \in \square^t \textbf{ for some } s, t$$

Then, by a reduction similar to (Y7), conclude:

$$\left[\tilde{s} \in \tilde{\square}^t \textbf{ and } s \mathrel{R} \tilde{s}\right] \textbf{ for some } \tilde{s}$$

(Y9) By Prop. 36, conclude $\varphi_1 \preceq_{R_1} \tilde{\varphi}_1$. Then, by Prop. 40, conclude:

$$(S_1, \blacksquare_1, \square_1, \text{—}\!\!\text{I}_1) \preceq_{R_1} \tilde{\varphi}_1$$

Then, by Prop. 41, conclude $(S_1, \blacksquare_1, \square_1, \text{—}\!\!\text{I}_1) \preceq_{R_1} (\tilde{S}_1, \tilde{\blacksquare}_1, \tilde{\square}_1, \overset{\sim}{\text{—}}\!\!\text{I}_1)$. Then, by Def. 5 of $\preceq$, conclude $R_1 \subseteq S_1 \times \tilde{S}_1$.

(Y0) Suppose:

$$s \mathrel{R_1} \tilde{s} \textbf{ for some } s, \tilde{s}$$

Then, by (Y9), conclude $\left[s \mathrel{R_1} \tilde{s} \textbf{ and } R_1 \subseteq S_1 \times \tilde{S}_1\right]$. Then, conclude $(s, \tilde{s}) \in S_1 \times \tilde{S}_1$. Then, conclude $\left[s \in S_1 \textbf{ and } \tilde{s} \in \tilde{S}_1\right]$.

(X1) Suppose:

$$s\ R\ \tilde{s}\ \textbf{ for some }\ s, \tilde{s}$$

Then, by Prop. 39, conclude $(s, \tilde{s}) \in R_1 \cup \{(\hat{s}, \hat{s}) \mid \hat{s} \in S_2\}$. Then, conclude $\big[s\ R_1\ \tilde{s}\ \textbf{ or }\ (s, \tilde{s}) \in \{(\hat{s}, \hat{s}) \mid \hat{s} \in S_2\}\big]$. Then, by (Y0), conclude:

$$\big[s \in S_1\ \textbf{ and }\ \tilde{s} \in \tilde{S}_1\big]\ \textbf{ or }\ (s, \tilde{s}) \in \{(\hat{s}, \hat{s}) \mid \hat{s} \in S_2\}$$

Then, conclude $\big[\big[s \in S_1\ \textbf{ and }\ \tilde{s} \in \tilde{S}_1\big]\ \textbf{ or }\ s, \tilde{s} \in S_2\big]$. Then, conclude:

$$\big[s \in S_1 \cup S_2\ \textbf{ and }\ \tilde{s} \in \tilde{S}_1 \cup S_2\big]\ \textbf{ or }\ \big[s \in S_1 \cup S_2\ \textbf{ and }\ \tilde{s} \in \tilde{S}_1 \cup S_2\big]$$

Then, conclude $\big[s \in S_1 \cup S_2\ \textbf{ and }\ \tilde{s} \in \tilde{S}_1 \cup S_2\big]$. Then, by Prop. 45, conclude $\big[s \in S\ \textbf{ and }\ \tilde{s} \in \tilde{S}_1 \cup S_2\big]$. Then, by Prop. 46, conclude $\big[s \in S\ \textbf{ and }\ \tilde{s} \in \tilde{S}\big]$. Then, conclude $(s, \tilde{s}) \in S \times \tilde{S}$.

(X2) By (X1), conclude $\big[\big[s\ R\ \tilde{s}\ \textbf{ implies }\ (s, \tilde{s}) \in S \times \tilde{S}\big]\ \textbf{ for all }\ s, \tilde{s}\big]$. Then, conclude $R \subseteq S \times \tilde{S}$.

(X3) By Prop. 36, conclude $\varphi_1 \preceq_{R_1} \tilde{\varphi}_1$. Then, by Prop. 40, conclude:

$$(S_1, \blacksquare_1, \square_1, \text{—}\!\!\text{I}_1) \preceq_{R_1} \tilde{\varphi}_1$$

Then, by Prop. 41, conclude $(S_1, \blacksquare_1, \square_1, \text{—}\!\!\text{I}_1) \preceq_{R_1} (\tilde{S}_1, \tilde{\blacksquare}_1, \tilde{\square}_1, \tilde{\text{—}}\!\!\text{I}_1)$. Then, by Def. 5 of $\preceq$, conclude $S_1 \subseteq \mathrm{Dom}(R_1)$.

(X4) Suppose:

$$s \in S_1\ \textbf{ for some }\ s$$

Then, by (X3), conclude $\big[s \in S_1\ \textbf{ and }\ S_1 \subseteq \mathrm{Dom}(R_1)\big]$. Then, conclude $s \in \mathrm{Dom}(R_1)$.

(X5) Suppose:

$$s \in S\ \textbf{ for some }\ s$$

Then, by Prop. 45, conclude $s \in S_1 \cup S_2$. Then, conclude:

$$s \in S_1\ \textbf{ or }\ s \in S_2$$

Then, by (X4), conclude $\big[s \in \mathrm{Dom}(R_1)\ \textbf{ or }\ s \in S_2\big]$. Then, conclude:

$$s \in \mathrm{Dom}(R_1)\ \textbf{ or }\ s \in \mathrm{Dom}(\{(\hat{s}, \hat{s}) \mid \hat{s} \in S_2\})$$

Then, conclude:

$$\begin{array}{r} s \in \mathrm{Dom}(R_1) \cup \mathrm{Dom}(\{(\hat{s}, \hat{s}) \mid \hat{s} \in S_2\}) \\ \textbf{or }\ s \in \mathrm{Dom}(R_1) \cup \mathrm{Dom}(\{(\hat{s}, \hat{s}) \mid \hat{s} \in S_2\}) \end{array}$$

Then, conclude $s \in \mathrm{Dom}(R_1) \cup \mathrm{Dom}(\{(\hat{s}, \hat{s}) \mid \hat{s} \in S_2\})$. Then, conclude $s \in \mathrm{Dom}(R_1 \cup \{(\hat{s}, \hat{s}) \mid \hat{s} \in S_2\})$. Then, by Prop. 39, conclude $s \in \mathrm{Dom}(R)$.

Ⓧ6 By Ⓧ5, conclude $\big[\big[s \in S \textbf{ implies } s \in \mathrm{Dom}(R)\big] \textbf{ for all } s\big]$. Then, conclude $S \subseteq \mathrm{Dom}(R)$.

Prove the lemma by the following reduction. By Ⓩ1, conclude:

$$\Big[s \xrightarrow{t}_{\mathtt{I}}{}^{+} s' \textbf{ implies } \left[\begin{array}{c}[\tilde{s} \xrightarrow{\sim t}_{\mathtt{I}}{}^{+} \tilde{s}' \textbf{ and } s\ R\ \tilde{s} \textbf{ and } s'\ R\ \tilde{s}'] \\ \textbf{for some } \tilde{s}, \tilde{s}'\end{array}\right]\Big] \textbf{ for all } s, s', t$$

Then, by applying ⓎⓎ7Ⓨ8, conclude:

$$\Big[\Big[s \xrightarrow{t}_{\mathtt{I}}{}^{+} s' \textbf{ implies } \left[\begin{array}{c}[\tilde{s} \xrightarrow{\sim t}_{\mathtt{I}}{}^{+} \tilde{s}' \textbf{ and } s\ R\ \tilde{s} \textbf{ and } s'\ R\ \tilde{s}'] \\ \textbf{for some } \tilde{s}, \tilde{s}'\end{array}\right]\Big] \textbf{ for all } s, s', t\Big]$$
$$\textbf{and } \Big[\Big[s \in \blacksquare^t \textbf{ implies } \left[\begin{array}{c}[\tilde{s} \in \tilde{\blacksquare}^t \textbf{ and } s\ R\ \tilde{s}] \\ \textbf{for some } \tilde{s}\end{array}\right]\Big] \textbf{ for all } s, t\Big]$$
$$\textbf{and } \Big[\Big[s \in \square^t \textbf{ implies } \left[\begin{array}{c}[\tilde{s} \in \tilde{\square}^t \textbf{ and } s\ R\ \tilde{s}] \\ \textbf{for some } \tilde{s}\end{array}\right]\Big] \textbf{ for all } s, t\Big]$$

Then, by applying Ⓧ2, conclude:

$$\Big[\Big[s \xrightarrow{t}_{\mathtt{I}}{}^{+} s' \textbf{ implies } \left[\begin{array}{c}[\tilde{s} \xrightarrow{\sim t}_{\mathtt{I}}{}^{+} \tilde{s}' \textbf{ and } s\ R\ \tilde{s} \textbf{ and } s'\ R\ \tilde{s}'] \\ \textbf{for some } \tilde{s}, \tilde{s}'\end{array}\right]\Big] \textbf{ for all } s, s', t\Big]$$
$$\textbf{and } \Big[\Big[s \in \blacksquare^t \textbf{ implies } \left[\begin{array}{c}[\tilde{s} \in \tilde{\blacksquare}^t \textbf{ and } s\ R\ \tilde{s}] \\ \textbf{for some } \tilde{s}\end{array}\right]\Big] \textbf{ for all } s, t\Big]$$
$$\textbf{and } \Big[\Big[s \in \square^t \textbf{ implies } \left[\begin{array}{c}[\tilde{s} \in \tilde{\square}^t \textbf{ and } s\ R\ \tilde{s}] \\ \textbf{for some } \tilde{s}\end{array}\right]\Big] \textbf{ for all } s, t\Big]$$
$$\textbf{and } R \subseteq S \times \tilde{S}$$

Then, by applying Ⓧ6, conclude:

$$\Big[\Big[s \xrightarrow{t}_{\mathtt{I}}{}^{+} s' \textbf{ implies } \left[\begin{array}{c}[\tilde{s} \xrightarrow{\sim t}_{\mathtt{I}}{}^{+} \tilde{s}' \textbf{ and } s\ R\ \tilde{s} \textbf{ and } s'\ R\ \tilde{s}'] \\ \textbf{for some } \tilde{s}, \tilde{s}'\end{array}\right]\Big] \textbf{ for all } s, s', t\Big]$$
$$\textbf{and } \Big[\Big[s \in \blacksquare^t \textbf{ implies } \left[\begin{array}{c}[\tilde{s} \in \tilde{\blacksquare}^t \textbf{ and } s\ R\ \tilde{s}] \\ \textbf{for some } \tilde{s}\end{array}\right]\Big] \textbf{ for all } s, t\Big]$$
$$\textbf{and } \Big[\Big[s \in \square^t \textbf{ implies } \left[\begin{array}{c}[\tilde{s} \in \tilde{\square}^t \textbf{ and } s\ R\ \tilde{s}] \\ \textbf{for some } \tilde{s}\end{array}\right]\Big] \textbf{ for all } s, t\Big]$$
$$\textbf{and } R \subseteq S \times \tilde{S} \textbf{ and } S \subseteq \mathrm{Dom}(R)$$

Then, by Def. 5 of $\preceq$, conclude $(S, \blacksquare, \square, \xrightarrow{}_{\mathtt{I}}) \preceq_R (\tilde{S}, \tilde{\blacksquare}, \tilde{\square}, \xrightarrow{\sim}_{\mathtt{I}})$. Then, by Prop. 43, conclude $\varphi_1 \oplus \varphi_2 \preceq_R (\tilde{S}, \tilde{\blacksquare}, \tilde{\square}, \xrightarrow{\sim}_{\mathtt{I}})$. Then, by Prop. 44, conclude:

$$\varphi_1 \oplus \varphi_2 \preceq_R \tilde{\varphi}_1 \oplus \varphi_2$$

□