A Predicate Transformer for Choreographies (Technical Report)

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Sung-Shik Jongmans[1,2] and Petra van den Bos[3]

[1] Open University, Heerlen, the Netherlands
[2] CWI, Amsterdam, the Netherlands
[3] University of Twente, Enschede, the Netherlands

Abstract. This technical report contains detailed definitions, auxiliary lemmas, main theorems, and proofs related to the paper *A Predicate Transformer for Choreographies*, published in the proceedings of ESOP 2022. The aim of this document is to provide a comprehensive reference guide for the theory presented in the paper. In contrast to the paper, we do not split the presentation into a base calculus, an extension with blocking if/while-statements, and an extension with non-blocking if/while-statements. Instead, we present the whole calculus of global programs, including all extensions, at once.

We now present a detailed mapping of definitions, lemmas, and theorems in the paper into this document, section by section. Along the way, we also mention some cosmetic differences.

Setting the Stage: Data and Conditions (§3)

- **Data (§3.1)**
  - The grammar of roles, variables, values, and expressions is given in Defn. 1.
  - An excerpt of the definition of the evaluation function is given in Defn. 4.
- **Conditions (§3.2)**
  - The grammar of conditions is given in Defn. 5. The extra syntactic constraints are given in Rem. 1.
  - The definition of the interpretation function is given in Defn. 13.

Global Programs: Base Calculus (§4)

- **Syntax and Semantics (§4.1)**
  - The grammar of global actions is given in Defn. 14. The extra syntactic constraints are given in Rem. 2. For simplicity, we also allow τ to be a global action.
    We note that collections of private decisions together have an extra subscript. This makes it possible to capture the operational semantics of both global programs and local programs uniformly in one set of rules. See below for details.
  - The grammar of global programs is given in Defn. 21. In this definition, ξ ranges over conjunctions; see also Defn. 5.
    We note that blocking if/while-statements have an additional "$R.$", and that non-blocking if/while statements have an additional "$|_0$". As a result, blocking and non-blocking if/while-statements in global programs have the same grammatical form as in local programs. This makes it possible to capture the operational semantics of both global programs and local programs uniformly in one set of rules. See below for details.
  - The subjects function is given in Defn. 24. In this definition, A ranges over both global programs and local programs; see also Defn. 21.
  - The abstract termination rules for global programs are given in Defn. 28. In this definition, A ranges over both global programs and local programs; see also Defn. 21.
  - The abstract reduction rules for global programs are given in Defn. 31. In this definition, A ranges over both global programs and local programs; see also Defn. 21. Furthermore, ξ^+ and ξ^- denote the positive and negative versions of ξ; see also Defn. 12. Furthermore, $\xi \restriction r$ denotes the projection of ξ onto r; see also Defn. 11.
    We note that rule [$\rightarrow$1-NIF1] should be used only with local programs; this is ensured because, grammatically, n=0 for global programs.
  - The concrete termination rule for families of local programs is given in Defn. 30. In this definition, $\mathcal{A}$ ranges over both global programs and families of local programs.

- The concrete reduction rules for global programs are given in Defn. 33. In this definition, $\mathcal{A}$ ranges over both global programs and families of local programs, while $\dot{\gamma}$ ranges over concrete global actions; see also Defn. 14. Furthermore, $\mathsf{con}_{\mathcal{S}}(\gamma)$ denotes the concretisation of γ; see also Defn. 19, and $\mathsf{effect}(\dot{\gamma}, \mathcal{S})$ denotes the effect of $\dot{\gamma}$; see also Defn. 20.
- The well-formedness relation is given in Defn. 27.

– **Predicate Transformer (§4.2)**
  - The predicate transformer function is given in Defn. 38.

– **Deadlock Freedom and Functional Correctness (§4.3)**
  - Lemma 1 in the paper corresponds to Lem. 88; it also extends the result to the concrete layer of the operational semantics.
  - Lemma 2 in the paper corresponds to Lem. 91; it also extends the result to the concrete layer of the operational semantics.
  - Lemma 3 in the paper corresponds to Lem. 89; it also extends the result to the concrete layer of the operational semantics.
  - Lemma 4 in the paper corresponds to Lem. 93; it also extends the result to the concrete layer of the operational semantics.
  - Theorem 1 in the paper corresponds to Thm. 1 and Thm. 2. In their formulation, $\xrightarrow{\mathbf{r}}^*$ denotes reachability using any concrete global action; see also Defn. 36.

Global Programs: If/While-Statements (§5) – Similar to §4.
Global Programs: Non-Blocking If/While-Statements (§6) – Similar to §4.

Local Programs and Projection (§7)

– **Syntax and Semantics (§7.1)**
  - The grammar of local actions is given in Defn. 14. The extra syntactic constraints are given in Rem. 2.
  - The grammar of local programs is given in Defn. 21. In this definition, ξ ranges over conjunctions; see also Defn. 5.
    We note that while-statements have an additional "`{true}`". As a result, while-statements in local programs have the same grammatical form as in global programs. This makes it possible to capture the operational semantics of both global programs and local programs uniformly in one set of rules.
  - The abstract termination rules for local programs are given in Defn. 28. In this definition, A ranges over both global programs and local programs; see also Defn. 21.
  - The abstract reduction rules for local programs are given in Defn. 31. In this definition, A ranges over both global programs and local programs; see also Defn. 21. Furthermore, ξ^+ and ξ^- denote the positive and negative versions of ξ; see also Defn. 12. Furthermore, $\xi \upharpoonright r$ denotes the projection of ξ onto r; see also Defn. 11.
  - The abstract termination rule for families of local programs is given in Defn. 29.
  - The abstract reduction rules for families of local programs are given in Defn. 32.
  - The concrete termination rule for families of local programs is given in Defn. 30. In this definition, $\mathcal{A}$ ranges over both global programs and families of local programs.
  - The concrete reduction rules for families of local programs are given in Defn. 33. In this definition, $\mathcal{A}$ ranges over both global programs and families of local programs, while $\dot{\gamma}$ ranges over concrete global actions; see also Defn. 14. Furthermore, $\mathsf{con}_{\mathcal{S}}(\gamma)$ denotes the concretisation of γ; see also Defn. 19, and $\mathsf{effect}(\dot{\gamma}, \mathcal{S})$ denotes the effect of $\dot{\gamma}$; see also Defn. 20.
  - The projection function is given in Defn. 26.

– **Operational Equivalence (§7.2)**
  - Lemma 7 in the paper corresponds to Lem. 59.
  - Lemma 8 in the paper corresponds to Lem. 64.
  - Theorem 6 in the paper corresponds to Thm. 3.

– **Additional Proof Steps of Thm. 6 (§C)**
  - Relation $\Rrightarrow\!\Lleftarrow$ is given in Defn. 40.
  - The correspondence relation is given in Defn. 41.
  - Weak bisimilarity is given in Defn. 39.
  - Lemma 9 in the paper corresponds to Lem. 97
  - Lemma 10 in the paper corresponds to Lem. 96

Part I

Definitions, Propositions, Lemmas, Theorems

1 Data

Definition 1 (Grammar). *Excerpt:*

$$
\begin{aligned}
p, q, r &::= \mathsf{A} \mid \mathsf{B} \mid \mathsf{C} \mid \cdots && (\mathbb{R})\\
x, y, z &::= \texttt{x} \mid \texttt{y} \mid \texttt{z} \mid \cdots && (\mathbb{X})\\
t, u, v &::= \texttt{error} \mid \texttt{true} \mid \texttt{false} \mid \mathsf{0} \mid \mathsf{1} \mid \mathsf{2} \mid \cdots && (\mathbb{V})\\
c, d, e &::= r.x \mid v \mid e_1\texttt{==}e_2 \mid e_1\texttt{<}e_2 \mid e_1\texttt{\&\&}e_2 \mid \texttt{!}e \mid e_1\texttt{+}e_2 \mid \cdots && (\mathbb{E})
\end{aligned}
$$

Definition 2 (Substitution). *Excerpt:*

$$
\begin{aligned}
r.x[d/q.y] &= \begin{cases} d & \textbf{if} : r.x = q.y \\ q.y & \textbf{else} \end{cases}\\
v[d/q.y] &= v\\
e_1\texttt{==}e_2[d/q.y] &= (e_1[d/q.y])\texttt{==}(e_2[d/q.y])
\end{aligned}
$$

$$
\mathcal{S}[u/q.y] = \{r \mapsto \mathcal{S}(r) \mid q \neq r\} \cup \{q \mapsto \{x \mapsto \mathcal{S}(q)(x) \mid x \neq y\} \cup \{y \mapsto u\}\}
$$

Definition 3 (Role-qualified variables). *Excerpt:*

$$
\begin{aligned}
\mathsf{rx}(r.x) &= \{r.x\}\\
\mathsf{rx}(v) &= \emptyset\\
\mathsf{rx}(e_1\texttt{==}e_2) &= \mathsf{rx}(e_1) \cup \mathsf{rx}(e_2)
\end{aligned}
$$

Proposition 1. $q.y \notin \mathsf{rx}(e)$ **implies** $e[d/q.y] = e$

Proposition 2.

$$
\begin{bmatrix} q_1.y_1 \neq q_2.y_2 \\ \textbf{and } q_1.y_1 \notin \mathsf{rx}(d_2) \\ \textbf{and } q_2.y_2 \notin \mathsf{rx}(d_1) \end{bmatrix} \textbf{ implies } e[d_1/q_1.y_1][d_2/q_2.y_2] = e[d_2/q_2.y_2][d_1/q_1.y_1]
$$

Definition 4 (Evaluation). *Excerpt:*

$$
\begin{aligned}
\mathsf{eval}_{\mathcal{S}}(r.x) &= \begin{cases} \mathcal{S}(r)(x) & \textbf{if} : \mathcal{S}(r)(x) \in \mathbb{V} \\ \texttt{nil} & \textbf{else} \end{cases}\\
\mathsf{eval}_{\mathcal{S}}(v) &= v\\
\mathsf{eval}_{\mathcal{S}}(e_1\texttt{==}e_2) &= \begin{cases} \texttt{true} & \textbf{if} : \mathsf{eval}_{\mathcal{S}}(e_1) = \mathsf{eval}_{\mathcal{S}}(e_2) \\ \texttt{false} & \textbf{else} \end{cases}
\end{aligned}
$$

Proposition 3. $\mathsf{eval}_{\mathcal{S}}(e[d/q.y]) = \mathsf{eval}_{\mathcal{S}[\mathsf{eval}_{\mathcal{S}}(d)/q.y]}(e)$

2 Conditions

Definition 5 (Grammar).

$$\phi, \chi, \psi ::= e \mid \neg\psi \mid \psi_1 \wedge \psi_2 \mid \forall\psi \qquad (\boldsymbol{\Psi})$$
$$\xi ::= \texttt{true} \mid \bigwedge\{e_r\}_{r\in R} \mid \bigwedge\{\neg e_r\}_{r\in R} \qquad (\boldsymbol{\Xi})$$

Remark 1.

- $\xi = \bigwedge\{e_r\}_{r\in R}$ **implies** $\left[\{\hat{r} \mid \hat{r}.\hat{x} \in \mathsf{rx}(e_r)\} = \{r\} \textbf{ for every } r \in R \neq \emptyset\right]$
- $\xi = \bigwedge\{\neg e_r\}_{r\in R}$ **implies** $\left[\{\hat{r} \mid \hat{r}.\hat{x} \in \mathsf{rx}(e_r)\} = \{r\} \textbf{ for every } r \in R \neq \emptyset\right]$

Definition 6 (Macros).

$$\psi_1 \Rightarrow \psi_2 \stackrel{\text{def}}{=} \neg(\psi_1 \wedge \neg\psi_2)$$

Definition 7 (Substitution).

$$\begin{aligned} e[d/q.y] &= \textit{Defn. 2} \\ (\neg\psi)[d/q.y] &= \neg(\psi[d/q.y]) \\ (\psi_1 \wedge \psi_2)[d/q.y] &= \psi_1[d/q.y] \wedge \psi_2[d/q.y] \\ (\forall\psi)[d/q.y] &= \forall\psi \end{aligned}$$

Definition 8 (Role-qualified variables).

$$\begin{aligned} \mathsf{rx}(e) &= \textit{Defn. 3} \\ \mathsf{rx}(\neg\psi) &= \mathsf{rx}(\psi) \\ \mathsf{rx}(\psi_1 \wedge \psi_2) &= \mathsf{rx}(\psi_1) \cup \mathsf{rx}(\psi_2) \\ \mathsf{rx}(\forall\psi) &= \emptyset \end{aligned}$$

Lemma 1. $q.y \notin \mathsf{rx}(\psi)$ **implies** $\psi[d/q.y] = \psi$

Proof. See page 32. □

Lemma 2.

$$\begin{bmatrix} q_1.y_1 \neq q_2.y_2 \\ \textbf{and } q_1.y_1 \notin \mathsf{rx}(d_2) \\ \textbf{and } q_2.y_2 \notin \mathsf{rx}(d_1) \end{bmatrix} \textbf{ implies } \psi[d_1/q_1.y_1][d_2/q_2.y_2] = \psi[d_2/q_2.y_2][d_1/q_1.y_1]$$

Proof. See page 32. □

Definition 9 (Reads/writes).

$$\mathsf{read}(\psi) = \mathsf{rx}(\psi) \qquad\qquad \mathsf{write}(\psi) = \emptyset$$

Lemma 3.

1. $\mathsf{read}(\texttt{false}) = \emptyset = \mathsf{read}(\texttt{true})$
2. $\mathsf{read}(\neg\psi) = \mathsf{read}(\psi)$
3. $\mathsf{read}(\psi_1 \wedge \psi_2) = \mathsf{read}(\psi_1) \cup \mathsf{read}(\psi_2)$
4. $\mathsf{read}(\psi_1 \Rightarrow \psi_2) = \mathsf{read}(\psi_1) \cup \mathsf{read}(\psi_2)$
5. $\mathsf{write}(\neg\psi) = \mathsf{write}(\psi)$
6. $\mathsf{write}(\psi_1 \wedge \psi_2) = \mathsf{write}(\psi_1) \cup \mathsf{write}(\psi_2)$
7. $\mathsf{write}(\psi_1 \Rightarrow \psi_2) = \mathsf{write}(\psi_1) \cup \mathsf{write}(\psi_2)$

Proof. See page 32. □

Definition 10 (Subjects).

$$\mathsf{subj}(\psi) = \{r \mid r.x \in \mathsf{rx}(\psi)\}$$

Lemma 4.

1. $\mathsf{subj}(\mathtt{false}) = \emptyset = \mathsf{subj}(\mathtt{true})$
2. $\mathsf{subj}(\neg\psi) = \mathsf{subj}(\psi)$
3. $\mathsf{subj}(\psi_1 \wedge \psi_2) = \mathsf{subj}(\psi_1) \cup \mathsf{subj}(\psi_2)$
4. $\mathsf{subj}(\psi_1 \Rightarrow \psi_2) = \mathsf{subj}(\psi_1) \cup \mathsf{subj}(\psi_2)$

Proof. See page 33. □

Lemma 5.

1. $\mathsf{read}(\psi) \subseteq \{r.x \mid r \in \mathsf{subj}(\psi)\}$
2. $\mathsf{write}(\psi) \subseteq \{r.x \mid r \in \mathsf{subj}(\psi)\}$

Proof. See page 33. □

Definition 11 (Projection).

$$\mathtt{true} \restriction r = \mathtt{true}$$

$$\bigwedge\{e_{\hat{r}}\}_{\hat{r}\in R} \restriction r = \begin{cases} e_r & \textbf{if} : r \in R \\ \mathtt{true} & \textbf{else} \end{cases}$$

$$\bigwedge\{\neg e_{\hat{r}}\}_{\hat{r}\in R} \restriction r = \begin{cases} \neg e_r & \textbf{if} : r \in R \\ \mathtt{true} & \textbf{else} \end{cases}$$

Lemma 6. $(\xi \restriction r) \restriction r = \xi \restriction r$

Proof. See page 33. □

Lemma 7. $\mathsf{rx}(\xi \restriction r) \subseteq \mathsf{rx}(\xi)$

Proof. See page 33. □

Lemma 8.

1. $\mathsf{read}(\xi \restriction r) \subseteq \mathsf{read}(\xi)$
2. $\mathsf{write}(\xi \restriction r) \subseteq \mathsf{write}(\xi)$

Proof. See page 34. □

Lemma 9.

1. $r \in \mathsf{subj}(\xi)$ **implies** $\mathsf{subj}(\xi \restriction r) = \{r\}$
2. $r \notin \mathsf{subj}(\xi)$ **implies** $\mathsf{subj}(\xi \restriction r) = \emptyset$

Proof. See page 34. □

Lemma 10.

1. $\mathsf{subj}(\xi \restriction r) \subseteq \mathsf{subj}(\xi)$
2. $\mathsf{subj}(\xi \restriction r) \subseteq \{r\}$

Proof. See page 34. □

Definition 12 (Derivatives).

$$\xi^{+} = \xi \qquad \begin{aligned} \texttt{true}^{-} &= \texttt{true} \\ \bigwedge\{e_r\}^{-}_{r\in R} &= \bigwedge\{\neg e_r\}_{r\in R} \\ \bigwedge\{\neg e_r\}^{-}_{r\in R} &= \bigwedge\{e_r\}_{r\in R} \end{aligned} \qquad \begin{aligned} \texttt{true}^{\equiv} &= \texttt{true} \\ \bigwedge\{e_r\}^{\equiv}_{r\in R} &= \bigwedge\{e_{r_1} \Rightarrow e_{r_2}\}_{r_1,r_2\in R} \\ \bigwedge\{\neg e_r\}^{\equiv}_{r\in R} &= \bigwedge\{\neg e_{r_1} \Rightarrow \neg e_{r_2}\}_{r_1,r_2\in R} \end{aligned}$$

Lemma 11.

1. $(\xi \restriction r)^{+} = \xi^{+} \restriction r$
2. $(\xi \restriction r)^{-} = \xi^{-} \restriction r$

Proof. See page 35. □

Lemma 12.

1. $\mathsf{read}(\xi^{+}) = \mathsf{read}(\xi)$
2. $\mathsf{read}(\xi^{-}) = \mathsf{read}(\xi)$
3. $\mathsf{read}(\xi^{\equiv}) = \mathsf{read}(\xi)$
4. $\mathsf{write}(\xi^{+}) = \mathsf{write}(\xi)$
5. $\mathsf{write}(\xi^{-}) = \mathsf{write}(\xi)$
6. $\mathsf{write}(\xi^{\equiv}) = \mathsf{write}(\xi)$

Proof. See page 35. □

Lemma 13.

1. $\mathsf{subj}(\xi^{+}) = \mathsf{subj}(\xi)$
2. $\mathsf{subj}(\xi^{-}) = \mathsf{subj}(\xi)$
3. $\mathsf{subj}(\xi^{\equiv}) = \mathsf{subj}(\xi)$

Proof. See page 36. □

Definition 13 (Satisfaction).

$$\begin{aligned} [\![e]\!] &= \{\mathcal{S} \mid \mathsf{eval}_{\mathcal{S}}(e) = \texttt{true}\} \\ [\![\neg\psi]\!] &= [\![\texttt{true}]\!] \setminus [\![\psi]\!] \\ [\![\psi_1 \wedge \psi_2]\!] &= [\![\psi_1]\!] \cap [\![\psi_2]\!] \\ [\![\forall\psi]\!] &= \begin{cases} [\![\texttt{true}]\!] & \textbf{if}: [\![\psi]\!] = [\![\texttt{true}]\!] \\ [\![\texttt{false}]\!] & \textbf{else} \end{cases} \end{aligned}$$

Lemma 14.

1. $\mathcal{S} \in [\![\psi[d/q.y]]\!]$ **implies** $\mathcal{S}[\mathsf{eval}_{\mathcal{S}}(d)/q.y] \in [\![\psi]\!]$
2. $\mathcal{S} \notin [\![\psi[d/q.y]]\!]$ **implies** $\mathcal{S}[\mathsf{eval}_{\mathcal{S}}(d)/q.y] \notin [\![\psi]\!]$

Proof. See page 36. □

Lemma 15. $\emptyset = [\![\texttt{false}]\!] \subseteq [\![\psi]\!] \subseteq [\![\texttt{true}]\!]$

Proof. See page 37. □

Lemma 16.

1. $[\![\forall(\psi_1 \wedge \psi_2)]\!] = [\![\forall\psi_1]\!] \cap [\![\forall\psi_2]\!]$
2. $[\![\forall(\psi_1 \wedge \psi_2)]\!] = [\![\forall\psi_1 \wedge \forall\psi_2]\!]$

Proof. See page 37. □

Lemma 17. $[\![\psi_1 \Rightarrow \psi_2]\!] = [\![\neg\psi_2 \Rightarrow \neg\psi_1]\!]$

Proof. See page 38. □

Lemma 18.

1. $[\![\psi \Rightarrow (\psi_1 \wedge \psi_2)]\!] = [\![\psi \Rightarrow \psi_1]\!] \cap [\![\psi \Rightarrow \psi_2]\!]$
2. $[\![\psi \Rightarrow (\psi_1 \wedge \psi_2)]\!] = [\![(\psi \Rightarrow \psi_1) \wedge (\psi \Rightarrow \psi_2)]\!]$

Proof. See page 38. □

Lemma 19.

1. $[\![\psi_1]\!] \cap [\![\psi_1 \Rightarrow \psi_2]\!] = [\![\psi_1]\!] \cap [\![\psi_2]\!]$
2. $[\![\psi_1 \wedge (\psi_1 \Rightarrow \psi_2)]\!] = [\![\psi_1 \wedge \psi_2]\!]$

Proof. See page 38. □

Lemma 20. $[\![\psi_1]\!] \subseteq [\![\psi_2]\!]$ **implies** $[\![\psi_1[d/q.y]]\!] \subseteq [\![\psi_2[d/q.y]]\!]$

Proof. See page 38. □

Lemma 21.

1. $[\![\psi_1]\!] \subseteq [\![\psi_2]\!]$ **implies** $[\![\forall\psi_1]\!] \subseteq [\![\forall\psi_2]\!]$
2. $[\![\psi_1]\!] = [\![\psi_2]\!]$ **implies** $[\![\forall\psi_1]\!] = [\![\forall\psi_2]\!]$

Proof. See page 38. □

Lemma 22.

1. $[\![\psi_1]\!] \subseteq [\![\psi_2]\!]$ **implies** $[\![\psi \Rightarrow \psi_1]\!] \subseteq [\![\psi \Rightarrow \psi_2]\!]$
2. $[\![\psi_1]\!] = [\![\psi_2]\!]$ **implies** $[\![\psi \Rightarrow \psi_1]\!] = [\![\psi \Rightarrow \psi_2]\!]$

Proof. See page 39. □

Lemma 23. $[\![\psi_1 \Rightarrow \psi_2]\!] = [\![\mathtt{true}]\!]$ **implies** $[\![\psi_1]\!] \subseteq [\![\psi_2]\!]$

Proof. See page 39. □

Lemma 24.

1. $[\![\forall\psi]\!] \subseteq [\![\psi]\!]$
2. $[\![\psi]\!] \cap \Sigma_1 \subseteq \Sigma_2$ **implies** $[\![\forall\psi]\!] \cap \Sigma_1 \subseteq \Sigma_2$
3. $\Sigma_1 \subseteq [\![\forall\psi]\!] \cap \Sigma_2$ **implies** $\Sigma_1 \subseteq [\![\psi]\!] \cap \Sigma_2$

Proof. See page 39. □

Lemma 25.

1. $[\![\psi_2]\!] \subseteq [\![\psi_1 \Rightarrow \psi_2]\!]$
2. $[\![\psi_1 \Rightarrow \psi_2]\!] \cap \Sigma_1 \subseteq \Sigma_2$ **implies** $[\![\psi_2]\!] \cap \Sigma_1 \subseteq \Sigma_2$
3. $\Sigma_1 \subseteq [\![\psi_2]\!] \cap \Sigma_2$ **implies** $\Sigma_1 \subseteq [\![\psi_1 \Rightarrow \psi_2]\!] \cap \Sigma_2$

Proof. See page 39. □

Lemma 26.

1. $[\![\xi]\!] = [\![\bigwedge\{\xi \restriction r\}_{r \in \mathsf{subj}(\xi)}]\!]$
2. $\mathsf{subj}(\xi) \subseteq R$ **implies** $[\![\xi]\!] = [\![\bigwedge\{\xi \restriction r\}_{r \in R}]\!]$

Proof. See page 40. □

Lemma 27.

1. $\mathcal{S} \in [\![\xi^{\equiv}]\!]$ **implies** $\left[\mathcal{S} \in [\![\xi^{+}]\!]\right.$ **or** $\left.\mathcal{S} \in [\![\xi^{-}]\!]\right]$
2. $\mathcal{S} \in [\![\xi^{\equiv}]\!]$ **implies** $\left[\mathcal{S} \in [\![\xi^{+} \upharpoonright r]\!]\right.$ **or** $\left.\mathcal{S} \in [\![\xi^{-} \upharpoonright r]\!]\right]$

Proof. See page 40. □

Lemma 28.

1. $r \in \mathsf{subj}(\xi)$ **implies** $[\![\xi^{+} \upharpoonright r]\!] \cap [\![\xi^{-} \upharpoonright r]\!] = \emptyset$.
2. $r \in \mathsf{subj}(\xi)$ **implies** $[\![\xi^{\equiv}]\!] \cap [\![\xi^{+} \upharpoonright r]\!] \subseteq [\![\xi^{+}]\!]$
3. $r \in \mathsf{subj}(\xi)$ **implies** $[\![\xi^{\equiv}]\!] \cap [\![\xi^{+} \upharpoonright r]\!] \subseteq [\![\bigwedge\{\xi^{+} \upharpoonright \hat{r}\}_{\hat{r} \in \mathsf{subj}(\xi) \setminus \{r\}}]\!]$
4. $r \in \mathsf{subj}(\xi)$ **implies** $[\![\xi^{\equiv}]\!] \cap [\![\xi^{-} \upharpoonright r]\!] \subseteq [\![\xi^{-}]\!]$
5. $r \in \mathsf{subj}(\xi)$ **implies** $[\![\xi^{\equiv}]\!] \cap [\![\xi^{-} \upharpoonright r]\!] \subseteq [\![\bigwedge\{\xi^{-} \upharpoonright \hat{r}\}_{\hat{r} \in \mathsf{subj}(\xi) \setminus \{r\}}]\!]$

Proof. See page 41. □

Lemma 29.

1. $r \in R \subseteq \mathsf{subj}(\xi)$ **implies** $[\![\bigwedge\{\xi^{-} \upharpoonright \hat{r}\}_{\hat{r} \in R}]\!] \cap [\![\xi^{+} \upharpoonright r]\!] = [\![\mathtt{false}]\!]$
2. $r \in R \subseteq \mathsf{subj}(\xi)$ **implies** $[\![\bigwedge\{\xi^{+} \upharpoonright \hat{r}\}_{\hat{r} \in R}]\!] \cap [\![\xi^{-} \upharpoonright r]\!] = [\![\mathtt{false}]\!]$

Proof. See page 43. □

3 Actions

Definition 14 (Grammar).

$$\alpha ::= \tau \mid q.y{:=}e \mid p.e \twoheadrightarrow q.y \mid i_R^R \mid pq!e \mid pq?y \mid i_{\{r\}}^R \quad (\mathbf{A})$$
$$\gamma ::= \tau \mid q.y{:=}e \mid p.e \twoheadrightarrow q.y \mid i_R^R \quad (\mathbf{\Gamma})$$
$$\lambda ::= \tau \mid q.y{:=}e \mid pq!e \mid pq?y \mid i_{\{r\}}^R \quad (\mathbf{\Lambda})$$

Remark 2.

- $\alpha = q.y{:=}e$ **implies** $\{r \mid r.x \in \mathsf{rx}(e)\} \subseteq \{q\}$
- $\alpha = p.e \twoheadrightarrow q.y$ **implies** $[p \neq q$ **and** $\{r \mid r.x \in \mathsf{rx}(e)\} \subseteq \{p\}]$
- $\alpha = i_R^R$ **implies** $R \neq \emptyset$
- $\alpha = pq!e$ **implies** $[p \neq q$ **and** $\{r \mid r.x \in \mathsf{rx}(e)\} \subseteq \{q\}]$
- $\alpha = pq?y$ **implies** $p \neq q$
- $\alpha = i_{\{r\}}^R$ **implies** $r \in R$
- $\gamma = q.y{:=}e$ **implies** $\{r \mid r.x \in \mathsf{rx}(e)\} \subseteq \{q\}$
- $\gamma = p.e \twoheadrightarrow q.y$ **implies** $[p \neq q$ **and** $\{r \mid r.x \in \mathsf{rx}(e)\} \subseteq \{p\}]$
- $\gamma = i_R^R$ **implies** $R \neq \emptyset$
- $\lambda = q.y{:=}e$ **implies** $\{r \mid r.x \in \mathsf{rx}(e)\} \subseteq \{q\}$
- $\lambda = pq!e$ **implies** $[p \neq q$ **and** $\{r \mid r.x \in \mathsf{rx}(e)\} \subseteq \{q\}]$
- $\lambda = pq?y$ **implies** $p \neq q$
- $\lambda = i_{\{r\}}^R$ **implies** $r \in R$

Definition 15 (Reads/writes).

$$\begin{aligned}
\mathsf{read}(\tau) &= \emptyset & \mathsf{write}(\tau) &= \emptyset\\
\mathsf{read}(q.y{:=}e) &= \mathsf{rx}(e) & \mathsf{write}(q.y{:=}e) &= \{q.y\}\\
\mathsf{read}(p.e \twoheadrightarrow q.y) &= \mathsf{rx}(e) & \mathsf{write}(p.e \twoheadrightarrow q.y) &= \{q.y\}\\
\mathsf{read}(i_R^R) &= \emptyset & \mathsf{write}(i_R^R) &= \emptyset\\
\mathsf{read}(pq!e) &= \mathsf{rx}(e) & \mathsf{write}(pq!e) &= \emptyset\\
\mathsf{read}(pq?y) &= \emptyset & \mathsf{write}(pq?y) &= \{q.y\}\\
\mathsf{read}(i_{\{r\}}^R) &= \emptyset & \mathsf{write}(i_{\{r\}}^R) &= \emptyset
\end{aligned}$$

Definition 16 (Subjects).

$$\begin{aligned}
\mathsf{subj}(\tau) &= \emptyset\\
\mathsf{subj}(q.y{:=}e) &= \{q\}\\
\mathsf{subj}(p.e \twoheadrightarrow q.y) &= \{p, q\}\\
\mathsf{subj}(i_R^R) &= R\\
\mathsf{subj}(pq!e) &= \{p\}\\
\mathsf{subj}(pq?y) &= \{q\}\\
\mathsf{subj}(i_{\{r\}}^R) &= \{r\}
\end{aligned}$$

Lemma 30. $\alpha \neq \tau$ **implies** $\mathsf{subj}(\gamma) \neq \emptyset$

Proof. See page 44. □

Lemma 31.

1. $\mathsf{read}(\alpha) \subseteq \{r.x \mid r \in \mathsf{subj}(\alpha)\}$
2. $\mathsf{write}(\alpha) \subseteq \{r.x \mid r \in \mathsf{subj}(\alpha)\}$

Proof. See page 44. □

Definition 17 (Channels).

$$\begin{aligned}
\mathsf{chan}(\tau) &= \emptyset\\
\mathsf{chan}(q.y := e) &= \emptyset\\
\mathsf{chan}(p.e \twoheadrightarrow q.y) &= \{pq\}\\
\mathsf{chan}(i^R_R) &= \{R\}\\
\mathsf{chan}(pq!e) &= \{pq\}\\
\mathsf{chan}(pq?y) &= \{pq\}\\
\mathsf{chan}(i^R_{\{r\}}) &= \{R\}
\end{aligned}$$

Definition 18 (Projection).

$$\tau \upharpoonright r = \tau$$

$$q.y := e \upharpoonright r = \begin{cases} q.y := e & \textbf{if} : r = q\\ \tau & \textbf{else} \end{cases}$$

$$p.e \twoheadrightarrow q.y \upharpoonright r = \begin{cases} pq!e & \textbf{if} : p = r \neq q\\ pq?y & \textbf{if} : p \neq r = q\\ \tau & \textbf{else} \end{cases}$$

$$i^R_R \upharpoonright r = \begin{cases} i^R_{\{r\}} & \textbf{if} : r \in R\\ \tau & \textbf{else} \end{cases}$$

Lemma 32.

1. $r \in \mathsf{subj}(\gamma)$ **implies** $\mathsf{subj}(\gamma \upharpoonright r) = \{r\}$
2. $r \notin \mathsf{subj}(\gamma)$ **implies** $\mathsf{subj}(\gamma \upharpoonright r) = \emptyset$

Proof. See page 44. □

Lemma 33. $\mathsf{subj}(\gamma \upharpoonright r) \subseteq \mathsf{subj}(\gamma)$

Proof. See page 45. □

Lemma 34. $r \notin \mathsf{subj}(\gamma)$ **implies** $\gamma \upharpoonright r = \tau$

Proof. See page 45. □

Lemma 35. $\mathsf{chan}(\gamma \upharpoonright r) \subseteq \mathsf{chan}(\gamma)$

Proof. See page 45. □

Definition 19 (Concretisation).

$$\begin{aligned}
\mathsf{con}_{\mathcal{S}}(\tau) &= \tau\\
\mathsf{con}_{\mathcal{S}}(q.y := e) &= q.y := \mathsf{eval}_{\mathcal{S}}(e)\\
\mathsf{con}_{\mathcal{S}}(p.e \twoheadrightarrow q.y) &= p.\mathsf{eval}_{\mathcal{S}}(e) \twoheadrightarrow q.y\\
\mathsf{con}_{\mathcal{S}}(i^R_R) &= i^R_R
\end{aligned}$$

Definition 20 (Effects).

$$\begin{aligned}
\mathsf{effect}(\tau, \mathcal{S}) &= \mathcal{S}\\
\mathsf{effect}(q.y := v, \mathcal{S}) &= \mathcal{S}[v/q.y]\\
\mathsf{effect}(p.v \twoheadrightarrow q.y, \mathcal{S}) &= \mathcal{S}[v/q.y]\\
\mathsf{effect}(i^R_R, \mathcal{S}) &= \mathcal{S}
\end{aligned}$$

4 Programs

Definition 21 **(Grammar).**

$$
\begin{aligned}
\circ &::= \ ; \ \big| \ \| \\
A &::= \xi \ \big| \ \alpha \ \big| \ \textbf{skip} \ \big| \ A_1 \circ A_2 \ \big| \ R.\textbf{if}\ \xi\ A_1\ A_2 \ \big| \ R.\textbf{while}\ \xi\ \{\psi\}\ A \ \big| && (\mathbb{A}) \\
&\quad\ \ \textbf{if}\ \xi|_n\ A_1|_{R_1}\ A_2|_{R_2} \ \big| \ \textbf{while}\ \xi|_n\ \{\psi\}\ A|_\emptyset \\
G &::= \xi \ \big| \ \gamma \ \big| \ \textbf{skip} \ \big| \ G_1 \circ G_2 \ \big| \ R.\textbf{if}\ \xi\ G_1\ G_2 \ \big| \ R.\textbf{while}\ \xi\ \{\psi\}\ G \ \big| && (\mathbb{G}) \\
&\quad\ \ \textbf{if}\ \xi|_0\ G_1|_{R_1}\ G_2|_{R_2} \ \big| \ \textbf{while}\ \xi|_0\ \{\psi\}\ G|_\emptyset \\
L &::= \xi \ \big| \ \lambda \ \big| \ \textbf{skip} \ \big| \ L_1 \circ L_2 \ \big| \ R.\textbf{if}\ \xi\ L_1\ L_2 \ \big| \ R.\textbf{while}\ \xi\ \{\texttt{true}\}\ L \ \big| && (\mathbb{L}) \\
&\quad\ \ \textbf{if}\ \xi|_n\ L_1|_{R_1}\ L_2|_{R_2} \ \big| \ \textbf{while}\ \xi|_n\ \{\texttt{true}\}\ L|_\emptyset \\
\underline{G} &::= q.y := e \ \big| \ p.e \rightarrow q.y \ \big| \ \textbf{skip} \ \big| \ \underline{G}_1 \circ \underline{G}_2 \ \big| \ R.\textbf{if}\ \xi\ \underline{G}_1\ \underline{G}_2 \ \big| && (\underline{\mathbb{G}}) \\
&\quad\ \ \textbf{if}\ \xi|_0\ \underline{G}_1|_{R_1}\ \underline{G}_2|_{R_2} \ \big| \ \textbf{if}\ \xi|_0\ G_1|_\emptyset\ \underline{G}_2|_{R_2\cup\{r\}} \ \big| \ \textbf{if}\ \xi|_0\ \underline{G}_1|_{R_1\cup\{r\}}\ G_2|_\emptyset
\end{aligned}
$$

Definition 22 **(Reads/writes).**

$$
\begin{aligned}
\mathsf{read}(\xi) &= \textit{Defn. 9} \\
\mathsf{read}(\alpha) &= \textit{Defn. 15} \\
\mathsf{read}(\textbf{skip}) &= \emptyset \\
\mathsf{read}(A_1 \circ A_2) &= \mathsf{read}(A_1) \cup \mathsf{read}(A_2) \\
\mathsf{read}(R.\textbf{if}\ \xi\ A_1\ A_2) &= \mathsf{read}(\xi) \cup \mathsf{read}(A_1) \cup \mathsf{read}(A_2) \\
\mathsf{read}(R.\textbf{while}\ \xi\ \{\psi\}\ A) &= \mathsf{read}(\xi) \cup \mathsf{read}(A) \\
\mathsf{read}(\textbf{if}\ \xi|_n\ A_1|_{R_1}\ A_2|_{R_2}) &= (\mathsf{read}(\xi) \setminus \{r.x \mid r \in R_1 \cup R_2\}) \cup \\
&\quad\ (\mathsf{read}(A_1) \setminus \{r.x \mid r \in R_2\}) \cup (\mathsf{read}(A_2) \setminus \{r.x \mid r \in R_1\}) \\
\mathsf{read}(\textbf{while}\ \xi|_n\ \{\psi\}\ A|_\emptyset) &= \mathsf{read}(\xi) \cup \mathsf{read}(A)
\end{aligned}
$$

$$
\begin{aligned}
\mathsf{write}(\xi) &= \textit{Defn. 9} \\
\mathsf{write}(\alpha) &= \textit{Defn. 15} \\
\mathsf{write}(\textbf{skip}) &= \emptyset \\
\mathsf{write}(A_1 \circ A_2) &= \mathsf{write}(A_1) \cup \mathsf{write}(A_2) \\
\mathsf{write}(R.\textbf{if}\ \xi\ A_1\ A_2) &= \mathsf{write}(\xi) \cup \mathsf{write}(A_1) \cup \mathsf{write}(A_2) \\
\mathsf{write}(R.\textbf{while}\ \xi\ \{\psi\}\ A) &= \mathsf{write}(\xi) \cup \mathsf{write}(A) \\
\mathsf{write}(\textbf{if}\ \xi|_n\ A_1|_{R_1}\ A_2|_{R_2}) &= (\mathsf{write}(\xi) \setminus \{r.x \mid r \in R_1 \cup R_2\}) \cup \\
&\quad\ (\mathsf{write}(A_1) \setminus \{r.x \mid r \in R_2\}) \cup (\mathsf{write}(A_2) \setminus \{r.x \mid r \in R_1\}) \\
\mathsf{write}(\textbf{while}\ \xi|_n\ \{\psi\}\ A|_\emptyset) &= \mathsf{write}(\xi) \cup \mathsf{write}(A)
\end{aligned}
$$

Lemma 36.

1. $R_1 = \emptyset = R_2$ **implies** $\mathsf{read}(\textbf{if}\ \xi|_n\ A_1|_{R_1}\ A_2|_{R_2}) = \mathsf{read}(\xi) \cup \mathsf{read}(A_1) \cup \mathsf{read}(A_2)$
2. $R_1 = \emptyset \neq R_2$ **implies** $\left[\begin{array}{c}\mathsf{read}(\textbf{if}\ \xi|_n\ A_1|_{R_1}\ A_2|_{R_2}) = \\ (\mathsf{read}(\xi) \setminus \{r.x \mid r \in R_2\}) \cup (\mathsf{read}(A_1) \setminus \{r.x \mid r \in R_2\}) \cup \mathsf{read}(A_2)\end{array}\right]$
3. $R_1 \neq \emptyset = R_2$ **implies** $\left[\begin{array}{c}\mathsf{read}(\textbf{if}\ \xi|_n\ A_1|_{R_1}\ A_2|_{R_2}) = \\ (\mathsf{read}(\xi) \setminus \{r.x \mid r \in R_1\}) \cup \mathsf{read}(A_1) \cup (\mathsf{read}(A_2) \setminus \{r.x \mid r \in R_1\})\end{array}\right]$
4. $R_1 = \emptyset = R_2$ **implies** $\mathsf{write}(\textbf{if}\ \xi|_n\ A_1|_{R_1}\ A_2|_{R_2}) = \mathsf{write}(\xi) \cup \mathsf{write}(A_1) \cup \mathsf{write}(A_2)$
5. $R_1 = \emptyset \neq R_2$ **implies** $\left[\begin{array}{c}\mathsf{write}(\textbf{if}\ \xi|_n\ A_1|_{R_1}\ A_2|_{R_2}) = \\ (\mathsf{write}(\xi) \setminus \{r.x \mid r \in R_2\}) \cup (\mathsf{write}(A_1) \setminus \{r.x \mid r \in R_2\}) \cup \mathsf{write}(A_2)\end{array}\right]$
6. $R_1 \neq \emptyset = R_2$ **implies** $\left[\begin{array}{c}\mathsf{write}(\textbf{if}\ \xi|_n\ A_1|_{R_1}\ A_2|_{R_2}) = \\ (\mathsf{write}(\xi) \setminus \{r.x \mid r \in R_1\}) \cup \mathsf{write}(A_1) \cup (\mathsf{write}(A_2) \setminus \{r.x \mid r \in R_1\})\end{array}\right]$

Proof. See page 47. □

Definition 23 (Non-interference).

$$\frac{\mathsf{read}(A_1) \cap \mathsf{read}(A_2) = \emptyset \quad \mathsf{read}(A_1) \cap \mathsf{write}(A_2) = \emptyset \qquad \mathsf{write}(A_1) \cap \mathsf{read}(A_2) = \emptyset \quad \mathsf{write}(A_1) \cap \mathsf{write}(A_2) = \emptyset}{A_1 \# A_2}\ [\#]$$

Lemma 37. $A_1 \# A_2$ **implies** $A_2 \# A_1$

Proof. See page 47. □

Lemma 38.

1. $A_1 \circ A_2 \# A$ **implies** $[A_1 \# A$ **and** $A_2 \# A]$
2. $R.\mathbf{if}\ \xi\ A_1\ A_2 \# A$ **implies** $[\xi \# A$ **and** $A_1 \# A$ **and** $A_2 \# A]$
3. $R.\mathbf{while}\ \xi\ \{\psi\}\ \hat{A} \# A$ **implies** $[\xi \# A$ **and** $\hat{A} \# A]$
4. $[\mathbf{if}\ \xi|_n\ A_1|_{R_1}\ A_2|_{R_2} \# A$ **and** $R_1 = \emptyset = R_2]$ **implies** $[\xi \# A$ **and** $A_1 \# A$ **and** $A_2 \# A]$
5. $[\mathbf{if}\ \xi|_n\ A_1|_{R_1}\ A_2|_{R_2} \# A$ **and** $R_1 = \emptyset \neq R_2]$ **implies** $\left[\begin{bmatrix}\xi \upharpoonright r \# A \textbf{ for every} \\ r \in \mathsf{subj}(\xi) \setminus R_2\end{bmatrix} \textbf{ and } A_2 \# A\right]$
6. $[\mathbf{if}\ \xi|_n\ A_1|_{R_1}\ A_2|_{R_2} \# A$ **and** $R_1 \neq \emptyset = R_2]$ **implies** $\left[\begin{bmatrix}\xi \upharpoonright r \# A \textbf{ for every} \\ r \in \mathsf{subj}(\xi) \setminus R_1\end{bmatrix} \textbf{ and } A_1 \# A\right]$
7. $\mathbf{while}\ \xi|_n\ \{\psi\}\ \hat{A}|_\emptyset \# A$ **implies** $[\xi \# A$ **and** $\hat{A} \# A]$

Proof. See page 47. □

Lemma 39.

1. $[\xi_r \# A$ **for every** $r \in R]$ **implies** $\bigwedge\{\xi_r^+\}_{r\in R} \# A$
2. $[\xi_r \# A$ **for every** $r \in R]$ **implies** $\bigwedge\{\xi_r^-\}_{r\in R} \# A$

Proof. See page 48. □

Definition 24 (Subjects).

$$\begin{aligned}
\mathsf{subj}(\xi) &= \textit{Defn. 10} \\
\mathsf{subj}(\alpha) &= \textit{Defn. 16} \\
\mathsf{subj}(\mathbf{skip}) &= \emptyset \\
\mathsf{subj}(A_1 \circ A_2) &= \mathsf{subj}(A_1) \cup \mathsf{subj}(A_2) \\
\mathsf{subj}(R.\mathbf{if}\ \xi\ A_1\ A_2) &= \mathsf{subj}(\xi) \cup \mathsf{subj}(A_1) \cup \mathsf{subj}(A_2) \\
\mathsf{subj}(R.\mathbf{while}\ \xi\ \{\psi\}\ A) &= \mathsf{subj}(\xi) \cup \mathsf{subj}(A) \\
\mathsf{subj}(\mathbf{if}\ \xi|_n\ A_1|_{R_1}\ A_2|_{R_2}) &= (\mathsf{subj}(\xi) \setminus (R_1 \cup R_2)) \cup (\mathsf{subj}(A_1) \setminus R_2) \cup (\mathsf{subj}(A_2) \setminus R_1) \\
\mathsf{subj}(\mathbf{while}\ \xi|_n\ \{\psi\}\ A|_\emptyset) &= \mathsf{subj}(\xi) \cup \mathsf{subj}(A)
\end{aligned}$$

Lemma 40.

1. $\mathsf{read}(A) \subseteq \{r.x \mid r \in \mathsf{subj}(A)\}$
2. $\mathsf{write}(A) \subseteq \{r.x \mid r \in \mathsf{subj}(A)\}$

Proof. See page 49. □

Lemma 41. $\mathsf{subj}(A_1) \cap \mathsf{subj}(A_2) = \emptyset$ **implies** $A_1 \# A_2$

Proof. See page 49. □

Definition 25 (Channels).

$$
\begin{aligned}
\mathsf{chan}(\xi) &= \emptyset \\
\mathsf{chan}(\alpha) &= \textit{Defn. 17} \\
\mathsf{chan}(\mathbf{skip}) &= \emptyset \\
\mathsf{chan}(A_1 \circ A_2) &= \mathsf{chan}(A_1) \cup \mathsf{chan}(A_2) \\
\mathsf{chan}(R.\mathbf{if}\ \xi\ A_1\ A_2) &= \{R\} \cup \mathsf{chan}(A_1) \cup \mathsf{chan}(A_2) \\
\mathsf{chan}(R.\mathbf{while}\ \xi\ \{\psi\}\ A) &= \{R\} \cup \mathsf{chan}(A) \\
\mathsf{chan}(\mathbf{if}\ \xi|_n\ A_1|_{R_1}\ A_2|_{R_2}) &= \{\{r\} \mid r \in \mathsf{subj}(\xi)\} \cup \mathsf{chan}(A_1) \cup \mathsf{chan}(A_2) \\
\mathsf{chan}(\mathbf{while}\ \xi|_n\ \{\psi\}\ A|_\emptyset) &= \{\{r\} \mid r \in \mathsf{subj}(\xi)\} \cup \mathsf{chan}(A)
\end{aligned}
$$

Definition 26 (Projection).

$$
\begin{aligned}
\xi \upharpoonright r &= \textit{Defn. 11} \\
\gamma \upharpoonright r &= \textit{Defn. 18} \\
\mathbf{skip} \upharpoonright r &= \mathbf{skip} \\
G_1 \circ G_2 \upharpoonright r &= (G_1 \upharpoonright r) \circ (G_2 \upharpoonright r) \\
R.\mathbf{if}\ \xi\ G_1\ G_2 \upharpoonright r &= R.\mathbf{if}\ (\xi \upharpoonright r)\ (G_1 \upharpoonright r)\ (G_2 \upharpoonright r) \\
R.\mathbf{while}\ \xi\ \{\psi\}\ G \upharpoonright r &= R.\mathbf{while}\ (\xi \upharpoonright r)\ \{\mathtt{true}\}\ (G \upharpoonright r) \\
\mathbf{if}\ \xi|_0\ G_1|_{R_1}\ G_2|_{R_2} \upharpoonright r &= \mathbf{if}\ (\xi \upharpoonright r)|_{|\mathsf{subj}(\xi)\setminus(R_1 \cup R_2 \cup \{r\})|}\ (G_1 \upharpoonright r)|_{R_1 \cap \{r\}}\ (G_2 \upharpoonright r)|_{R_2 \cap \{r\}} \\
\mathbf{while}\ \xi|_0\ \{\psi\}\ G|_\emptyset \upharpoonright r &= \mathbf{while}\ (\xi \upharpoonright r)|_{|\mathsf{subj}(\xi)\setminus\{r\}|}\ \{\mathtt{true}\}\ (G \upharpoonright r)|_\emptyset
\end{aligned}
$$

$$\{G\} \upharpoonright R = \{G \upharpoonright r\}_{r \in R}$$

Lemma 42.

1. $r \in \mathsf{subj}(G)$ **implies** $\mathsf{subj}(G \upharpoonright r) = \{r\}$
2. $r \notin \mathsf{subj}(G)$ **implies** $\mathsf{subj}(G \upharpoonright r) = \emptyset$

Proof. See page 50. □

Lemma 43. $\mathsf{subj}(G \upharpoonright r) \subseteq \mathsf{subj}(G)$

Proof. See page 56. □

Lemma 44. $r \in \mathsf{subj}(G)$ **implies** $r \in \mathsf{subj}(G \upharpoonright r)$

Proof. See page 56. □

Lemma 45. $\mathsf{chan}(G \upharpoonright r) \subseteq \mathsf{chan}(G)$

Proof. See page 56. □

Definition 27 (Well-formedness).

$$\frac{q \in R}{\checkmark_R(q.y := e)}\ [\checkmark\text{1-Act1}] \qquad \frac{p, q \in R}{\checkmark_R(p.e \rightarrow q.y)}\ [\checkmark\text{1-Act2}] \qquad \frac{R \neq \emptyset}{\checkmark_R(\textbf{skip})}\ [\checkmark\text{1-Skip}]$$

$$\frac{\checkmark_R(G_1) \quad \checkmark_R(G_2)}{\checkmark_R(G_1 \,;\, G_2)}\ [\checkmark\text{1-Seq}] \qquad \frac{\checkmark_R(G_1) \quad \checkmark_R(G_2) \quad \mathsf{chan}(G_1) \cap \mathsf{chan}(G_2) = \emptyset}{\checkmark_R(G_1 \parallel G_2)}\ [\checkmark\text{1-Par}]$$

$$\frac{\checkmark_R(G_1) \quad \checkmark_R(G_2) \quad R = \mathsf{subj}(\xi)}{\checkmark_R(R.\textbf{if}\ \xi\ G_1\ G_2)}\ [\checkmark\text{1-If}] \qquad \frac{\checkmark_R(G) \quad R = \mathsf{subj}(\xi)}{\checkmark_R(R.\textbf{while}\ \xi\ \{\psi\}\ G)}\ [\checkmark\text{1-While}]$$

$$\frac{\checkmark_R(G_1) \quad \checkmark_R(G_2) \quad R = \mathsf{subj}(\xi) \quad R_1, R_2 \subseteq \mathsf{subj}(\xi) \quad R_1 \neq \emptyset \textbf{ implies } R_2 = \emptyset \quad R_2 \neq \emptyset \textbf{ implies } R_1 = \emptyset}{\checkmark_R(\textbf{if}\ \xi|_0\ G_1|_{R_1}\ G_2|_{R_2})}\ [\checkmark\text{1-NIf}] \qquad \frac{\checkmark_R(G) \quad R = \mathsf{subj}(\xi)}{\checkmark_R(\textbf{while}\ \xi|_0\ \{\psi\}\ G|_\emptyset)}\ [\checkmark\text{1-NWhile}]$$

$$\frac{\checkmark_R(G)}{\checkmark_R(\{G\})}\ [\checkmark 2]$$

Lemma 46. $\checkmark_R(\textbf{while}\ \xi|_0\ \{\psi\}\ G|_\emptyset)$ **implies** $\checkmark_R(\textbf{if}\ \xi|_0\ (G\,;\,\textbf{while}\ \xi|_0\ \{\psi\}\ G|_\emptyset)|_\emptyset\ \textbf{skip}|_\emptyset)$

Proof. See page 57. □

Lemma 47. $\checkmark_R(G)$ **implies** $\mathsf{subj}(G) \subseteq R \neq \emptyset$

Proof. See page 57. □

5 Termination

Definition 28 (Termination, part 1).

$$\frac{}{\textbf{skip}\downarrow}\,[\downarrow\text{1-Skip}] \qquad \frac{A_1\downarrow \quad A_2\downarrow}{A_1\,;A_2\downarrow}\,[\downarrow\text{1-Seq}] \qquad \frac{A_1\downarrow \quad A_2\downarrow}{A_1 \parallel A_2\downarrow}\,[\downarrow\text{1-Par}]$$

$$\frac{\mathsf{subj}(\xi) = R_1 \cup R_2 \quad R_1 \neq \emptyset \textbf{ implies } A_1\downarrow \quad R_2 \neq \emptyset \textbf{ implies } A_2\downarrow}{\textbf{if } \xi|_0\; A_1|_{R_1}\; A_2|_{R_2}\downarrow}\,[\downarrow\text{1-NIf}]$$

Lemma 48. $\left[G\downarrow \textbf{ and } \checkmark_R(G)\right]$ **implies** $\mathsf{subj}(G) = \emptyset$

Proof. See page 58. □

Lemma 49. $\left[G\downarrow \textbf{ and } \checkmark_R(G)\right]$ **implies** $(G \upharpoonright r)\downarrow$

Proof. See page 58. □

Definition 29 (Termination, part 2).

$$\frac{G\downarrow}{\{G\}\downarrow}\,[\downarrow\text{2-Glob}] \qquad \frac{L_r\downarrow \textbf{ for every } r \in R}{\{L_r\}_{r\in R}\downarrow}\,[\downarrow\text{2-Locs}]$$

Definition 30 (Termination, part 3).

$$\frac{\mathcal{A}\downarrow}{(\mathcal{A},\mathcal{S})\downarrow}\,[\downarrow 3]$$

6 Reduction

Definition 31 (Reduction, part 1).

$$\frac{}{\alpha \xrightarrow{\texttt{true},\alpha} \mathbf{skip}}\ [\to 1\text{-}\textsc{Act}] \qquad \frac{A_1 \xrightarrow{\xi,\alpha} A_1'}{A_1 \,;\, A_2 \xrightarrow{\xi,\alpha} A_1' \,;\, A_2}\ [\to 1\text{-}\textsc{Seq1}]$$

$$\frac{\mathsf{subj}(A_1) \cap (\mathsf{subj}(\xi) \cup \mathsf{subj}(\alpha)) = \emptyset \quad A_2 \xrightarrow{\xi,\alpha} A_2'}{A_1 \,;\, A_2 \xrightarrow{\xi,\alpha} A_1 \,;\, A_2'}\ [\to 1\text{-}\textsc{Seq2}]$$

$$\frac{A_1 \xrightarrow{\xi,\alpha} A_1'}{A_1 \parallel A_2 \xrightarrow{\xi,\alpha} A_1' \parallel A_2}\ [\to 1\text{-}\textsc{Par1}] \qquad \frac{A_2 \xrightarrow{\xi,\alpha} A_2'}{A_1 \parallel A_2 \xrightarrow{\xi,\alpha} A_1 \parallel A_2'}\ [\to 1\text{-}\textsc{Par2}]$$

$$\frac{}{R.\mathbf{if}\ \xi\ A_1\ A_2 \xrightarrow{\xi^+,1^R_{\mathsf{subj}(\xi)}} A_1}\ [\to 1\text{-}\textsc{If1}] \qquad \frac{}{R.\mathbf{if}\ \xi\ A_1\ A_2 \xrightarrow{\xi^-,2^R_{\mathsf{subj}(\xi)}} A_2}\ [\to 1\text{-}\textsc{If2}]$$

$$\frac{}{R.\mathbf{while}\ \xi\ \{\psi\}\ A \xrightarrow{\xi^+,1^R_{\mathsf{subj}(\xi)}} A \,;\, R.\mathbf{while}\ \xi\ \{\psi\}\ A}\ [\to 1\text{-}\textsc{While1}]$$

$$\frac{}{R.\mathbf{while}\ \xi\ \{\psi\}\ A \xrightarrow{\xi^-,2^R_{\mathsf{subj}(\xi)}} \mathbf{skip}}\ [\to 1\text{-}\textsc{While2}]$$

$$\frac{n > 0}{\mathbf{if}\ \xi|_n\ A_1|_{R_1}\ A_2|_{R_2} \xrightarrow{\texttt{true},\tau} \mathbf{if}\ \xi|_{n-1}\ A_1|_{R_1}\ A_2|_{R_2}}\ [\to 1\text{-}\textsc{NIf1}]$$

$$\frac{r \in \mathsf{subj}(\xi) \setminus (R_1 \cup R_2)}{\mathbf{if}\ \xi|_n\ A_1|_{R_1}\ A_2|_{R_2} \xrightarrow{\xi^+\restriction r,1^{\{r\}}_{\{r\}}} \mathbf{if}\ \xi|_n\ A_1|_{R_1 \cup \{r\}}\ A_2|_{R_2}}\ [\to 1\text{-}\textsc{NIf2}]$$

$$\frac{r \in \mathsf{subj}(\xi) \setminus (R_1 \cup R_2)}{\mathbf{if}\ \xi|_n\ A_1|_{R_1}\ A_2|_{R_2} \xrightarrow{\xi^-\restriction r,2^{\{r\}}_{\{r\}}} \mathbf{if}\ \xi|_n\ A_1|_{R_1}\ A_2|_{R_2 \cup \{r\}}}\ [\to 1\text{-}\textsc{NIf3}]$$

$$\frac{A_1 \xrightarrow{\hat{\xi},\alpha} A_1' \quad \mathsf{subj}(\hat{\xi}) \cup \mathsf{subj}(\alpha) \subseteq R_1 \setminus R_2}{\mathbf{if}\ \xi|_n\ A_1|_{R_1}\ A_2|_{R_2} \xrightarrow{\hat{\xi},\alpha} \mathbf{if}\ \xi|_n\ A_1'|_{R_1}\ A_2|_{R_2}}\ [\to 1\text{-}\textsc{NIf4}]$$

$$\frac{A_2 \xrightarrow{\hat{\xi},\alpha} A_2' \quad \mathsf{subj}(\hat{\xi}) \cup \mathsf{subj}(\alpha) \subseteq R_2 \setminus R_1}{\mathbf{if}\ \xi|_n\ A_1|_{R_1}\ A_2|_{R_2} \xrightarrow{\hat{\xi},\alpha} \mathbf{if}\ \xi|_n\ A_1|_{R_1}\ A_2'|_{R_2}}\ [\to 1\text{-}\textsc{NIf5}]$$

$$\frac{\mathbf{if}\ \xi|_n\ (A \,;\, \mathbf{while}\ \xi|_n\ \{\psi\}\ A|_\emptyset)|_\emptyset\ \mathbf{skip}|_\emptyset \xrightarrow{\hat{\xi},\alpha} A'}{\mathbf{while}\ \xi|_n\ \{\psi\}\ A|_\emptyset \xrightarrow{\hat{\xi},\alpha} A'}\ [\to 1\text{-}\textsc{NWhile}]$$

Lemma 50. $\left[A \xrightarrow{\xi,\alpha} A' \text{ and } (\xi,\alpha) \neq (\texttt{true},\tau)\right]$ **implies** $\alpha \neq \tau$

Proof. See page 60. □

Lemma 51.

1. $A \xrightarrow{\xi,\alpha} A'$ **implies** $\mathsf{read}(\xi) \cup \mathsf{read}(\alpha) \cup \mathsf{read}(A') \subseteq \mathsf{read}(A)$
2. $A \xrightarrow{\xi,\alpha} A'$ **implies** $\mathsf{read}(\xi) \cup \mathsf{write}(\alpha) \cup \mathsf{write}(A') \subseteq \mathsf{write}(A)$

Proof. See page 60. □

Lemma 52.

1. $\left[A_1 \# A_2 \text{ and } A_2 \xrightarrow{\xi,\alpha} A_2'\right]$ **implies** $A_1 \# \xi$

2. $\left[A_2 \# A_1 \textbf{ and } A_2 \xrightarrow{\xi,\alpha} A_2'\right]$ **implies** $\xi \# A_1$
3. $\left[A_1 \# A_2 \textbf{ and } A_2 \xrightarrow{\xi,\alpha} A_2'\right]$ **implies** $A_1 \# \alpha$
4. $\left[A_2 \# A_1 \textbf{ and } A_2 \xrightarrow{\xi,\alpha} A_2'\right]$ **implies** $\alpha \# A_1$
5. $\left[A_1 \# A_2 \textbf{ and } A_2 \xrightarrow{\xi,\alpha} A_2'\right]$ **implies** $A_1 \# A_2'$
6. $\left[A_2 \# A_1 \textbf{ and } A_2 \xrightarrow{\xi,\alpha} A_2'\right]$ **implies** $A_2' \# A_1$

Proof. See page 62. □

Lemma 53.

1. $A \xrightarrow{\xi,\alpha} A'$ **implies** $\mathsf{subj}(\xi) \cup \mathsf{subj}(\alpha) \cup \mathsf{subj}(A') \subseteq \mathsf{subj}(A)$
2. $A \xrightarrow{\xi,\alpha} A'$ **implies** $\mathsf{subj}(\xi) \subseteq \mathsf{subj}(\alpha)$
3. $A \xrightarrow{\mathtt{true},\tau} A'$ **implies** $\mathsf{subj}(A) = \mathsf{subj}(A')$

Proof. See page 63. □

Lemma 54. $A \xrightarrow{\xi,\alpha} A'$ **implies** $\mathsf{chan}(\alpha) \cup \mathsf{chan}(A') \subseteq \mathsf{chan}(A)$

Proof. See page 66. □

Lemma 55. $\left[A\!\downarrow \textbf{ and } A \xrightarrow{\xi,\alpha} A'\right]$ **implies** $\left[A'\!\downarrow \textbf{ and } \xi = \mathtt{true} \textbf{ and } \alpha = \tau\right]$

Proof. See page 67. □

Lemma 56.

1. $\left[A \xrightarrow{\mathtt{true},\tau} A_1' \textbf{ and } A \xrightarrow{\xi,\alpha} A_2'\right]$ **implies** $\begin{bmatrix}\left[\left[A_1' \xrightarrow{\xi,\alpha} A'' \textbf{ and } A_2' \xrightarrow{\mathtt{true},\tau} A''\right] \textbf{ for some } A''\right] \\ \textbf{or } \left[\xi = \mathtt{true} \textbf{ and } \alpha = \tau \textbf{ and } A_1' = A_2'\right]\end{bmatrix}$
2. $A \xrightarrow{\mathtt{true},\tau} A_1' \xrightarrow{\xi,\alpha} A''$ **implies** $\left[A \xrightarrow{\xi,\alpha} A_2' \xrightarrow{\mathtt{true},\tau} A'' \textbf{ for some } A_2'\right]$

Proof. See page 67. □

Lemma 57. $\checkmark_R(G)$ **implies** $\left[G\!\downarrow \textbf{ or } \left[G \xrightarrow{\xi,\gamma} G' \textbf{ for some } G', \xi, \gamma\right]\right]$

Proof. See page 72. □

Lemma 58.

1. $\left[G \xrightarrow{\xi,\gamma} G' \textbf{ and } \checkmark_R(G)\right]$ **implies** $\mathsf{subj}(\gamma) \neq \emptyset$
2. $\left[G \xrightarrow{\xi,\gamma} G' \textbf{ and } \checkmark_R(G)\right]$ **implies** $\gamma \neq \tau$

Proof. See page 74. □

Lemma 59.

1. $\left[G \xrightarrow{\xi,\gamma} G' \textbf{ and } \checkmark_R(G) \textbf{ and } r \in R \cap \mathsf{subj}(\gamma)\right]$ **implies** $(G \upharpoonright r) \xrightarrow{\xi \upharpoonright r, \gamma \upharpoonright r} (G' \upharpoonright r)$
2. $\left[G \xrightarrow{\xi,\gamma} G' \textbf{ and } \checkmark_R(G) \textbf{ and } r \in R \setminus \mathsf{subj}(\gamma)\right]$ **implies** $(G \upharpoonright r) \xrightarrow{\mathtt{true},\tau} (G' \upharpoonright r)$

Proof. See page 74. □

Lemma 60. $\begin{bmatrix}(G \upharpoonright r) \xrightarrow{\xi_r,\lambda_r} L_r' \textbf{ and} \\ \checkmark_R(G) \textbf{ and } \lambda_r \neq \tau\end{bmatrix}$ **implies** $\begin{bmatrix}\left[\mathsf{subj}(\gamma) \subseteq \mathsf{subj}(G) \textbf{ and } \lambda_r = \gamma \upharpoonright r\right] \\ \textbf{for some } \gamma\end{bmatrix}$

Proof. See page 78. □

Lemma 61. $\begin{bmatrix}(G \upharpoonright q) \xrightarrow{\xi_q, q.y := e} L_q' \\ \textbf{and } \checkmark_R(G)\end{bmatrix}$ **implies** $\begin{bmatrix}\begin{bmatrix}G \xrightarrow{\xi, q.y := e} G' \textbf{ and} \\ L_q' = G' \upharpoonright q \textbf{ and } \xi_q = \xi \upharpoonright q\end{bmatrix} \\ \textbf{for some } G', \xi\end{bmatrix}$

Proof. See page 80. □

Lemma 62.

$$\begin{bmatrix}(G \upharpoonright p) \xrightarrow{\xi_p, pq!e} L'_p \textbf{ and}\\ (G \upharpoonright q) \xrightarrow{\xi_q, pq?y} L'_q \textbf{ and}\\ \checkmark_R(G)\end{bmatrix} \textbf{ implies } \begin{bmatrix}\begin{bmatrix}G \xrightarrow{\xi, p.e \twoheadrightarrow q.y} G' \textbf{ and } L'_p = G' \upharpoonright p \textbf{ and } \xi_p = \xi \upharpoonright p\\ \textbf{and } L'_q = G' \upharpoonright q \textbf{ and } \xi_q = \xi \upharpoonright q\end{bmatrix}\\ \textbf{for some } G', \xi\end{bmatrix}$$

Proof. See page 82. □

Lemma 63.

$$\begin{bmatrix}(G \upharpoonright r) \xrightarrow{\xi_r, i^{\hat{R}}_{\{r\}}} L'_r\\ \textbf{and } r \in \hat{R} \textbf{ and}\\ \begin{bmatrix}(G \upharpoonright \hat{r}) \xrightarrow{\xi_{\hat{r}}, i^{\hat{R}}_{\{\hat{r}\}}} L'_{\hat{r}}\\ \textbf{for every } \hat{r} \in \hat{R}\end{bmatrix}\\ \textbf{and } \checkmark_R(G)\end{bmatrix} \textbf{ implies } \begin{bmatrix}\begin{bmatrix}G \xrightarrow{\xi, i^{\hat{R}}_{\hat{R}}} G' \textbf{ and } \begin{bmatrix}L'_{\hat{r}} = G' \upharpoonright \hat{r}\\ \textbf{for every}\\ \hat{r} \in \hat{R}\end{bmatrix} \textbf{ and } \begin{bmatrix}\xi_{\hat{r}} = \xi \upharpoonright \hat{r}\\ \textbf{for every}\\ \hat{r} \in \hat{R}\end{bmatrix}\end{bmatrix}\\ \textbf{for some } G', \xi\end{bmatrix}$$

Proof. See page 86. □

Lemma 64.

$$\begin{bmatrix}\begin{bmatrix}(G \upharpoonright r) \xrightarrow{\xi_r, \gamma \upharpoonright r} L'_r\\ \textbf{for every}\\ r \in \mathsf{subj}(\gamma)\end{bmatrix}\\ \textbf{and } \checkmark_R(G) \textbf{ and } \gamma \neq \tau\end{bmatrix} \textbf{ implies } \begin{bmatrix}\begin{bmatrix}G \xrightarrow{\xi, \gamma} G' \textbf{ and } \begin{bmatrix}L'_r = G' \upharpoonright r\\ \textbf{for every}\\ r \in \mathsf{subj}(\gamma)\end{bmatrix} \textbf{ and } \begin{bmatrix}\xi_r = \xi \upharpoonright r\\ \textbf{for every}\\ r \in \mathsf{subj}(\gamma)\end{bmatrix}\end{bmatrix}\\ \textbf{for some } G', \xi\end{bmatrix}$$

Proof. See page 91. □

Definition 32 (Reduction, part 2).

$$\frac{G \xrightarrow{\xi, \gamma} G'}{\{G\} \xrightarrow{\xi, \gamma} \{G'\}} \; [\to 2\text{-}\textsc{Glob}]$$

$$\frac{L_{\hat{r}} \xrightarrow{\xi, \tau} L'_{\hat{r}} \quad \hat{r} \in R}{\{L_r\}_{r \in R} \xrightarrow{\xi, \tau} \{L_r\}_{r \in R \setminus \{\hat{r}\}} \cup \{L'_r\}_{r \in \{\hat{r}\}}} \; [\to 2\text{-}\textsc{Locs1}]$$

$$\frac{L_q \xrightarrow{\xi, q.y := e} L'_q \quad q \in R}{\{L_r\}_{r \in R} \xrightarrow{\xi, q.y := e} \{L_r\}_{r \in R \setminus \{q\}} \cup \{L'_r\}_{r \in \{q\}}} \; [\to 2\text{-}\textsc{Locs2}]$$

$$\frac{L_p \xrightarrow{\xi_p, pq!e} L'_p \quad L_q \xrightarrow{\xi_q, pq?y} L'_q \quad p, q \in R}{\{L_r\}_{r \in R} \xrightarrow{\xi_p \wedge \xi_q, p.e \twoheadrightarrow q.y} \{L_r\}_{r \in R \setminus \{p,q\}} \cup \{L'_r\}_{r \in \{p,q\}}} \; [\to 2\text{-}\textsc{Locs3}]$$

$$\frac{[L_r \xrightarrow{\xi_r, i^{\hat{R}}_{\{r\}}} L'_r \textbf{ for every } r \in \hat{R}] \quad \hat{R} \subseteq R}{\{L_r\}_{r \in R} \xrightarrow{\bigwedge\{\xi_r\}_{r \in \hat{R}}, i^{\hat{R}}_{\hat{R}}} \{L_r\}_{r \in R \setminus \hat{R}} \cup \{L'_r\}_{r \in \hat{R}}} \; [\to 2\text{-}\textsc{Locs4}]$$

Lemma 65. $\left[\mathcal{A} \xrightarrow{\xi, \gamma} \mathcal{A}' \textbf{ and } (\xi, \gamma) \neq (\mathtt{true}, \tau)\right]$ **implies** $\gamma \neq \tau$

Proof. See page 91. □

Lemma 66. $\{L_r\}_{r \in R} \xrightarrow{\xi, \gamma} \{L'_r\}_{r \in R}$ **implies** $\mathsf{subj}(\gamma) \subseteq R$

Proof. See page 92. □

Lemma 67.

1. $\{L_r\}_{r\in R} \xrightarrow{\xi,\tau} \{L'_r\}_{r\in R}$ **implies** $\left[\begin{array}{c}\left[\begin{array}{c} L_{\hat{r}} \xrightarrow{\xi,\tau} L'_{\hat{r}} \textbf{ and } \hat{r} \in R \textbf{ and} \\ \left[L_r = L'_r \textbf{ for every } r \in R \setminus \{\hat{r}\}\right]\end{array}\right] \\ \textbf{for some } \hat{r}\end{array}\right]$

2. $\left[\{L_r\}_{r\in R} \xrightarrow{\xi,\gamma} \{L'_r\}_{r\in R} \textbf{ and } \gamma \neq \tau\right]$ **implies** $\left[\begin{array}{c}\left[\begin{array}{c}\left[L_r \xrightarrow{\xi_r,\gamma\restriction r} L'_r \textbf{ for every } r \in \mathsf{subj}(\gamma)\right] \\ \textbf{and } \xi = \bigwedge\{\xi_r\}_{r\in\mathsf{subj}(\gamma)} \textbf{ and} \\ \left[L_r = L'_r \textbf{ for every } r \in R \setminus \mathsf{subj}(\gamma)\right]\end{array}\right] \\ \textbf{for some } \{\xi_r\}_{r\in\mathsf{subj}(\gamma)}\end{array}\right]$

Proof. See page 92. □

Lemma 68.

$$\left[\begin{array}{c}\left[L_r \xrightarrow{\xi_r,\gamma\restriction r} L'_r \textbf{ for every } r \in \mathsf{subj}(\gamma) \subseteq R\right] \\ \textbf{and } \left[L_r = L'_r \textbf{ for every } r \in R \setminus \mathsf{subj}(\gamma)\right] \\ \textbf{and } \gamma \neq \tau\end{array}\right] \textbf{ implies } \{L_r\}_{r\in R} \xrightarrow{\bigwedge\{\xi_r\}_{r\in\mathsf{subj}(\gamma)},\gamma} \{L'_r\}_{r\in R}$$

Proof. See page 92. □

Definition 33 (Reduction, part 3).

$$\frac{\mathcal{A} \xrightarrow{\xi,\gamma} \mathcal{A}' \quad \mathcal{S} \in [\![\xi]\!] \quad \dot{\gamma} = \mathsf{con}_{\mathcal{S}}(\gamma)}{(\mathcal{A},\mathcal{S}) \xrightarrow{\dot{\gamma}} (\mathcal{A}', \mathsf{effect}(\dot{\gamma},\mathcal{S}))}\ [\rightarrow 3]$$

7 Reachability

Definition 34 (Reachability, part 1).

$$\frac{}{A \xrightarrow{\Pi}{}^{*} A}\,[\to^{*}\text{1-Base}] \qquad \frac{A \xrightarrow{\Pi}{}^{*} A^{*} \xrightarrow{\pi} A^{\dagger} \quad \pi \in \Pi}{A \xrightarrow{\Pi}{}^{*} A^{\dagger}}\,[\to^{*}\text{1-Step}] \qquad \frac{A \xrightarrow{\{(\mathtt{true},\tau)\}}{}^{*} A^{\dagger}}{A \Longrightarrow A^{\dagger}}\,[\Rightarrow 1]$$

Lemma 69.

1. $\left[A \xrightarrow{\{(\mathtt{true},\tau)\}}{}^{*} A^{\dagger} \textbf{ and } A{\downarrow}\right]$ **implies** $A^{\dagger}{\downarrow}$
2. $\begin{bmatrix} A \xrightarrow{\{(\mathtt{true},\tau)\}}{}^{*} A^{\dagger}_1 \\ \textbf{and } A \xrightarrow{\xi,\alpha} A'_2 \end{bmatrix}$ **implies** $\left[\begin{bmatrix} A^{\dagger}_1 \xrightarrow{\xi,\alpha} A^{\ddagger} \textbf{ and} \\ A'_2 \xrightarrow{\{(\mathtt{true},\tau)\}}{}^{*} A^{\ddagger} \end{bmatrix} \textbf{for some } A^{\ddagger}\right]$
3. $A \xrightarrow{\{(\mathtt{true},\tau)\}}{}^{*} A^{\dagger}_1 \xrightarrow{\xi,\alpha} A^{\ddagger}$ **implies** $\left[A \xrightarrow{\xi,\alpha} A'_2 \xrightarrow{\{(\mathtt{true},\tau)\}}{}^{*} A^{\ddagger} \textbf{ for some } A'_2\right]$

Proof. See page 94. □

Lemma 70.

1. $A^{\dagger} \Longleftarrow A{\downarrow}$ **implies** $A^{\dagger}{\downarrow}$
2. $A_1 \Longrightarrow A^{\dagger} \Longleftarrow A_2 \xrightarrow{\mathtt{true},\tau} A'_2$ **implies** $\left[A_1 \Longrightarrow A^{\ddagger} \Longleftarrow A'_2 \textbf{ for some } A^{\ddagger}\right]$
3. $A_1 \Longrightarrow A^{\dagger} \Longleftarrow A_2 \xrightarrow{\xi,\alpha} A'_2$ **implies** $\left[A_1 \xrightarrow{\xi,\alpha} A'_1 \Longrightarrow A^{\ddagger} \Longleftarrow A'_2 \textbf{ for some } A'_1, A^{\ddagger}\right]$

Proof. See page 94. □

Definition 35 (Reachability, part 2).

$$\frac{}{\mathcal{A} \xrightarrow{\Pi}{}^{*} \mathcal{A}}\,[\to^{*}\text{2-Base}] \qquad \frac{\mathcal{A} \xrightarrow{\Pi}{}^{*} \mathcal{A}^{*} \xrightarrow{\pi} \mathcal{A}^{\dagger} \quad \pi \in \Pi}{\mathcal{A} \xrightarrow{\Pi}{}^{*} \mathcal{A}^{\dagger}}\,[\to^{*}\text{2-Step}] \qquad \frac{\mathcal{A} \xrightarrow{\{(\mathtt{true},\tau)\}}{}^{*} \mathcal{A}^{\dagger}}{\mathcal{A} \Longrightarrow \mathcal{A}^{\dagger}}\,[\Rightarrow 2]$$

Lemma 71. $\mathcal{A} \xrightarrow{\Pi}{}^{*} \mathcal{A}^{\dagger} \xrightarrow{\Pi}{}^{*} \mathcal{A}^{\S}$ **implies** $\mathcal{A} \xrightarrow{\Pi}{}^{*} \mathcal{A}^{\S}$

Proof. See page 95. □

Lemma 72.

1. $\left[L_{\hat{r}} \xrightarrow{\{(\mathtt{true},\tau)\}}{}^{*} L^{\dagger}_{\hat{r}} \textbf{ and } \hat{r} \in R\right]$ **implies** $\{L_r\}_{r \in R} \xrightarrow{\{(\mathtt{true},\tau)\}}{}^{*} \{L_r\}_{r \in R \setminus \{\hat{r}\}} \cup \{L^{\dagger}_r\}_{r \in \{\hat{r}\}}$
2. $\left[L_r \xrightarrow{\{(\mathtt{true},\tau)\}}{}^{*} L^{\dagger}_r \textbf{ for every } r \in \hat{R} \subseteq R\right]$ **implies** $\{L_r\}_{r \in R} \xrightarrow{\{(\mathtt{true},\tau)\}}{}^{*} \{L_r\}_{r \in R \setminus \hat{R}} \cup \{L^{\dagger}_r\}_{r \in \hat{R}}$
3. $\left[L_r \xrightarrow{\{(\mathtt{true},\tau)\}}{}^{*} L^{\dagger}_r \textbf{ for every } r \in R\right]$ **implies** $\{L_r\}_{r \in R} \xrightarrow{\{(\mathtt{true},\tau)\}}{}^{*} \{L^{\dagger}_r\}_{r \in R}$

Proof. See page 95. □

Definition 36 (Reachability, part 3).

$$\frac{}{(\mathcal{A},\mathcal{S}) \xrightarrow{\Pi}{}^{*} (\mathcal{A},\mathcal{S})}\,[\to^{*}\text{3-Base}] \qquad \frac{(\mathcal{A},\mathcal{S}) \xrightarrow{\Pi}{}^{*} (\mathcal{A}^{*},\mathcal{S}^{*}) \xrightarrow{\pi} (\mathcal{A}^{\dagger},\mathcal{S}^{\dagger}) \quad \pi \in \Pi}{(\mathcal{A},\mathcal{S}) \xrightarrow{\Pi}{}^{*} (\mathcal{A}^{\dagger},\mathcal{S}^{\dagger})}\,[\to^{*}\text{3-Step}]$$

$$\frac{(\mathcal{A},\mathcal{S}) \xrightarrow{\{\tau\}}{}^{*} (\mathcal{A}^{\dagger},\mathcal{S}^{\dagger})}{(\mathcal{A},\mathcal{S}) \Longrightarrow (\mathcal{A}^{\dagger},\mathcal{S}^{\dagger})}\,[\Rightarrow 3]$$

Lemma 73. $\mathcal{A} \xrightarrow{\{(\mathtt{true},\tau)\}}{}^{*} \mathcal{A}^{\dagger}$ **implies** $(\mathcal{A},\mathcal{S}) \xrightarrow{\{\tau\}}{}^{*} (\mathcal{A}^{\dagger},\mathcal{S})$

Proof. See page 96. □

Lemma 74. $\mathcal{A} \Longrightarrow \mathcal{A}^{\dagger}$ **implies** $(\mathcal{A},\mathcal{S}) \Longrightarrow (\mathcal{A}^{\dagger},\mathcal{S})$

Proof. See page 96. □

8 Theorems: Deadlock Freedom and Functional Correctness

Definition 37 (Preconditions of actions).

$$\begin{aligned}
\phi(\tau, \chi) &= \chi \\
\phi(q.y := e, \chi) &= \chi[e/q.y] \\
\phi(p.e \twoheadrightarrow q.y, \chi) &= \chi[e/q.y] \\
\phi(i_R^R, \chi) &= \chi
\end{aligned}$$

Lemma 75.

1. $\phi(\gamma, \neg\chi) = \neg\phi(\gamma, \chi)$
2. $\phi(\gamma, \chi_1 \wedge \chi_2) = \phi(\gamma, \chi_1) \wedge \phi(\gamma, \chi_2)$
3. $\phi(\gamma, \chi_1 \Rightarrow \chi_2) = \phi(\gamma, \chi_1) \Rightarrow \phi(\gamma, \chi_2)$

Proof. See page 97. □

Lemma 76. $\mathsf{write}(\gamma) \cap \mathsf{read}(\chi) = \emptyset$ **implies** $\phi(\gamma, \chi) = \chi$

Proof. See page 97. □

Lemma 77.

$$\begin{bmatrix} \mathsf{read}(\gamma_1) \cap \mathsf{write}(\gamma_2) = \emptyset \textbf{ and} \\ \mathsf{write}(\gamma_1) \cap \mathsf{read}(\gamma_2) = \emptyset \textbf{ and} \\ \mathsf{write}(\gamma_1) \cap \mathsf{write}(\gamma_2) = \emptyset \end{bmatrix} \textbf{ implies } \phi(\gamma_1, \phi(\gamma_2, \chi)) = \phi(\gamma_2, \phi(\gamma_1, \chi))$$

Proof. See page 97. □

Lemma 78. $[\![\chi_1]\!] \subseteq [\![\chi_2]\!]$ **implies** $[\![\phi(\gamma, \chi_1)]\!] \subseteq [\![\phi(\gamma, \chi_2)]\!]$

Proof. See page 98. □

Lemma 79. $\mathcal{S} \in [\![\phi(\gamma, \chi)]\!]$ **implies** $\mathsf{effect}(\mathsf{con}_{\mathcal{S}}(\gamma), \mathcal{S}) \in [\![\chi]\!]$

Proof. See page 98. □

Definition 38 (Preconditions of programs).

$$\begin{aligned}
\phi(\xi, \chi) &= \xi \wedge \chi \\
\phi(\gamma, \chi) &= \textit{Defn. 37} \\
\phi(\textbf{skip}, \chi) &= \chi \\
\phi(G_1 \,;\, G_2, \chi) &= \phi(G_1, \phi(G_2, \chi)) \\
\phi(G_1 \parallel G_2, \chi) &= \begin{cases} \phi(G_1, \phi(G_2, \chi)) & \textbf{if} : G_1, G_2 \in \underline{\mathbb{G}} \textbf{ and } G_1 \# G_2 \\ \texttt{false} & \textbf{else} \end{cases} \\
\phi(R.\textbf{if}\, \xi\, G_1\, G_2, \chi) &= (\xi^+ \Rightarrow \phi(G_1, \chi)) \wedge (\xi^- \Rightarrow \phi(G_2, \chi)) \wedge \xi^{\equiv} \\
\phi(R.\textbf{while}\, \xi\, \{\psi\}\, G, \chi) &= \psi \wedge \forall(\psi \Rightarrow ((\xi^+ \Rightarrow \phi(G, \psi)) \wedge (\xi^- \Rightarrow \chi) \wedge \xi^{\equiv})) \\
\phi(\textbf{if}\, \xi|_0\, G_1|_{R_1}\, G_2|_{R_2}, \chi) &= \begin{cases} (\xi^+ \Rightarrow \phi(G_1, \chi)) \wedge (\xi^- \Rightarrow \phi(G_2, \chi)) \wedge \xi^{\equiv} & \textbf{if} : R_1 = \emptyset = R_2 \\ \phi(G_2, \chi) \wedge \bigwedge\{\xi^- \upharpoonright r\}_{r \in \mathsf{subj}(\xi) \setminus R_2} & \textbf{if} : R_1 = \emptyset \neq R_2 \\ \phi(G_1, \chi) \wedge \bigwedge\{\xi^+ \upharpoonright r\}_{r \in \mathsf{subj}(\xi) \setminus R_1} & \textbf{if} : R_1 \neq \emptyset = R_2 \\ \texttt{false} & \textbf{if} : R_1 \neq \emptyset \neq R_2 \end{cases} \\
\phi(\textbf{while}\, \xi|_0\, \{\psi\}\, G|_{\emptyset}, \chi) &= \psi \wedge \forall(\psi \Rightarrow ((\xi^+ \Rightarrow \phi(G, \psi)) \wedge (\xi^- \Rightarrow \chi) \wedge \xi^{\equiv})) \\
\phi(\{G\}, \chi) &= \phi(G, \chi)
\end{aligned}$$

Lemma 80. $[\![\phi(\mathbf{while}\,\xi|_0\,\{\psi\}\,G|_\emptyset,\chi)]\!] \subseteq [\![\phi(\mathbf{if}\,\xi|_0\,(G\,;\mathbf{while}\,\xi|_0\,\{\psi\}\,G|_\emptyset)|_\emptyset\,\mathbf{skip}|_\emptyset,\chi)]\!]$

Proof. See page 98. □

Lemma 81.

1. $[\![\chi_1]\!] \subseteq [\![\chi_2]\!]$ **implies** $[\![\phi(G,\chi_1)]\!] \subseteq [\![\phi(G,\chi_2)]\!]$
2. $[\![\chi_1]\!] = [\![\chi_2]\!]$ **implies** $[\![\phi(G,\chi_1)]\!] = [\![\phi(G,\chi_2)]\!]$

Proof. See page 99. □

Lemma 82. $[\![\phi(G,\chi_1 \wedge \chi_2)]\!] = [\![\phi(G,\chi_1) \wedge \phi(G,\chi_2)]\!]$

Proof. See page 100. □

Lemma 83. $[\![\phi(\underline{G},\texttt{false})]\!] = [\![\texttt{false}]\!]$

Proof. See page 103. □

Lemma 84. $[\![\phi(\underline{G},\chi)]\!] \neq \emptyset$ **implies** $[\![\chi]\!] \neq \emptyset$

Proof. See page 104. □

Lemma 85. $\underline{G} \,\#\, \xi$ **implies** $[\![\phi(\underline{G},\chi) \wedge \xi]\!] \subseteq [\![\phi(\underline{G},\xi)]\!]$

Proof. See page 105. □

Lemma 86. $\underline{G} \,\#\, \gamma$ **implies** $[\![\phi(\underline{G},\phi(\gamma,\chi))]\!] = [\![\phi(\gamma,\phi(\underline{G},\chi))]\!]$

Proof. See page 108. □

Lemma 87. $\big[\checkmark_R(G)$ **and** $[\![\phi(G,\chi)]\!] \neq \emptyset\big]$ **implies** $\big[\mathsf{subj}(G) = R$ **or** $G \in \underline{\mathbb{G}}\big]$

Proof. See page 110. □

Lemma 88.

1. $\big[\checkmark_R(G)$ **and** $G\!\downarrow\big]$ **implies** $[\![\phi(G,\chi)]\!] \subseteq [\![\chi]\!]$
2. $\big[\checkmark_R(\mathcal{G})$ **and** $\mathcal{G}\!\downarrow\big]$ **implies** $[\![\phi(\mathcal{G},\chi)]\!] \subseteq [\![\chi]\!]$
3. $\big[\checkmark_R(\mathcal{G})$ **and** $\mathcal{S} \in [\![\phi(\mathcal{G},\chi)]\!]$ **and** $(\mathcal{G},\mathcal{S})\!\downarrow\big]$ **implies** $\mathcal{S} \in [\![\chi]\!]$

Proof. See page 112. □

Lemma 89.

1. $\big[\checkmark_R(G)$ **and** $[\![\phi(G,\chi) \wedge \xi]\!] \neq \emptyset$ **and** $G \xrightarrow{\xi,\gamma} G'\big]$ **implies** $\checkmark_R(G')$
2. $\big[\checkmark_R(\mathcal{G})$ **and** $[\![\phi(\mathcal{G},\chi) \wedge \xi]\!] \neq \emptyset$ **and** $\mathcal{G} \xrightarrow{\xi,\gamma} \mathcal{G}'\big]$ **implies** $\checkmark_R(\mathcal{G}')$
3. $\big[\checkmark_R(\mathcal{G})$ **and** $\mathcal{S} \in [\![\phi(\mathcal{G},\chi) \wedge \xi]\!]$ **and** $(\mathcal{G},\mathcal{S}) \xrightarrow{\gamma} (\mathcal{G}',\mathcal{S}')\big]$ **implies** $\checkmark_R(\mathcal{G}')$

Proof. See page 113. □

Lemma 90. $\big[G \in \underline{\mathbb{G}}$ **and** $\checkmark_R(G)$ **and** $G \xrightarrow{\xi,\gamma} G'\big]$ **implies** $\big[G' \in \underline{\mathbb{G}}$ **or** $[\![\phi(G,\chi) \wedge \xi]\!] = \emptyset\big]$

Proof. See page 116. □

Lemma 91.

1. $\big[\checkmark_R(G)$ **and** $G \xrightarrow{\xi,\gamma} G'\big]$ **implies** $[\![\phi(G,\chi) \wedge \xi]\!] \subseteq [\![\phi(\gamma,\phi(G',\chi))]\!]$
2. $\big[\checkmark_R(\mathcal{G})$ **and** $\mathcal{G} \xrightarrow{\xi,\gamma} \mathcal{G}'\big]$ **implies** $[\![\phi(\mathcal{G},\chi) \wedge \xi]\!] \subseteq [\![\phi(\gamma,\phi(\mathcal{G}',\chi))]\!]$
3. $\big[\checkmark_R(\mathcal{G})$ **and** $\mathcal{S} \in [\![\phi(\mathcal{G},\chi)]\!]$ **and** $(\mathcal{G},\mathcal{S}) \xrightarrow{\gamma} (\mathcal{G}',\mathcal{S}')\big]$ **implies** $\mathcal{S}' \in [\![\phi(\mathcal{G}',\chi)]\!]$

Proof. See page 118. □

Lemma 92. $\begin{bmatrix} \checkmark_R(\mathcal{G}) \textbf{ and } \mathcal{S} \in [\![\phi(\mathcal{G}, \chi)]\!] \\ \textbf{and } (\mathcal{G}, \mathcal{S}) \xrightarrow{\dot{\Gamma}}{}^{*} (\mathcal{G}^\dagger, \mathcal{S}^\dagger) \end{bmatrix}$ **implies** $\left[\checkmark_R(\mathcal{G}^\dagger) \textbf{ and } \mathcal{S}^\dagger \in [\![\phi(G^\dagger, \chi)]\!]\right]$

Proof. See page 124. □

Lemma 93.

1. $\left[\checkmark_R(G) \textbf{ and } \mathcal{S} \in [\![\phi(G, \chi)]\!]\right]$ **implies** $\begin{bmatrix} G\downarrow \textbf{ or } \begin{bmatrix} \left[G \xrightarrow{\xi, \gamma} G' \textbf{ and } \mathcal{S} \in [\![\xi]\!]\right] \\ \textbf{for some } G', \xi, \gamma \end{bmatrix} \end{bmatrix}$
2. $\left[\checkmark_R(\mathcal{G}) \textbf{ and } \mathcal{S} \in [\![\phi(\mathcal{G}, \chi)]\!]\right]$ **implies** $\begin{bmatrix} \mathcal{G}\downarrow \textbf{ or } \begin{bmatrix} \left[\mathcal{G} \xrightarrow{\xi, \gamma} \mathcal{G}' \textbf{ and } \mathcal{S} \in [\![\xi]\!]\right] \\ \textbf{for some } \mathcal{G}', \xi, \gamma \end{bmatrix} \end{bmatrix}$
3. $\left[\checkmark_R(\mathcal{G}) \textbf{ and } \mathcal{S} \in [\![\phi(\mathcal{G}, \chi)]\!]\right]$ **implies** $\begin{bmatrix} (\mathcal{G}, \mathcal{S})\downarrow \textbf{ or } \begin{bmatrix} (\mathcal{G}, \mathcal{S}) \xrightarrow{\dot{\gamma}} (\mathcal{G}', \mathcal{S}') \\ \textbf{for some } \mathcal{G}', \mathcal{S}', \dot{\gamma} \end{bmatrix} \end{bmatrix}$

Proof. See page 124. □

Theorem 1 (Deadlock freedom).

$$\begin{bmatrix} \checkmark_R(\mathcal{G}) \textbf{ and } \mathcal{S} \in [\![\phi(\mathcal{G}, \chi)]\!] \\ \textbf{and } (\mathcal{G}, \mathcal{S}) \xrightarrow{\dot{\Gamma}}{}^{*} (\mathcal{G}^\dagger, \mathcal{S}^\dagger) \end{bmatrix} \textbf{ implies } \begin{bmatrix} (\mathcal{G}^\dagger, \mathcal{S}^\dagger)\downarrow \textbf{ or } \begin{bmatrix} (\mathcal{G}^\dagger, \mathcal{S}^\dagger) \xrightarrow{\dot{\gamma}} (\mathcal{G}^\ddagger, \mathcal{S}^\ddagger) \\ \textbf{for some } \mathcal{G}^\ddagger, \mathcal{S}^\ddagger, \dot{\gamma} \end{bmatrix} \end{bmatrix}$$

Proof. See page 127. □

Theorem 2 (Functional correctness).

$$\left[\checkmark_R(\mathcal{G}) \textbf{ and } \mathcal{S} \in [\![\phi(\mathcal{G}, \chi)]\!] \textbf{ and } (\mathcal{G}, \mathcal{S}) \xrightarrow{\dot{\Gamma}}{}^{*} (\mathcal{G}^\dagger, \mathcal{S}^\dagger)\downarrow\right] \textbf{ implies } \mathcal{S}^\dagger \in [\![\chi]\!]$$

Proof. See page 127. □

9 Theorem: Operational Equivalence

Definition 39 (Weak bisimilarity).

$$\frac{\begin{array}{c}
\left[\begin{array}{c}\left[\begin{array}{c}(\mathcal{A}_2,\mathcal{S}_2) \Longrightarrow (\mathcal{A}_2^\dagger,\mathcal{S}_2^\dagger)\downarrow \\ \textbf{for some } \mathcal{A}_2^\dagger,\mathcal{S}_2^\dagger\end{array}\right] \\ \textbf{for every } (\mathcal{A}_1,\mathcal{S}_1)\downarrow\end{array}\right] \quad \left[\begin{array}{c}(\mathcal{A}_1',\mathcal{S}_1') \approx (\mathcal{A}_2,\mathcal{S}_2) \\ \textbf{for every} \\ (\mathcal{A}_1,\mathcal{S}_1) \xrightarrow{\tau} (\mathcal{A}_1',\mathcal{S}_1')\end{array}\right] \\
\left[\begin{array}{c}\left[\begin{array}{c}\left[\begin{array}{c}(\mathcal{A}_1',\mathcal{S}_1') \approx (\mathcal{A}_2^\P,\mathcal{S}_2^\P)\ \textbf{and} \\ (\mathcal{A}_2,\mathcal{S}_2) \Longrightarrow (\mathcal{A}_2^\dagger,\mathcal{S}_2^\dagger) \xrightarrow{\dot{\gamma}} (\mathcal{A}_2^\ddagger,\mathcal{S}_2^\ddagger) \Longrightarrow (\mathcal{A}_2^\P,\mathcal{S}_2^\P)\end{array}\right] \\ \textbf{for some } \mathcal{A}_2^\dagger,\mathcal{A}_2^\ddagger,\mathcal{A}_2^\P,\mathcal{S}_2^\dagger,\mathcal{S}_2^\ddagger,\mathcal{S}_2^\P\end{array}\right] \\ \textbf{for every } \left[(\mathcal{A}_1,\mathcal{S}_1) \xrightarrow{\dot{\gamma}} (\mathcal{A}_1',\mathcal{S}_1')\ \textbf{and}\ \dot{\gamma}\neq\tau\right]\end{array}\right] \\
\left[\begin{array}{c}\left[\begin{array}{c}(\mathcal{A}_1,\mathcal{S}_1) \Longrightarrow (\mathcal{A}_1^\dagger,\mathcal{S}_1^\dagger)\downarrow \\ \textbf{for some } \mathcal{A}_1^\dagger,\mathcal{S}_1^\dagger\end{array}\right] \\ \textbf{for every } (\mathcal{A}_2,\mathcal{S}_2)\downarrow\end{array}\right] \quad \left[\begin{array}{c}(\mathcal{A}_2',\mathcal{S}_2') \approx (\mathcal{A}_1,\mathcal{S}_1) \\ \textbf{for every} \\ (\mathcal{A}_2,\mathcal{S}_2) \xrightarrow{\tau} (\mathcal{A}_2',\mathcal{S}_2')\end{array}\right] \\
\left[\begin{array}{c}\left[\begin{array}{c}\left[\begin{array}{c}(\mathcal{A}_1^\P,\mathcal{S}_1^\P) \approx (\mathcal{A}_2',\mathcal{S}_2')\ \textbf{and} \\ (\mathcal{A}_1,\mathcal{S}_1) \Longrightarrow (\mathcal{A}_1^\dagger,\mathcal{S}_1^\dagger) \xrightarrow{\dot{\gamma}} (\mathcal{A}_1^\ddagger,\mathcal{S}_1^\ddagger) \Longrightarrow (\mathcal{A}_1^\P,\mathcal{S}_1^\P)\end{array}\right] \\ \textbf{for some } \mathcal{A}_1^\dagger,\mathcal{A}_1^\ddagger,\mathcal{A}_1^\P,\mathcal{S}_1^\dagger,\mathcal{S}_1^\ddagger,\mathcal{S}_1^\P\end{array}\right] \\ \textbf{for every } \left[(\mathcal{A}_2,\mathcal{S}_2) \xrightarrow{\dot{\gamma}} (\mathcal{A}_2',\mathcal{S}_2')\ \textbf{and}\ \dot{\gamma}\neq\tau\right]\end{array}\right]
\end{array}}{(\mathcal{A}_1,\mathcal{S}_1)\approx(\mathcal{A}_2,\mathcal{S}_2)}\ [\approx]$$

Definition 40 (Biconvergence).

$$\frac{L_1 \Longrightarrow L^\dagger \Longleftarrow L_2\ \textbf{for some}\ L^\dagger}{L_1 \Rightarrow\!\!\Leftarrow L_2}\ [\Rightarrow\!\!\Leftarrow]$$

Lemma 94. $L_1 \Rightarrow\!\!\Leftarrow L_2$ **implies** $L_2 \Rightarrow\!\!\Leftarrow L_1$

Proof. See page 128. □

Lemma 95.

1. $L_1 \Rightarrow\!\!\Leftarrow L_2\downarrow$ **implies** $\left[L_1 \Longrightarrow L^\dagger\downarrow\ \textbf{for some}\ L^\dagger\right]$
2. $L_1 \Rightarrow\!\!\Leftarrow L_2 \xrightarrow{\texttt{true},\tau} L_2'$ **implies** $L_1 \Rightarrow\!\!\Leftarrow L_2'$
3. $L_1 \Rightarrow\!\!\Leftarrow L_2 \xrightarrow{\xi,\lambda} L_2'$ **implies** $\left[L_1 \xrightarrow{\xi,\lambda} L_1' \Rightarrow\!\!\Leftarrow L_2'\ \textbf{for some}\ L_1'\right]$

Proof. See page 128. □

Definition 41 (Correspondence).

$$\frac{\checkmark_R(G) \quad \mathcal{S}\in[\![\phi(G,\texttt{true})]\!] \quad (G\upharpoonright r) \Rightarrow\!\!\Leftarrow L_r\ \textbf{for every}\ r\in R}{(\{G\},\mathcal{S}) \bowtie (\{L_r\}_{r\in R},\mathcal{S})}\ [\bowtie]$$

Lemma 96. $\left[\checkmark_R(\mathcal{G})\ \textbf{and}\ \mathcal{S}\in[\![\phi(\mathcal{G},\texttt{true})]\!]\right]$ **implies** $(\mathcal{G},\mathcal{S}) \bowtie (\mathcal{G}\upharpoonright R,\mathcal{S})$

Proof. See page 128. □

Lemma 97.

1. $\left[\begin{array}{l}(\mathcal{G},\mathcal{S}) \bowtie (\mathcal{L},\mathcal{S})\ \textbf{and} \\ (\mathcal{G},\mathcal{S})\downarrow\end{array}\right]$ **implies** $\left[(\mathcal{L},\mathcal{S}) \Longrightarrow (\mathcal{L}^\dagger,\mathcal{S}^\dagger)\downarrow\ \textbf{for some}\ \mathcal{L}^\dagger,\mathcal{S}^\dagger\right]$
2. $\left[\begin{array}{l}(\mathcal{G},\mathcal{S}) \bowtie (\mathcal{L},\mathcal{S})\ \textbf{and} \\ (\mathcal{G},\mathcal{S}) \xrightarrow{\tau} (\mathcal{G}',\mathcal{S}')\end{array}\right]$ **implies** $(\mathcal{G}',\mathcal{S}') \bowtie (\mathcal{L},\mathcal{S})$

3. $\begin{bmatrix}(\mathcal{G},\mathcal{S}) \bowtie (\mathcal{L},\mathcal{S}) \textbf{ and} \\ (\mathcal{G},\mathcal{S}) \xrightarrow{\dot{\gamma}} (\mathcal{G}',\mathcal{S}') \textbf{ and } \dot{\gamma} \neq \tau\end{bmatrix}$ **implies** $\left[\begin{matrix}\left[\begin{matrix}(\mathcal{G}',\mathcal{S}') \bowtie (\mathcal{L}^{¶},\mathcal{S}^{¶}) \textbf{ and} \\ (\mathcal{L},\mathcal{S}) \Longrightarrow (\mathcal{L}^{\dagger},\mathcal{S}^{\dagger}) \xrightarrow{\dot{\gamma}} (\mathcal{L}^{\ddagger},\mathcal{S}^{\ddagger}) \Longrightarrow (\mathcal{L}^{¶},\mathcal{S}^{¶})\end{matrix}\right] \\ \textbf{for some } \mathcal{L}^{\dagger},\mathcal{L}^{\ddagger},\mathcal{L}^{¶},\mathcal{S}^{\dagger},\mathcal{S}^{\ddagger},\mathcal{S}^{¶}\end{matrix}\right]$

4. $\begin{bmatrix}(\mathcal{G},\mathcal{S}) \bowtie (\mathcal{L},\mathcal{S}) \textbf{ and} \\ (\mathcal{L},\mathcal{S})\downarrow\end{bmatrix}$ **implies** $\left[(\mathcal{G},\mathcal{S}) \Longrightarrow (\mathcal{G}^{\dagger},\mathcal{S}^{\dagger})\downarrow \textbf{ for some } \mathcal{G}^{\dagger},\mathcal{S}^{\dagger}\right]$

5. $\begin{bmatrix}(\mathcal{G},\mathcal{S}) \bowtie (\mathcal{L},\mathcal{S}) \textbf{ and} \\ (\mathcal{L},\mathcal{S}) \xrightarrow{\tau} (\mathcal{L}',\mathcal{S}')\end{bmatrix}$ **implies** $(\mathcal{G},\mathcal{S}) \bowtie (\mathcal{L}',\mathcal{S}')$

6. $\begin{bmatrix}(\mathcal{G},\mathcal{S}) \bowtie (\mathcal{L},\mathcal{S}) \textbf{ and} \\ (\mathcal{L},\mathcal{S}) \xrightarrow{\dot{\gamma}} (\mathcal{L}',\mathcal{S}') \textbf{ and } \dot{\gamma} \neq \tau\end{bmatrix}$ **implies** $\left[\begin{matrix}\left[\begin{matrix}(\mathcal{G}^{¶},\mathcal{S}^{¶}) \bowtie (\mathcal{L}',\mathcal{S}') \textbf{ and} \\ (\mathcal{G},\mathcal{S}) \Longrightarrow (\mathcal{G}^{\dagger},\mathcal{S}^{\dagger}) \xrightarrow{\dot{\gamma}} (\mathcal{G}^{\ddagger},\mathcal{S}^{\ddagger}) \Longrightarrow (\mathcal{G}^{¶},\mathcal{S}^{¶})\end{matrix}\right] \\ \textbf{for some } \mathcal{G}^{\dagger},\mathcal{G}^{\ddagger},\mathcal{G}^{¶},\mathcal{S}^{\dagger},\mathcal{S}^{\ddagger},\mathcal{S}^{¶}\end{matrix}\right]$

Proof. See page 128. □

Corollary 1. $(\mathcal{G},\mathcal{S}) \bowtie (\mathcal{L},\mathcal{S})$ **implies** $(\mathcal{G},\mathcal{S}) \approx (\mathcal{L},\mathcal{S})$

Theorem 3. $\left[\checkmark_R(\mathcal{G}) \textbf{ and } \mathcal{S} \in [\![\phi(\mathcal{G},\texttt{true})]\!]\right]$ **implies** $(\mathcal{G},\mathcal{S}) \approx (\mathcal{G} \restriction R,\mathcal{S})$

Proof. See page 131. □

Part II

Proofs

A Data

B Conditions

Proof (of Lem. 1). By Defn. 5:

- **Base:** $\psi = e$, for some e.
  Recall $q.y \notin \mathsf{rx}(\psi)$. Then, $q.y \notin \mathsf{rx}(e)$. Then, by Prop. 1, $e[d/q.y] = e$. Then, $\psi[d/q.y] = \psi$.
- **Step:** $\psi = \neg\hat{\psi}$, for some $\hat{\psi}$.
  Recall $q.y \notin \mathsf{rx}(\psi)$. Then, $q.y \notin \mathsf{rx}(\neg\hat{\psi})$. Then, by Defn. 8, $q.y \notin \mathsf{rx}(\hat{\psi})$. Then, by induction, $\hat{\psi}[d/q.y] = \hat{\psi}$. Then, $\neg(\hat{\psi}[d/q.y]) = \neg\hat{\psi}$. Then, by Defn. 7, $(\neg\hat{\psi})[d/q.y] = \neg\hat{\psi}$. Then, $\psi[d/q.y] = \psi$.
- **Step:** $\psi = \psi_1 \wedge \psi_2$, for some ψ_1, ψ_2.
  Recall $q.y \notin \mathsf{rx}(\psi)$. Then, $q.y \notin \mathsf{rx}(\psi_1 \wedge \psi_2)$. Then, by Defn. 8, $q.y \notin \mathsf{rx}(\psi_1) \cup \mathsf{rx}(\psi_2)$. Then, $q.y \notin \mathsf{rx}(\psi_1)$ and $q.y \notin \mathsf{rx}(\psi_2)$. Then, by induction, $\psi_1[d/q.y] = \psi_1$ and $\psi_2[d/q.y] = \psi_2$. Then, $\psi_1[d/q.y] \wedge \psi_2[d/q.y] = \psi_1 \wedge \psi_2$. Then, by Defn. 7, $(\psi_1 \wedge \psi_2)[d/q.y] = \psi_1 \wedge \psi_2$. Then, $\psi[d/q.y] = \psi$.
- **Step:** $\psi = \forall\hat{\psi}$, for some $\hat{\psi}$.
  By Defn. 7, $(\forall\hat{\psi})[d/q.y] = \forall\hat{\psi}$. Then, $\psi[d/q.y] = \psi$. □

Proof (of Lem. 2). By Defn. 5:

- **Base:** $\psi = e$, for some e.
  Recall $q_1.y_1 \neq q_2.y_2$ and $q_1.y_1 \notin \mathsf{rx}(d_2)$ and $q_2.y_2 \notin \mathsf{rx}(d_1)$. Then, by Prop. 2, $e[d_1/q_1.y_1][d_2/q_2.y_2] = e[d_2/q_2.y_2][d_1/q_1.y_1]$. Then, $\psi[d_1/q_1.y_1][d_2/q_2.y_2] = \psi[d_2/q_2.y_2][d_1/q_1.y_1]$.
- **Step:** $\psi = \neg\hat{\psi}$, for some $\neg\psi$.
  Recall $q_1.y_1 \neq q_2.y_2$ and $q_1.y_1 \notin \mathsf{rx}(d_2)$ and $q_2.y_2 \notin \mathsf{rx}(d_1)$. Then, by induction, $\hat{\psi}[d_1/q_1.y_1][d_2/q_2.y_2] = \hat{\psi}[d_2/q_2.y_2][d_1/q_1.y_1]$. Then, $\neg(\hat{\psi}[d_1/q_1.y_1][d_2/q_2.y_2]) = \neg(\hat{\psi}[d_2/q_2.y_2][d_1/q_1.y_1])$. Then, by Defn. 7, $(\neg(\hat{\psi}[d_1/q_1.y_1]))[d_2/q_2.y_2] = (\neg(\hat{\psi}[d_2/q_2.y_2]))[d_1/q_1.y_1]$. Then, by Defn. 7, $(\neg\hat{\psi})[d_1/q_1.y_1][d_2/q_2.y_2] = (\neg\hat{\psi})[d_2/q_2.y_2][d_1/q_1.y_1]$. Then, $\psi[d_1/q_1.y_1][d_2/q_2.y_2] = \psi[d_2/q_2.y_2][d_1/q_1.y_1]$.
- **Step:** $\psi = \psi_1 \wedge \psi_2$, for some ψ_1, ψ_2.
  Recall $q_1.y_1 \neq q_2.y_2$ and $q_1.y_1 \notin \mathsf{rx}(d_2)$ and $q_2.y_2 \notin \mathsf{rx}(d_1)$. Then, by induction, $\psi_1[d_1/q_1.y_1][d_2/q_2.y_2] = \psi_1[d_2/q_2.y_2][d_1/q_1.y_1]$ and $\psi_2[d_1/q_1.y_1][d_2/q_2.y_2] = \psi_2[d_2/q_2.y_2][d_1/q_1.y_1]$. Then, $\psi_1[d_1/q_1.y_1][d_2/q_2.y_2] \wedge \psi_2[d_1/q_1.y_1][d_2/q_2.y_2] = \psi_1[d_2/q_2.y_2][d_1/q_1.y_1] \wedge \psi_2[d_2/q_2.y_2][d_1/q_1.y_1]$. Then, by Defn. 7, $(\psi_1[d_1/q_1.y_1] \wedge \psi_2[d_1/q_1.y_1])[d_2/q_2.y_2] = (\psi_1[d_2/q_2.y_2] \wedge \psi_2[d_2/q_2.y_2])[d_1/q_1.y_1]$. Then, by Defn. 7, $(\psi_1 \wedge \psi_2)[d_1/q_1.y_1][d_2/q_2.y_2] = (\psi_1 \wedge \psi_2)[d_2/q_2.y_2][d_1/q_1.y_1]$. Then, $\psi[d_1/q_1.y_1][d_2/q_2.y_2] = \psi[d_2/q_2.y_2][d_1/q_1.y_1]$.
- **Step:** $\psi = \forall\hat{\psi}$, for some $\hat{\psi}$.
  Recall $\forall\hat{\psi} = \forall\hat{\psi}$. Then, by Defn. 7, $(\forall\hat{\psi})[d_2/q_2.y_2] = (\forall\hat{\psi})[d_1/q_1.y_1]$. Then, by Defn. 7, $(\forall\hat{\psi})[d_1/q_1.y_1][d_2/q_2.y_2] = (\forall\hat{\psi})[d_2/q_2.y_2][d_1/q_1.y_1]$. Then, $\psi[d_1/q_1.y_1][d_2/q_2.y_2] = \psi[d_2/q_2.y_2][d_1/q_1.y_1]$. □

Proof (of Lem. 3).

1. By Defn. 3, $\mathsf{rx}(\texttt{false}) = \emptyset = \mathsf{rx}(\texttt{true})$. Then, by Defn. 9, $\mathsf{read}(\texttt{false}) = \emptyset = \mathsf{read}(\texttt{true})$.
2. By Defn. 9, $\mathsf{read}(\neg\psi) = \mathsf{rx}(\neg\psi)$. Then, by Defn. 8, $\mathsf{read}(\neg\psi) = \mathsf{rx}(\psi)$. Then, by Defn. 9, $\mathsf{read}(\neg\psi) = \mathsf{read}(\psi)$. □
3. Similar to 2. □
4. Recall $\mathsf{read}(\psi_1 \Rightarrow \psi_2) = \mathsf{read}(\psi_1 \Rightarrow \psi_2)$. Then, by Defn. 6, $\mathsf{read}(\psi_1 \Rightarrow \psi_2) = \mathsf{read}(\neg(\psi_1 \wedge \neg\psi_2))$. Then, by 1, $\mathsf{read}(\psi_1 \Rightarrow \psi_2) = \mathsf{read}(\psi_1 \wedge \neg\psi_2)$. Then, by 2, $\mathsf{read}(\psi_1 \Rightarrow \psi_2) = \mathsf{read}(\psi_1) \cup \mathsf{read}(\neg\psi_2)$. Then, by 1, $\mathsf{read}(\psi_1 \Rightarrow \psi_2) = \mathsf{read}(\psi_1) \cup \mathsf{read}(\psi_2)$. □

5. By Defn. 9, $\mathsf{write}(\mathtt{false}) = \emptyset = \mathsf{write}(\mathtt{true})$.
6. Recall $\emptyset = \emptyset$. Then, by Defn. 9, $\mathsf{write}(\neg\psi) = \mathsf{write}(\psi)$. □
7. Similar to 6. □
8. Recall $\emptyset = \emptyset \cup \emptyset$. Then, by Defn. 9, $\mathsf{write}(\neg(\psi_1 \wedge \neg\psi_2)) = \mathsf{write}(\psi_1) \cup \mathsf{write}(\psi_2)$. Then, by Defn. 6, $\mathsf{write}(\psi_1 \Rightarrow \psi_2) = \mathsf{write}(\psi_1) \cup \mathsf{write}(\psi_2)$. □

Proof (of Lem. 4).

1. Recall $\{r \mid r.x \in \emptyset\} = \emptyset = \{r \mid r.x \in \emptyset\}$. Then, by Defn. 3, $\{r \mid r.x \in \mathsf{rx}(\mathtt{false})\} = \emptyset = \{r \mid r.x \in \mathsf{rx}(\mathtt{true})\}$. Then, by Defn. 10, $\mathsf{subj}(\mathtt{false}) = \emptyset = \mathsf{subj}(\mathtt{true})$. □
2. By Defn. 10, $\mathsf{subj}(\neg\psi) = \{r \mid r.x \in \mathsf{rx}(\neg\psi)\}$. Then, by Defn. 8, $\mathsf{subj}(\neg\psi) = \{r \mid r.x \in \mathsf{rx}(\psi)\}$. Then, by Defn. 10, $\mathsf{subj}(\neg\psi) = \mathsf{subj}(\psi)$. □
3. By Defn. 10, $\mathsf{subj}(\psi_1 \wedge \psi_2) = \{r \mid r.x \in \mathsf{rx}(\psi_1 \wedge \psi_2)\}$. Then, by Defn. 8, $\mathsf{subj}(\psi_1 \wedge \psi_2) = \{r \mid r.x \in \mathsf{rx}(\psi_1) \cup \mathsf{rx}(\psi_2)\}$. Then, $\mathsf{subj}(\psi_1 \wedge \psi_2) = \{r \mid r.x \in \mathsf{rx}(\psi_1)\} \cup \{r \mid r.x \in \mathsf{rx}(\psi_2)\}$. Then, by Defn. 10, $\mathsf{subj}(\psi_1 \wedge \psi_2) = \mathsf{subj}(\psi_1) \cup \mathsf{subj}(\psi_2)$. □
4. Recall $\mathsf{subj}(\psi_1 \Rightarrow \psi_2) = \mathsf{subj}(\psi_1 \Rightarrow \psi_2)$. Then, by Defn. 6, $\mathsf{subj}(\psi_1 \Rightarrow \psi_2) = \mathsf{subj}(\neg(\psi_1 \wedge \neg\psi_2))$. Then, by 1, $\mathsf{subj}(\psi_1 \Rightarrow \psi_2) = \mathsf{subj}(\psi_1 \wedge \neg\psi_2)$. Then, by 2, $\mathsf{subj}(\psi_1 \Rightarrow \psi_2) = \mathsf{subj}(\psi_1) \cup \mathsf{subj}(\neg\psi_2)$. Then, by 1, $\mathsf{subj}(\psi_1 \Rightarrow \psi_2) = \mathsf{subj}(\psi_1) \cup \mathsf{subj}(\psi_2)$. □

Proof (of Lem. 5).

1. Recall $\mathsf{rx}(\psi) \subseteq \mathsf{rx}(\psi)$. Then, $\mathsf{rx}(\psi) \subseteq \{r.x \mid r \in \{\hat{r} \mid \hat{r}.\hat{x} \in \mathsf{rx}(\psi)\}\}$. Then, by Defn. 9, $\mathsf{read}(\psi) \subseteq \{r.x \mid r \in \{\hat{r} \mid \hat{r}.\hat{x} \in \mathsf{rx}(\psi)\}\}$. Then, by Defn. 10, $\mathsf{read}(\psi) \subseteq \{r.x \mid r \in \mathsf{subj}(\psi)\}$. □
2. Recall $\emptyset \subseteq \{r.x \mid r \in \mathsf{subj}(\psi)\}$. Then, by Defn. 9, $\mathsf{write}(\psi) \subseteq \{r.x \mid r \in \mathsf{subj}(\psi)\}$. □

Proof (of Lem. 6). By Defn. 5:

- **Base:** $\xi = \mathtt{true}$.
  Recall $(\xi \restriction r) \restriction r = (\xi \restriction r) \restriction r$. Then, $(\xi \restriction r) \restriction r = (\mathtt{true} \restriction r) \restriction r$. Then, by Defn. 11, $(\xi \restriction r) \restriction r = \mathtt{true} \restriction r$. Then, $(\xi \restriction r) \restriction r = \xi \restriction r$.
- **Base:** $\xi = \bigwedge\{e_{\hat{r}}\}_{\hat{r}\in R}$, for some $\{e_{\hat{r}}\}_{\hat{r}\in R}$.
  By Defn. 11:
  - **Case:** $\bigwedge\{e_{\hat{r}}\}_{\hat{r}\in R} \restriction r = e_r$.
    Recall $r \in \{r\}$. Then, by Defn. 11, $e_r \restriction r = e_r$. Then, $(\bigwedge\{e_{\hat{r}}\}_{\hat{r}\in R} \restriction r) \restriction r = \bigwedge\{e_{\hat{r}}\}_{\hat{r}\in R} \restriction r$. Then, $(\xi \restriction r) \restriction r = \xi \restriction r$.
  - **Case:** $\bigwedge\{e_{\hat{r}}\}_{\hat{r}\in R} \restriction r = \mathtt{true}$.
    By Defn. 11, $\mathtt{true} \restriction r = \mathtt{true}$. Then, $(\bigwedge\{e_{\hat{r}}\}_{\hat{r}\in R} \restriction r) \restriction r = \bigwedge\{e_{\hat{r}}\}_{\hat{r}\in R} \restriction r$. Then, $(\xi \restriction r) \restriction r = \xi \restriction r$.
- **Base:** $\xi = \bigwedge\{\neg e_{\hat{r}}\}_{\hat{r}\in R}$, for some $\{e_{\hat{r}}\}_{\hat{r}\in R}$. Similar to case "$\xi = \bigwedge\{e_{\hat{r}}\}_{\hat{r}\in R}$, for some $\{e_{\hat{r}}\}_{\hat{r}\in R}$". □

Proof (of Lem. 7). By Defn. 5:

- **Base:** $\xi = \mathtt{true}$.
  Recall $\emptyset \subseteq \mathsf{rx}(\xi)$. Then, by Defn. 3, $\mathsf{rx}(\mathtt{true}) \subseteq \mathsf{rx}(\xi)$. Then, by Defn. 11, $\mathsf{rx}(\mathtt{true} \restriction r) \subseteq \mathsf{rx}(\xi)$. Then, $\mathsf{rx}(\xi \restriction r) \subseteq \mathsf{rx}(\xi)$.
- **Base:** $\xi = \bigwedge\{e_{\hat{r}}\}_{\hat{r}\in R}$, for some $\{e_{\hat{r}}\}_{\hat{r}\in R}$.
  By Defn. 11:
  - **Case:** $\bigwedge\{e_{\hat{r}}\}_{\hat{r}\in R} \restriction r = e_r$ and $r \in R$.
    * Recall $r \in R$. Then, $R = (R \setminus \{r\}) \cup \{r\}$.
    * Recall $\mathsf{rx}(e_r) \subseteq \bigcup\{\mathsf{rx}(e_{\hat{r}})\}_{\hat{r}\in R\setminus\{r\}} \cup \mathsf{rx}(e_r)$. Then, $\mathsf{rx}(e_r) \subseteq \bigcup\{\mathsf{rx}(e_{\hat{r}})\}_{\hat{r}\in(R\setminus\{r\})\cup\{r\}}$. Then, $\mathsf{rx}(e_r) \subseteq \bigcup\{\mathsf{rx}(e_{\hat{r}})\}_{\hat{r}\in R}$. Then, by Defn. 8, $\mathsf{rx}(e_r) \subseteq \mathsf{rx}(\bigwedge\{e_{\hat{r}}\}_{\hat{r}\in R})$. Then, $\mathsf{rx}(\bigwedge\{e_{\hat{r}}\}_{\hat{r}\in R} \restriction r) \subseteq \mathsf{rx}(\bigwedge\{e_{\hat{r}}\}_{\hat{r}\in R})$. Then, $\mathsf{rx}(\xi \restriction r) \subseteq \mathsf{rx}(\xi)$.

- **Case:** $\bigwedge\{e_{\hat{r}}\}_{\hat{r}\in R} \upharpoonright r = \mathtt{true}$.
  Recall $\emptyset \subseteq \mathsf{rx}(\xi)$. Then, by Defn. 3, $\mathsf{rx}(\mathtt{true}) \subseteq \mathsf{rx}(\xi)$. Then, $\mathsf{rx}(\bigwedge\{e_{\hat{r}}\}_{\hat{r}\in R} \upharpoonright r) \subseteq \mathsf{rx}(\xi)$. Then, $\mathsf{rx}(\xi \upharpoonright r) \subseteq \mathsf{rx}(\xi)$.

– **Base:** $\xi = \bigwedge\{\neg e_{\hat{r}}\}_{\hat{r}\in R}$, for some $\{e_{\hat{r}}\}_{\hat{r}\in R}$. Similar to case "$\xi = \bigwedge\{e_{\hat{r}}\}_{\hat{r}\in R}$, for some $\{e_{\hat{r}}\}_{\hat{r}\in R}$". □

Proof (of Lem. 8).

1. By Lem. 7, $\mathsf{rx}(\xi \upharpoonright r) \subseteq \mathsf{rx}(\xi)$. Then, by Defn. 9, $\mathsf{read}(\xi \upharpoonright r) \subseteq \mathsf{read}(\xi)$. □
2. Recall $\emptyset \subseteq \emptyset$. Then, by Defn. 9, $\mathsf{write}(\xi \upharpoonright r) \subseteq \mathsf{write}(\xi)$. □

Proof (of Lem. 9).

1. By Defn. 5:
   – **Base:** $\xi = \mathtt{true}$.
   Recall $r \in \mathsf{subj}(\xi)$. Then, $r \in \mathsf{subj}(\mathtt{true})$. Then, by Lem. 4, $r \in \emptyset$. Then, **false**.
   – **Base:** $\xi = \bigwedge\{e_{\hat{r}}\}_{\hat{r}\in R}$, for some $\{e_{\hat{r}}\}_{\hat{r}\in R}$.
   - Recall $\xi = \bigwedge\{e_{\hat{r}}\}_{\hat{r}\in R}$. Then, by Rem. 1, $\{\hat{\hat{r}} \mid \hat{\hat{r}}.\hat{\hat{x}} \in \mathsf{rx}(e_{\hat{r}})\} = \{\hat{r}\}$ for every $\hat{r} \in R$. Then, by Defn. 10, $\mathsf{subj}(e_{\hat{r}}) = \{\hat{r}\}$ for every $\hat{r} \in R$.
   - Recall $r \in \mathsf{subj}(\xi)$. Then, $r \in \mathsf{subj}(\bigwedge\{e_{\hat{r}}\}_{\hat{r}\in R})$. Then, by Lem. 4, $r \in \bigcup\{\mathsf{subj}(e_{\hat{r}})\}_{\hat{r}\in R}$. Then, $r \in \bigcup\{\{\hat{r}\}\}_{\hat{r}\in R}$. Then, $r \in \{\hat{r} \mid \hat{r} \in R\}$. Then, $r \in R$.
   - Recall $r \in R$. Then, by Defn. 11, $\bigwedge\{e_{\hat{r}}\}_{\hat{r}\in R} \upharpoonright r = e_r$.
   - Recall $r \in R$. Then, $\mathsf{subj}(e_r) = \{r\}$.
   - Recall $\mathsf{subj}(\xi \upharpoonright r) = \mathsf{subj}(\xi \upharpoonright r)$. Then, $\mathsf{subj}(\xi \upharpoonright r) = \mathsf{subj}(\bigwedge\{e_{\hat{r}}\}_{\hat{r}\in R} \upharpoonright r)$. Then, $\mathsf{subj}(\xi \upharpoonright r) = \mathsf{subj}(e_r)$. Then, $\mathsf{subj}(\xi \upharpoonright r) = \{r\}$.
   – **Base:** $\xi = \bigwedge\{\neg e_{\hat{r}}\}_{\hat{r}\in R}$, for some $\{e_{\hat{r}}\}_{\hat{r}\in R}$. Similar to case "$\xi = \bigwedge\{e_{\hat{r}}\}_{\hat{r}\in R}$, for some $\{e_{\hat{r}}\}_{\hat{r}\in R}$". □
2. By Defn. 5:
   – **Base:** $\xi = \mathtt{true}$.
   By Lem. 4, $\mathsf{subj}(\mathtt{true}) = \emptyset$. Then, by Defn. 11, $\mathsf{subj}(\mathtt{true} \upharpoonright r) = \emptyset$. Then, $\mathsf{subj}(\xi \upharpoonright r) = \emptyset$.
   – **Base:** $\xi = \bigwedge\{e_{\hat{r}}\}_{\hat{r}\in R}$, for some $\{e_{\hat{r}}\}_{\hat{r}\in R}$.
   - Recall $\xi = \bigwedge\{e_{\hat{r}}\}_{\hat{r}\in R}$. Then, by Rem. 1, $\{\hat{\hat{r}} \mid \hat{\hat{r}}.\hat{\hat{x}} \in \mathsf{rx}(e_{\hat{r}})\} = \{\hat{r}\}$ for every $\hat{r} \in R$. Then, by Defn. 10, $\mathsf{subj}(e_{\hat{r}}) = \{\hat{r}\}$ for every $\hat{r} \in R$.
   - Recall $r \notin \mathsf{subj}(\xi)$. Then, $r \notin \mathsf{subj}(\bigwedge\{e_{\hat{r}}\}_{\hat{r}\in R})$. Then, by Lem. 4, $r \notin \bigcup\{\mathsf{subj}(e_{\hat{r}})\}_{\hat{r}\in R}$. Then, $r \notin \bigcup\{\{\hat{r}\}\}_{\hat{r}\in R}$. Then, $r \notin \{\hat{r} \mid \hat{r} \in R\}$. Then, $r \notin R$. Then, by Defn. 11, $\bigwedge\{e_{\hat{r}}\}_{\hat{r}\in R} \upharpoonright r = \mathtt{true}$.
   - By Lem. 4, $\mathsf{subj}(\mathtt{true}) = \emptyset$. Then, by Defn. 11, $\mathsf{subj}(\bigwedge\{e_{\hat{r}}\}_{\hat{r}\in R} \upharpoonright r) = \emptyset$. Then, $\mathsf{subj}(\xi \upharpoonright r) = \emptyset$.
   – **Base:** $\xi = \bigwedge\{\neg e_{\hat{r}}\}_{\hat{r}\in R}$, for some $\{e_{\hat{r}}\}_{\hat{r}\in R}$. Similar to case "$\xi = \bigwedge\{e_{\hat{r}}\}_{\hat{r}\in R}$, for some $\{e_{\hat{r}}\}_{\hat{r}\in R}$". □

Proof (of Lem. 10).

1. Recall:
   – **Case:** $r \in \mathsf{subj}(\xi)$.
   - Recall $r \in \mathsf{subj}(\xi)$. Then, by Lem. 9, $\mathsf{subj}(\xi \upharpoonright r) = \{r\}$.
   - Recall $r \in \mathsf{subj}(\xi)$. Then, $\{r\} \subseteq \mathsf{subj}(\xi)$. Then, $\mathsf{subj}(\xi \upharpoonright r) \subseteq \mathsf{subj}(\xi)$.
   – **Case:** $r \notin \mathsf{subj}(\xi)$.
   - Recall $r \notin \mathsf{subj}(\xi)$. Then, by Lem. 9, $\mathsf{subj}(\xi \upharpoonright r) = \emptyset$.
   - Recall $\emptyset \subseteq \mathsf{subj}(\xi)$. Then, $\mathsf{subj}(\xi \upharpoonright r) \subseteq \mathsf{subj}(\xi)$. □
2. Recall:
   – **Case:** $r \in \mathsf{subj}(\xi)$.
   Recall $r \in \mathsf{subj}(\xi)$. Then, by Lem. 9, $\mathsf{subj}(\xi \upharpoonright r) = \{r\}$. Then, $\mathsf{subj}(\xi \upharpoonright r) \subseteq \{r\}$.
   – **Case:** $r \notin \mathsf{subj}(\xi)$.

- Recall $r \notin \mathsf{subj}(\xi)$. Then, by Lem. 9, $\mathsf{subj}(\xi \upharpoonright r) = \emptyset$.
- Recall $\emptyset \subseteq \{r\}$. Then, $\mathsf{subj}(\xi \upharpoonright r) \subseteq \{r\}$. □

Proof (of Lem. 11).

1. By Defn. 12, $(\xi \upharpoonright r)^+ = \xi \upharpoonright r = \xi^+ \upharpoonright r$. Then, $(\xi \upharpoonright r)^+ = \xi^+ \upharpoonright r$. □
2. By Defn. 5:
   - **Case:** $\xi = \texttt{true}$.
     Recall $(\xi \upharpoonright r)^- = (\xi \upharpoonright r)^-$. Then, $(\xi \upharpoonright r)^- = (\texttt{true} \upharpoonright r)^-$. Then, by Defn. 11, $(\xi \upharpoonright r)^- = \texttt{true}^-$. Then, by Defn. 12, $(\xi \upharpoonright r)^- = \texttt{true}$. Then, by Defn. 11, $(\xi \upharpoonright r)^- = \texttt{true} \upharpoonright r$. Then, by Defn. 12, $(\xi \upharpoonright r)^- = \texttt{true}^- \upharpoonright r$. Then, $(\xi \upharpoonright r)^- = \xi^- \upharpoonright r$.
   - **Case:** $\xi = \bigwedge\{e_{\hat{r}}\}_{\hat{r}\in R}$, for some $\{e_{\hat{r}}\}_{\hat{r}\in R}$.
     Recall:
     - **Case:** $r \in R$.
       * Recall $r \in R$. Then, by Defn. 11, $\bigwedge\{e_{\hat{r}}\}_{\hat{r}\in R} \upharpoonright r = e_r$ and $\bigwedge\{\neg e_{\hat{r}}\}_{\hat{r}\in R} \upharpoonright r = \neg e_r$.
       * Recall $(\xi \upharpoonright r)^- = (\xi \upharpoonright r)^-$. Then, $(\xi \upharpoonright r)^- = (\bigwedge\{e_{\hat{r}}\}_{\hat{r}\in R} \upharpoonright r)^-$. Then, $(\xi \upharpoonright r)^- = e_r^-$. Then, by Defn. 12, $(\xi \upharpoonright r)^- = \neg e_r$. Then, $(\xi \upharpoonright r)^- = \bigwedge\{\neg e_{\hat{r}}\}_{\hat{r}\in R} \upharpoonright r$. Then, by Defn. 12, $(\xi \upharpoonright r)^- = \bigwedge\{e_{\hat{r}}\}^-_{\hat{r}\in R} \upharpoonright r$. Then, $(\xi \upharpoonright r)^- = \xi^- \upharpoonright r$.
     - **Case:** $\bigwedge\{e_{\hat{r}}\}_{\hat{r}\in R} \upharpoonright r = \texttt{true}$.
       * Recall $r \notin R$. Then, by Defn. 11, $\bigwedge\{e_{\hat{r}}\}_{\hat{r}\in R} \upharpoonright r = \texttt{true}$ and $\bigwedge\{\neg e_{\hat{r}}\}_{\hat{r}\in R} \upharpoonright r = \texttt{true}$.
       * Recall $(\xi \upharpoonright r)^- = (\xi \upharpoonright r)^-$. Then, $(\xi \upharpoonright r)^- = (\bigwedge\{e_{\hat{r}}\}_{\hat{r}\in R} \upharpoonright r)^-$. Then, $(\xi \upharpoonright r)^- = \texttt{true}$. Then, $(\xi \upharpoonright r)^- = \bigwedge\{\neg e_{\hat{r}}\}_{\hat{r}\in R} \upharpoonright r$. Then, by Defn. 12, $(\xi \upharpoonright r)^- = \bigwedge\{e_{\hat{r}}\}^-_{\hat{r}\in R} \upharpoonright r$. Then, $(\xi \upharpoonright r)^- = \xi^- \upharpoonright r$.
   - **Case:** $\xi = \bigwedge\{\neg e_{\hat{r}}\}_{\hat{r}\in R}$, for some $\{e_{\hat{r}}\}_{\hat{r}\in R}$. Similar to case "$\xi = \bigwedge\{e_{\hat{r}}\}_{\hat{r}\in R}$, for some $\{e_{\hat{r}}\}_{\hat{r}\in R}$". □

Proof (of Lem. 12).

1. By Defn. 12, $\xi^+ = \xi$. Then, $\mathsf{read}(\xi^+) = \mathsf{read}(\xi)$. □
2. By Defn. 5:
   - **Base:** $\xi = \texttt{true}$.
     Recall $\mathsf{read}(\xi^-) = \mathsf{read}(\xi^-)$. Then, $\mathsf{read}(\xi^-) = \mathsf{read}(\texttt{true}^-)$. Then, by Defn. 12, $\mathsf{read}(\xi^-) = \mathsf{read}(\texttt{true})$. Then, $\mathsf{read}(\xi^-) = \mathsf{read}(\xi)$.
   - **Base:** $\xi = \bigwedge\{e_r\}_{r\in R}$, for some $\{e_r\}_{r\in R}$.
     Recall $\mathsf{read}(\xi^-) = \mathsf{read}(\xi^-)$. Then, $\mathsf{read}(\xi^-) = \mathsf{read}(\bigwedge\{e_r\}^-_{r\in R})$. Then, by Defn. 12, $\mathsf{read}(\xi^-) = \mathsf{read}(\bigwedge\{\neg e_r\}_{r\in R})$. Then, by Lem. 3, $\mathsf{read}(\xi^-) = \bigcup\{\mathsf{read}(\neg e_r)\}_{r\in R}$. Then, by Lem. 3, $\mathsf{read}(\xi^-) = \bigcup\{\mathsf{read}(e_r)\}_{r\in R}$. Then, by Lem. 3, $\mathsf{read}(\xi^-) = \mathsf{read}(\bigwedge\{e_r\}_{r\in R})$. Then, $\mathsf{read}(\xi^-) = \mathsf{read}(\xi)$.
   - **Base:** $\xi = \bigwedge\{\neg e_r\}_{r\in R}$, for some $\{e_r\}_{r\in R}$. Similar to case "$\xi = \bigwedge\{e_r\}_{r\in R}$, for some $\{e_r\}_{r\in R}$". □
3. By Defn. 5:
   - **Base:** $\xi = \texttt{true}$.
     Recall $\mathsf{read}(\xi^\equiv) = \mathsf{read}(\xi^\equiv)$. Then, $\mathsf{read}(\xi^\equiv) = \mathsf{read}(\texttt{true}^\equiv)$. Then, by Defn. 12, $\mathsf{read}(\xi^\equiv) = \mathsf{read}(\texttt{true})$. Then, $\mathsf{read}(\xi^\equiv) = \mathsf{read}(\xi)$.
   - **Base:** $\xi = \bigwedge\{e_r\}_{r\in R}$, for some $\{e_r\}_{r\in R}$.
     Recall $\mathsf{read}(\xi^\equiv) = \mathsf{read}(\xi^\equiv)$. Then, $\mathsf{read}(\xi^\equiv) = \mathsf{read}(\bigwedge\{e_r\}^\equiv_{r\in R})$. Then, by Defn. 12, $\mathsf{read}(\xi^\equiv) = \mathsf{read}(\bigwedge\{e_{r_1} \Rightarrow e_{r_2}\}_{r_1,r_2\in R})$. Then, by Lem. 3, $\mathsf{read}(\xi^\equiv) = \bigcup\{\mathsf{read}(e_{r_1} \Rightarrow e_{r_2})\}_{r_1,r_2\in R}$. Then, by Lem. 3, $\mathsf{read}(\xi^\equiv) = \bigcup\{\mathsf{read}(e_{r_1}) \cup \mathsf{read}(e_{r_2})\}_{r_1,r_2\in R}$. Then, $\mathsf{read}(\xi^\equiv) = \bigcup\{\mathsf{read}(e_r)\}_{r\in R}$. Then, by Lem. 3, $\mathsf{read}(\xi^\equiv) = \mathsf{read}(\bigwedge\{e_r\}_{r\in R})$. Then, $\mathsf{read}(\xi^\equiv) = \mathsf{read}(\xi)$.
   - **Base:** $\xi = \bigwedge\{\neg e_r\}_{r\in R}$, for some $\{e_r\}_{r\in R}$. Similar to case "$\xi = \bigwedge\{e_r\}_{r\in R}$, for some $\{e_r\}_{r\in R}$". □
4. By Defn. 9, $\mathsf{write}(\xi^+) = \emptyset = \mathsf{write}(\xi)$. Then, $\mathsf{write}(\xi^+) = \mathsf{write}(\xi)$. □
5. Similar to 4. □

6. Similar to 4. □

Proof (of Lem. 13).

1. By Defn. 12, $\xi^+ = \xi$. Then, $\mathsf{subj}(\xi^+) = \mathsf{subj}(\xi)$. □
2. By Defn. 5:
   - **Base:** $\xi = \mathtt{true}$.
     Recall $\mathsf{subj}(\xi^-) = \mathsf{subj}(\xi^-)$. Then, $\mathsf{subj}(\xi^-) = \mathsf{subj}(\mathtt{true}^-)$. Then, by Defn. 12, $\mathsf{subj}(\xi^-) = \mathsf{subj}(\mathtt{true})$. Then, $\mathsf{subj}(\xi^-) = \mathsf{subj}(\xi)$.
   - **Base:** $\xi = \bigwedge\{e_r\}_{r\in R}$, for some $\{e_r\}_{r\in R}$.
     Recall $\mathsf{subj}(\xi^-) = \mathsf{subj}(\xi^-)$. Then, $\mathsf{subj}(\xi^-) = \mathsf{subj}(\bigwedge\{e_r\}^-_{r\in R})$. Then, by Defn. 12, $\mathsf{subj}(\xi^-) = \mathsf{subj}(\bigwedge\{\neg e_r\}_{r\in R})$. Then, by Lem. 4, $\mathsf{subj}(\xi^-) = \bigcup\{\mathsf{subj}(\neg e_r)\}_{r\in R}$. Then, by Lem. 4, $\mathsf{subj}(\xi^-) = \bigcup\{\mathsf{subj}(e_r)\}_{r\in R}$. Then, by Lem. 4, $\mathsf{subj}(\xi^-) = \mathsf{subj}(\bigwedge\{e_r\}_{r\in R})$. Then, $\mathsf{subj}(\xi^-) = \mathsf{subj}(\xi)$.
   - **Base:** $\xi = \bigwedge\{\neg e_r\}_{r\in R}$, for some $\{e_r\}_{r\in R}$. Similar to case "$\xi = \bigwedge\{e_r\}_{r\in R}$, for some $\{e_r\}_{r\in R}$". □
3. By Defn. 5:
   - **Base:** $\xi = \mathtt{true}$.
     Recall $\mathsf{subj}(\xi^{\equiv}) = \mathsf{subj}(\xi^{\equiv})$. Then, $\mathsf{subj}(\xi^{\equiv}) = \mathsf{subj}(\mathtt{true}^{\equiv})$. Then, by Defn. 12, $\mathsf{subj}(\xi^{\equiv}) = \mathsf{subj}(\mathtt{true})$. Then, $\mathsf{subj}(\xi^{\equiv}) = \mathsf{subj}(\xi)$.
   - **Base:** $\xi = \bigwedge\{e_r\}_{r\in R}$, for some $\{e_r\}_{r\in R}$.
     Recall $\mathsf{subj}(\xi^{\equiv}) = \mathsf{subj}(\xi^{\equiv})$. Then, $\mathsf{subj}(\xi^{\equiv}) = \mathsf{subj}(\bigwedge\{e_r\}^{\equiv}_{r\in R})$. Then, by Defn. 12, $\mathsf{subj}(\xi^{\equiv}) = \mathsf{subj}(\bigwedge\{e_{r_1} \Rightarrow e_{r_2}\}_{r_1,r_2\in R})$. Then, by Lem. 4, $\mathsf{subj}(\xi^{\equiv}) = \bigcup\{\mathsf{subj}(e_{r_1} \Rightarrow e_{r_2})\}_{r_1,r_2\in R}$. Then, by Lem. 4, $\mathsf{subj}(\xi^{\equiv}) = \bigcup\{\mathsf{subj}(e_{r_1}) \cup \mathsf{subj}(e_{r_2})\}_{r_1,r_2\in R}$. Then, $\mathsf{subj}(\xi^{\equiv}) = \bigcup\{\mathsf{subj}(e_r)\}_{r\in R}$. Then, by Lem. 4, $\mathsf{subj}(\xi^{\equiv}) = \mathsf{subj}(\bigwedge\{e_r\}_{r\in R})$. Then, $\mathsf{subj}(\xi^{\equiv}) = \mathsf{subj}(\xi)$.
   - **Base:** $\xi = \bigwedge\{\neg e_r\}_{r\in R}$, for some $\{e_r\}_{r\in R}$. Similar to case "$\xi = \bigwedge\{e_r\}_{r\in R}$, for some $\{e_r\}_{r\in R}$". □

Proof (of Lem. 14). By Defn. 5:

- **Base:** $\psi = e$, for some e.
  1. Recall $\mathcal{S} \in [\![\psi[d/q.y]]\!]$. Then, $\mathcal{S} \in [\![e[d/q.y]]\!]$. Then, by Defn. 13, $\mathcal{S} \in \{\hat{\mathcal{S}} \mid \mathsf{eval}_{\hat{\mathcal{S}}}(e[d/q.y]) = \mathtt{true}\}$. Then, $\mathsf{eval}_{\mathcal{S}}(e[d/q.y]) = \mathtt{true}$. Then, by Prop. 3, $\mathsf{eval}_{\mathcal{S}[\mathsf{eval}_{\mathcal{S}}(d)/q.y]}(e) = \mathtt{true}$. Then, $\mathcal{S}[\mathsf{eval}_{\mathcal{S}}(d)/q.y] \in \{\hat{\mathcal{S}} \mid \mathsf{eval}_{\hat{\mathcal{S}}}(e) = \mathtt{true}\}$. Then, by Defn. 13, $\mathcal{S}[\mathsf{eval}_{\mathcal{S}}(d)/q.y] \in [\![e]\!]$. Then, $\mathcal{S}[\mathsf{eval}_{\mathcal{S}}(d)/q.y] \in [\![\psi]\!]$.
  2. Similar to case 1.
- **Step:** $\psi = \neg\hat{\psi}$, for some $\hat{\psi}$.
  1.
     - By Defn. 4, $\mathsf{eval}_{\mathcal{S}[\mathsf{eval}_{\mathcal{S}}(d)/q.y]}(\mathtt{true}) = \mathtt{true}$. Then, $\mathcal{S}[\mathsf{eval}_{\mathcal{S}}(d)/q.y] \in \{\hat{\mathcal{S}} \mid \mathsf{eval}_{\hat{\mathcal{S}}}(\mathtt{true}) = \mathtt{true}\}$. Then, by Defn. 13, $\mathcal{S}[\mathsf{eval}_{\mathcal{S}}(d)/q.y] \in [\![\mathtt{true}]\!]$.
     - Recall $\mathcal{S} \in [\![\psi[d/q.y]]\!]$. Then, $\mathcal{S} \in [\![(\neg\hat{\psi})[d/q.y]]\!]$. Then, by Defn. 7, $\mathcal{S} \in [\![\neg(\hat{\psi}[d/q.y])]\!]$. Then, by Defn. 13, $\mathcal{S} \in [\![\mathtt{true}]\!] \setminus [\![\hat{\psi}[d/q.y]]\!]$. Then, $\mathcal{S} \notin [\![\hat{\psi}[d/q.y]]\!]$. Then, by induction, $\mathcal{S}[\mathsf{eval}_{\mathcal{S}}(d)/q.y] \notin [\![\hat{\psi}]\!]$.
     - Recall $\mathcal{S}[\mathsf{eval}_{\mathcal{S}}(d)/q.y] \in [\![\mathtt{true}]\!]$ and $\mathcal{S}[\mathsf{eval}_{\mathcal{S}}(d)/q.y] \notin [\![\hat{\psi}]\!]$. Then, $\mathcal{S}[\mathsf{eval}_{\mathcal{S}}(d)/q.y] \in [\![\mathtt{true}]\!] \setminus [\![\hat{\psi}]\!]$. Then, by Defn. 13, $\mathcal{S}[\mathsf{eval}_{\mathcal{S}}(d)/q.y] \in [\![\neg\hat{\psi}]\!]$. Then, $\mathcal{S}[\mathsf{eval}_{\mathcal{S}}(d)/q.y] \in [\![\psi]\!]$.
  2. Recall $\mathcal{S} \notin [\![\psi[d/q.y]]\!]$. Then, $\mathcal{S} \notin [\![(\neg\hat{\psi})[d/q.y]]\!]$. Then, by Defn. 7, $\mathcal{S} \notin [\![\neg(\hat{\psi}[d/q.y])]\!]$. Then, by Defn. 13, $\mathcal{S} \notin [\![\mathtt{true}]\!] \setminus [\![\hat{\psi}[d/q.y]]\!]$. Then:
     - **Case:** $\mathcal{S} \notin [\![\mathtt{true}]\!]$.
       By Defn. 4, $\mathsf{eval}_{\mathcal{S}}(\mathtt{true}) = \mathtt{true}$. Then, $\mathcal{S} \in \{\hat{\mathcal{S}} \mid \mathsf{eval}_{\hat{\mathcal{S}}}(\mathtt{true}) = \mathtt{true}\}$. Then, by Defn. 13, $\mathcal{S} \in [\![\mathtt{true}]\!]$. Then, **false**.

- **Case:** $\mathcal{S} \in [\![\hat{\psi}[d/q.y]]\!]$. Then, by induction, $\mathcal{S}[d/q.y] \in [\![\hat{\psi}]\!]$. Then, $\mathcal{S}[d/q.y] \notin [\![\texttt{true}]\!] \setminus [\![\hat{\psi}]\!]$. Then, by Defn. 13, $\mathcal{S}[\mathsf{eval}_{\mathcal{S}}(d)/q.y] \notin [\![\neg\hat{\psi}]\!]$. Then, $\mathcal{S}[\mathsf{eval}_{\mathcal{S}}(d)/q.y] \notin [\![\psi]\!]$.

– **Step:** $\psi = \psi_1 \wedge \psi_2$, for some ψ_1, ψ_2.
1. Recall $\mathcal{S} \in [\![\psi[d/q.y]]\!]$. Then, $\mathcal{S} \in [\![(\psi_1 \wedge \psi_2)[d/q.y]]\!]$. Then, by Defn. 7, $\mathcal{S} \in [\![\psi_1[d/q.y] \wedge \psi_2[d/q.y]]\!]$. Then, by Defn. 13, $\mathcal{S} \in [\![\psi_1[d/q.y]]\!] \cap [\![\psi_2[d/q.y]]\!]$. Then, $\mathcal{S} \in [\![\psi_1[d/q.y]]\!]$ and $\mathcal{S} \in [\![\psi_2[d/q.y]]\!]$. Then, by induction, $\mathcal{S}[\mathsf{eval}_{\mathcal{S}}(d)/q.y] \in [\![\psi_1]\!]$ and $\mathcal{S}[\mathsf{eval}_{\mathcal{S}}(d)/q.y] \in [\![\psi_2]\!]$. Then, $\mathcal{S}[\mathsf{eval}_{\mathcal{S}}(d)/q.y] \in [\![\psi_1]\!] \cap [\![\psi_2]\!]$. Then, by Defn. 13, $\mathcal{S}[\mathsf{eval}_{\mathcal{S}}(d)/q.y] \in [\![\psi_1 \wedge \psi_2]\!]$. Then, $\mathcal{S}[\mathsf{eval}_{\mathcal{S}}(d)/q.y] \in [\![\psi]\!]$.
2. Similar to case 1.

– **Step:** $\psi = \forall\hat{\psi}$, for some $\hat{\psi}$.
1. By Defn. 13:
   - **Case:** $[\![\forall\hat{\psi}]\!] = [\![\texttt{true}]\!]$.
     By Lem. 15, $\{\mathcal{S}[\mathsf{eval}_{\mathcal{S}}(d)/q.y]\} \subseteq [\![\texttt{true}]\!]$. Then, $\mathcal{S}[\mathsf{eval}_{\mathcal{S}}(d)/q.y] \in [\![\texttt{true}]\!]$. Then, $\mathcal{S}[\mathsf{eval}_{\mathcal{S}}(d)/q.y] \in [\![\forall\hat{\psi}]\!]$. Then, $\mathcal{S}[\mathsf{eval}_{\mathcal{S}}(d)/q.y] \in [\![\psi]\!]$.
   - **Case:** $[\![\forall\hat{\psi}]\!] = [\![\texttt{false}]\!]$.
     Recall $\mathcal{S} \notin \emptyset$. Then, $\mathcal{S} \notin [\![\texttt{false}]\!]$. Then, $\mathcal{S} \notin [\![\forall\hat{\psi}]\!]$. Then, by Defn. 7, $\mathcal{S} \notin [\![(\forall\hat{\psi})[d/q.y]]\!]$. Then, $\mathcal{S} \notin [\![\psi[d/q.y]]\!]$. Then, **false**.
2. By Defn. 13:
   - **Case:** $[\![\forall\hat{\psi}]\!] = [\![\texttt{true}]\!]$.
     By Defn. 4, $\mathsf{eval}_{\mathcal{S}}(\texttt{true}) = \texttt{true}$. Then, $\mathcal{S} \in \{\hat{\mathcal{S}} \mid \mathsf{eval}_{\hat{\mathcal{S}}}(\texttt{true}) = \texttt{true}\}$. Then, by Defn. 13, $\mathcal{S} \in [\![\texttt{true}]\!]$. Then, $\mathcal{S} \in [\![\forall\hat{\psi}]\!]$. Then, by Defn. 7, $\mathcal{S} \in [\![(\forall\hat{\psi})[d/q.y]]\!]$. Then, $\mathcal{S} \in [\![\psi[d/q.y]]\!]$. Then, **false**.
   - **Case:** $[\![\forall\hat{\psi}]\!] = [\![\texttt{false}]\!]$.
     Recall $\mathcal{S}[\mathsf{eval}_{\mathcal{S}}(d)/q.y] \notin \emptyset$. Then, $\mathcal{S}[\mathsf{eval}_{\mathcal{S}}(d)/q.y] \notin [\![\texttt{false}]\!]$. Then, $\mathcal{S}[\mathsf{eval}_{\mathcal{S}}(d)/q.y] \notin [\![\forall\hat{\psi}]\!]$. Then, $\mathcal{S}[\mathsf{eval}_{\mathcal{S}}(d)/q.y] \notin [\![\psi]\!]$. □

Proof (of Lem. 15).

– By Defn. 4, $\mathsf{eval}_{\mathcal{S}}(\texttt{false}) \neq \texttt{true}$ for every $\mathcal{S}$. Then, $\mathcal{S} \notin \{\hat{\mathcal{S}} \mid \mathsf{eval}_{\hat{\mathcal{S}}}(\texttt{true}) = \texttt{true}\}$ for every $\mathcal{S}$. Then, by Defn. 13, $\mathcal{S} \notin [\![\texttt{false}]\!]$ for every $\mathcal{S}$. Then, $[\![\texttt{false}]\!] = \emptyset$.
– Recall $\emptyset \subseteq [\![\psi]\!]$. Then, $[\![\texttt{false}]\!] \subseteq [\![\psi]\!]$.
– By Defn. 4, $\mathsf{eval}_{\mathcal{S}}(\texttt{true}) = \texttt{true}$ for every $\mathcal{S}$. Then, $\mathcal{S} \in \{\hat{\mathcal{S}} \mid \mathsf{eval}_{\hat{\mathcal{S}}}(\texttt{true}) = \texttt{true}\}$ for every $\mathcal{S}$. Then, by Defn. 13, $\mathcal{S} \in [\![\texttt{true}]\!]$ for every $\mathcal{S}$. Then, $\mathcal{S} \in [\![\psi]\!]$ implies $\mathcal{S} \in [\![\texttt{true}]\!]$ for every $\mathcal{S}$. Then, $[\![\psi]\!] \subseteq [\![\texttt{true}]\!]$. □

Proof (of Lem. 16).

1. Recall:
   – **Case:** $[\![\psi_1 \wedge \psi_2]\!] = [\![\texttt{true}]\!]$.
     - Recall $[\![\psi_1 \wedge \psi_2]\!] = [\![\texttt{true}]\!]$. Then, by Defn. 13, $[\![\psi_1]\!] \cap [\![\psi_2]\!] = [\![\texttt{true}]\!]$. Then, $[\![\psi_1]\!] = [\![\texttt{true}]\!]$ and $[\![\psi_2]\!] = [\![\texttt{true}]\!]$. Then, by Defn. 13, $[\![\forall\psi_1]\!] = [\![\texttt{true}]\!]$ and $[\![\forall\psi_2]\!] = [\![\texttt{true}]\!]$. Then, $[\![\forall\psi_1]\!] \cap [\![\forall\psi_2]\!] = [\![\texttt{true}]\!]$.
     - Recall $[\![\psi_1 \wedge \psi_2]\!] = [\![\texttt{true}]\!]$. Then, by Defn. 13, $[\![\forall(\psi_1 \wedge \psi_2)]\!] = [\![\texttt{true}]\!]$. Then, $[\![\forall(\psi_1 \wedge \psi_2)]\!] = [\![\forall\psi_1]\!] \cap [\![\forall\psi_2]\!]$.
   – **Case:** $[\![\psi_1 \wedge \psi_2]\!] \neq [\![\texttt{true}]\!]$.
     - Recall $[\![\psi_1 \wedge \psi_2]\!] \neq [\![\texttt{true}]\!]$. Then, by Defn. 13, $[\![\psi_1]\!] \cap [\![\psi_2]\!] \neq [\![\texttt{true}]\!]$. Then, $[\![\psi_1]\!] \neq [\![\texttt{true}]\!]$ or $[\![\psi_2]\!] \neq [\![\texttt{true}]\!]$. Then, by Defn. 13, $[\![\forall\psi_1]\!] = [\![\texttt{false}]\!]$ or $[\![\forall\psi_2]\!] = [\![\texttt{false}]\!]$. Then, by Lem. 15, $[\![\forall\psi_1]\!] = \emptyset$ or $[\![\forall\psi_2]\!] = \emptyset$. Then, $[\![\forall\psi_1]\!] \cap [\![\forall\psi_2]\!] = \emptyset$. Then, by Lem. 15, Then, $[\![\forall\psi_1]\!] \cap [\![\forall\psi_2]\!] = [\![\texttt{false}]\!]$.
     - Recall $[\![\psi_1 \wedge \psi_2]\!] \neq [\![\texttt{true}]\!]$. Then, by Defn. 13, $[\![\forall(\psi_1 \wedge \psi_2)]\!] = [\![\texttt{false}]\!]$. Then, $[\![\forall(\psi_1 \wedge \psi_2)]\!] = [\![\forall\psi_1]\!] \cap [\![\forall\psi_2]\!]$. □
2. By 1, $[\![\forall(\psi_1 \wedge \psi_2)]\!] = [\![\forall\psi_1]\!] \cap [\![\forall\psi_2]\!]$. Then, by Defn. 13, $[\![\forall(\psi_1 \wedge \psi_2)]\!] = [\![\forall\psi_1 \wedge \forall\psi_2]\!]$. □

Proof (of Lem. 17). By Defn. 6, $\llbracket \psi_1 \Rightarrow \psi_2 \rrbracket = \llbracket \neg(\psi_1 \wedge \neg\psi_2) \rrbracket$. Then, by Defn. 13, $\llbracket \psi_1 \Rightarrow \psi_2 \rrbracket = \llbracket \texttt{true} \rrbracket \setminus \llbracket \psi_1 \wedge \neg\psi_2 \rrbracket$. Then, by Defn. 13, $\llbracket \psi_1 \Rightarrow \psi_2 \rrbracket = \llbracket \texttt{true} \rrbracket \setminus (\llbracket \psi_1 \rrbracket \cap \llbracket \neg\psi_2 \rrbracket)$. Then, $\llbracket \psi_1 \Rightarrow \psi_2 \rrbracket = \llbracket \texttt{true} \rrbracket \setminus ((\llbracket \texttt{true} \rrbracket \setminus (\llbracket \texttt{true} \rrbracket \setminus \llbracket \psi_1 \rrbracket)) \cap \llbracket \neg\psi_2 \rrbracket)$. Then, by Defn. 13, $\llbracket \psi_1 \Rightarrow \psi_2 \rrbracket = \llbracket \texttt{true} \rrbracket \setminus ((\llbracket \texttt{true} \rrbracket \setminus \llbracket \neg\psi_1 \rrbracket) \cap \llbracket \neg\psi_2 \rrbracket)$. Then, by Defn. 13, $\llbracket \psi_1 \Rightarrow \psi_2 \rrbracket = \llbracket \texttt{true} \rrbracket \setminus (\llbracket \neg\neg\psi_1 \rrbracket \cap \llbracket \neg\psi_2 \rrbracket)$. Then, by Defn. 13, $\llbracket \psi_1 \Rightarrow \psi_2 \rrbracket = \llbracket \texttt{true} \rrbracket \setminus \llbracket \neg\psi_2 \wedge \neg\neg\psi_1 \rrbracket$. Then, by Defn. 13, $\llbracket \psi_1 \Rightarrow \psi_2 \rrbracket = \llbracket \neg(\neg\psi_2 \wedge \neg\neg\psi_1) \rrbracket$. Then, by Defn. 6, $\llbracket \psi_1 \Rightarrow \psi_2 \rrbracket = \llbracket \neg\psi_2 \Rightarrow \neg\psi_1 \rrbracket$. □

Proof (of Lem. 18).

1. – By Lem. 15, $\llbracket \psi_1 \rrbracket, \llbracket \psi_2 \rrbracket \subseteq \llbracket \texttt{true} \rrbracket$. Then,
   $\llbracket \texttt{true} \rrbracket \setminus (\llbracket \psi_1 \rrbracket \cap \llbracket \psi_2 \rrbracket) = (\llbracket \texttt{true} \rrbracket \setminus \llbracket \psi_1 \rrbracket) \cup (\llbracket \texttt{true} \rrbracket \setminus \llbracket \psi_2 \rrbracket)$.
   – By Lem. 15, $\llbracket \psi \wedge \neg\psi_1 \rrbracket, \llbracket \psi \wedge \neg\psi_2 \rrbracket \subseteq \llbracket \texttt{true} \rrbracket$. Then,
   $\llbracket \texttt{true} \rrbracket \setminus (\llbracket \psi \wedge \neg\psi_1 \rrbracket \cup \llbracket \psi \wedge \neg\psi_2 \rrbracket) = (\llbracket \texttt{true} \rrbracket \setminus \llbracket \psi \wedge \neg\psi_1 \rrbracket) \cap (\llbracket \texttt{true} \rrbracket \setminus \llbracket \psi \wedge \neg\psi_2 \rrbracket)$.
   – Recall $\llbracket \psi \Rightarrow (\psi_1 \wedge \psi_2) \rrbracket = \llbracket \psi \Rightarrow (\psi_1 \wedge \psi_2) \rrbracket$. Then, by Defn. 6,
   $\llbracket \psi \Rightarrow (\psi_1 \wedge \psi_2) \rrbracket = \llbracket \neg(\psi \wedge \neg(\psi_1 \wedge \psi_2)) \rrbracket$. Then, by Defn. 13,
   $\llbracket \psi \Rightarrow (\psi_1 \wedge \psi_2) \rrbracket = \llbracket \texttt{true} \rrbracket \setminus \llbracket \psi \wedge \neg(\psi_1 \wedge \psi_2) \rrbracket$. Then, by Defn. 13,
   $\llbracket \psi \Rightarrow (\psi_1 \wedge \psi_2) \rrbracket = \llbracket \texttt{true} \rrbracket \setminus (\llbracket \psi \rrbracket \cap \llbracket \neg(\psi_1 \wedge \psi_2) \rrbracket)$. Then, by Defn. 13,
   $\llbracket \psi \Rightarrow (\psi_1 \wedge \psi_2) \rrbracket = \llbracket \texttt{true} \rrbracket \setminus (\llbracket \psi \rrbracket \cap (\llbracket \texttt{true} \rrbracket \setminus \llbracket \psi_1 \wedge \psi_2 \rrbracket))$. Then, by Defn. 13,
   $\llbracket \psi \Rightarrow (\psi_1 \wedge \psi_2) \rrbracket = \llbracket \texttt{true} \rrbracket \setminus (\llbracket \psi \rrbracket \cap (\llbracket \texttt{true} \rrbracket \setminus (\llbracket \psi_1 \rrbracket \cap \llbracket \psi_2 \rrbracket)))$. Then,
   $\llbracket \psi \Rightarrow (\psi_1 \wedge \psi_2) \rrbracket = \llbracket \texttt{true} \rrbracket \setminus (\llbracket \psi \rrbracket \cap ((\llbracket \texttt{true} \rrbracket \setminus \llbracket \psi_1 \rrbracket) \cup (\llbracket \texttt{true} \rrbracket \setminus \llbracket \psi_2 \rrbracket)))$. Then, by Defn. 13,
   $\llbracket \psi \Rightarrow (\psi_1 \wedge \psi_2) \rrbracket = \llbracket \texttt{true} \rrbracket \setminus (\llbracket \psi \rrbracket \cap (\llbracket \neg\psi_1 \rrbracket \cup \llbracket \neg\psi_2 \rrbracket))$. Then,
   $\llbracket \psi \Rightarrow (\psi_1 \wedge \psi_2) \rrbracket = \llbracket \texttt{true} \rrbracket \setminus ((\llbracket \psi \rrbracket \cap \llbracket \neg\psi_1 \rrbracket) \cup (\llbracket \psi \rrbracket \cap \llbracket \neg\psi_2 \rrbracket))$. Then, by Defn. 13,
   $\llbracket \psi \Rightarrow (\psi_1 \wedge \psi_2) \rrbracket = \llbracket \texttt{true} \rrbracket \setminus ((\llbracket \psi \wedge \neg\psi_1 \rrbracket) \cup (\llbracket \psi \wedge \neg\psi_2 \rrbracket))$. Then,
   $\llbracket \psi \Rightarrow (\psi_1 \wedge \psi_2) \rrbracket = (\llbracket \texttt{true} \rrbracket \setminus \llbracket \psi \wedge \neg\psi_1 \rrbracket) \cap (\llbracket \texttt{true} \rrbracket \setminus \llbracket \psi \wedge \neg\psi_2 \rrbracket)$. Then, by Defn. 13,
   $\llbracket \psi \Rightarrow (\psi_1 \wedge \psi_2) \rrbracket = \llbracket \neg(\psi \wedge \neg\psi_1) \rrbracket \cap \llbracket \neg(\psi \wedge \neg\psi_2) \rrbracket$. Then, by Defn. 6,
   $\llbracket \psi \Rightarrow (\psi_1 \wedge \psi_2) \rrbracket = \llbracket \psi \Rightarrow \psi_1 \rrbracket \cap \llbracket \psi \Rightarrow \psi_2 \rrbracket$. □
2. By 1, $\llbracket \psi \Rightarrow (\psi_1 \wedge \psi_2) \rrbracket = \llbracket \psi \Rightarrow \psi_1 \rrbracket \cap \llbracket \psi \Rightarrow \psi_2 \rrbracket$. Then, by Defn. 13,
   $\llbracket \psi \Rightarrow (\psi_1 \wedge \psi_2) \rrbracket = \llbracket (\psi \Rightarrow \psi_1) \wedge (\psi \Rightarrow \psi_2) \rrbracket$. □

Proof (of Lem. 19).

1. – By Lem. 14, $\llbracket \psi_1 \rrbracket \subseteq \llbracket \texttt{true} \rrbracket$. Then, $\llbracket \psi_1 \rrbracket \cap \llbracket \texttt{true} \rrbracket = \llbracket \psi_2 \rrbracket$.
   – Recall $\llbracket \psi_1 \rrbracket \cap \llbracket \psi_1 \Rightarrow \psi_2 \rrbracket = \llbracket \psi_1 \rrbracket \cap \llbracket \psi_1 \Rightarrow \psi_2 \rrbracket$. Then, by Defn. 6,
   $\llbracket \psi_1 \rrbracket \cap \llbracket \psi_1 \Rightarrow \psi_2 \rrbracket = \llbracket \psi_1 \rrbracket \cap \llbracket \neg(\psi_1 \wedge \neg\psi_2) \rrbracket$. Then, by Defn. 13,
   $\llbracket \psi_1 \rrbracket \cap \llbracket \psi_1 \Rightarrow \psi_2 \rrbracket = \llbracket \psi_1 \rrbracket \cap (\llbracket \texttt{true} \rrbracket \setminus \llbracket \psi_1 \wedge \neg\psi_2 \rrbracket)$. Then, by Defn. 13,
   $\llbracket \psi_1 \rrbracket \cap \llbracket \psi_1 \Rightarrow \psi_2 \rrbracket = \llbracket \psi_1 \rrbracket \cap (\llbracket \texttt{true} \rrbracket \setminus (\llbracket \psi_1 \rrbracket \cap \llbracket \neg\psi_2 \rrbracket))$. Then, by Defn. 13,
   $\llbracket \psi_1 \rrbracket \cap \llbracket \psi_1 \Rightarrow \psi_2 \rrbracket = \llbracket \psi_1 \rrbracket \cap (\llbracket \texttt{true} \rrbracket \setminus (\llbracket \psi_1 \rrbracket \cap (\llbracket \texttt{true} \rrbracket \setminus \llbracket \psi_2 \rrbracket)))$. Then,
   $\llbracket \psi_1 \rrbracket \cap \llbracket \psi_1 \Rightarrow \psi_2 \rrbracket = (\llbracket \psi_1 \rrbracket \cap \llbracket \texttt{true} \rrbracket) \setminus ((\llbracket \psi_1 \rrbracket \cap \llbracket \texttt{true} \rrbracket) \setminus \llbracket \psi_2 \rrbracket)$. Then,
   $\llbracket \psi_1 \rrbracket \cap \llbracket \psi_1 \Rightarrow \psi_2 \rrbracket = \llbracket \psi_1 \rrbracket \setminus (\llbracket \psi_1 \rrbracket \setminus \llbracket \psi_2 \rrbracket)$. Then, $\llbracket \psi_1 \rrbracket \cap \llbracket \psi_1 \Rightarrow \psi_2 \rrbracket = \llbracket \psi_1 \rrbracket \cap \llbracket \psi_2 \rrbracket$. □
2. By 1, $\llbracket \psi_1 \rrbracket \cap \llbracket \psi_1 \Rightarrow \psi_2 \rrbracket = \llbracket \psi_1 \rrbracket \cap \llbracket \psi_2 \rrbracket$. Then, by Defn. 13, $\llbracket \psi_1 \wedge (\psi_1 \Rightarrow \psi_2) \rrbracket = \llbracket \psi_1 \wedge \psi_2 \rrbracket$. □

Proof (of Lem. 20). Recall $\mathcal{S} \in \llbracket \psi_1[d/q.y] \rrbracket$ implies $\mathcal{S} \in \llbracket \psi_1[d/q.y] \rrbracket$ for every $\mathcal{S}$. Then, by Lem. 14, $\mathcal{S} \in \llbracket \psi_1[d/q.y] \rrbracket$ implies $\mathcal{S}[\mathsf{eval}_{\mathcal{S}}(d)/q.y] \in \llbracket \psi_1 \rrbracket$ for every $\mathcal{S}$. Then, $\mathcal{S} \in \llbracket \psi_1[d/q.y] \rrbracket$ implies $\mathcal{S}[\mathsf{eval}_{\mathcal{S}}(d)/q.y] \in \llbracket \psi_1 \rrbracket \subseteq \llbracket \psi_2 \rrbracket$ for every $\mathcal{S}$. Then, $\mathcal{S} \in \llbracket \psi_1[d/q.y] \rrbracket$ implies $\mathcal{S}[\mathsf{eval}_{\mathcal{S}}(d)/q.y] \in \llbracket \psi_2 \rrbracket$ for every $\mathcal{S}$. Then, by Lem. 14, $\mathcal{S} \in \llbracket \psi_1[d/q.y] \rrbracket$ implies $\mathcal{S} \in \llbracket \psi_2[d/q.y] \rrbracket$ for every $\mathcal{S}$. Then, $\llbracket \psi_1[d/q.y] \rrbracket \subseteq \llbracket \psi_2[d/q.y] \rrbracket$. □

Proof (of Lem. 21).

1. Recall:
   – **Case:** $\llbracket \psi_1 \rrbracket = \llbracket \texttt{true} \rrbracket = \llbracket \psi_2 \rrbracket$. Then, by Defn. 13, $\llbracket \forall\psi_1 \rrbracket = \llbracket \texttt{true} \rrbracket = \llbracket \forall\psi_2 \rrbracket$. Then,
   $\llbracket \forall\psi_1 \rrbracket = \llbracket \forall\psi_2 \rrbracket$. Then, $\llbracket \forall\psi_1 \rrbracket \subseteq \llbracket \forall\psi_2 \rrbracket$.
   – **Case:** $\llbracket \psi_1 \rrbracket = \llbracket \texttt{true} \rrbracket \neq \llbracket \psi_2 \rrbracket$.
     - Recall $\llbracket \psi_1 \rrbracket = \llbracket \texttt{true} \rrbracket \neq \llbracket \psi_2 \rrbracket$. Then, $\llbracket \psi_1 \rrbracket \neq \llbracket \psi_2 \rrbracket$.
     - Recall $\llbracket \psi_1 \rrbracket \subseteq \llbracket \psi_2 \rrbracket$ and $\llbracket \psi_1 \rrbracket \neq \llbracket \psi_2 \rrbracket$. Then, $\llbracket \psi_1 \rrbracket \not\supseteq \llbracket \psi_2 \rrbracket$. Then, $\llbracket \texttt{true} \rrbracket \not\supseteq \llbracket \psi_2 \rrbracket$.

- By Lem. 15, $[\![\texttt{true}]\!] \supseteq [\![\psi_2]\!]$. Then, **false**.

– **Case:** $[\![\psi_1]\!] \neq [\![\texttt{true}]\!] = [\![\psi_2]\!]$.
- Recall $[\![\psi_2]\!] = [\![\texttt{true}]\!]$. Then, by Defn. 13, $[\![\forall\psi_2]\!] = [\![\texttt{true}]\!]$.
- By Lem. 15, $[\![\forall\psi_1]\!] \subseteq [\![\texttt{true}]\!]$. Then, $[\![\forall\psi_1]\!] \subseteq [\![\forall\psi_2]\!]$.

– **Case:** $[\![\psi_1]\!] \neq [\![\texttt{true}]\!] \neq [\![\psi_2]\!]$. Similar to case $[\![\psi_1]\!] = [\![\texttt{true}]\!] = [\![\psi_2]\!]$. □

2. Recall $[\![\psi_1]\!] = [\![\psi_2]\!]$. Then, $[\![\psi_1]\!] \subseteq [\![\psi_2]\!]$ and $[\![\psi_1]\!] \supseteq [\![\psi_2]\!]$. Then, by 1, $[\![\forall\psi_1]\!] \subseteq [\![\forall\psi_2]\!]$ and $[\![\forall\psi_1]\!] \supseteq [\![\forall\psi_2]\!]$. Then, $[\![\forall\psi_1]\!] = [\![\forall\psi_2]\!]$. □

Proof (of Lem. 22).

1. Recall $[\![\psi_1]\!] \subseteq [\![\psi_2]\!]$. Then, $[\![\texttt{true}]\!] \setminus [\![\psi_2]\!] \subseteq [\![\texttt{true}]\!] \setminus [\![\psi_1]\!]$. Then, $[\![\psi]\!] \cap ([\![\texttt{true}]\!] \setminus [\![\psi_2]\!]) \subseteq [\![\psi]\!] \cap ([\![\texttt{true}]\!] \setminus [\![\psi_1]\!])$. Then, $[\![\texttt{true}]\!] \setminus ([\![\psi]\!] \cap ([\![\texttt{true}]\!] \setminus [\![\psi_1]\!])) \subseteq [\![\texttt{true}]\!] \setminus ([\![\psi]\!] \cap ([\![\texttt{true}]\!] \setminus [\![\psi_2]\!]))$. Then, by Defn. 13, $[\![\texttt{true}]\!] \setminus ([\![\psi]\!] \cap [\![\neg\psi_1]\!]) \subseteq [\![\texttt{true}]\!] \setminus ([\![\psi]\!] \cap [\![\neg\psi_2]\!])$. Then, by Defn. 13, $[\![\texttt{true}]\!] \setminus [\![\psi \wedge \neg\psi_1]\!] \subseteq [\![\texttt{true}]\!] \setminus [\![\psi \wedge \neg\psi_2]\!]$. Then, by Defn. 13, $[\![\neg(\psi \wedge \neg\psi_1)]\!] \subseteq [\![\neg(\psi \wedge \neg\psi_2)]\!]$. Then, by Defn. 6, $[\![\psi \Rightarrow \psi_1]\!] \subseteq [\![\psi \Rightarrow \psi_2]\!]$. □
2. Recall $[\![\psi_1]\!] = [\![\psi_2]\!]$. Then, $[\![\psi_1]\!] \subseteq [\![\psi_2]\!]$ and $[\![\psi_1]\!] \supseteq [\![\psi_2]\!]$. Then, by 1, $[\![\psi \Rightarrow \psi_1]\!] \subseteq [\![\psi \Rightarrow \psi_2]\!]$ and $[\![\psi \Rightarrow \psi_1]\!] \supseteq [\![\psi \Rightarrow \psi_2]\!]$. Then, $[\![\psi \Rightarrow \psi_1]\!] = [\![\psi \Rightarrow \psi_2]\!]$. □

Proof (of Lem. 23).

– By Lem. 15, $[\![\psi_1]\!] \subseteq [\![\texttt{true}]\!]$. Then, $[\![\psi_1]\!] \cup [\![\texttt{true}]\!] = [\![\texttt{true}]\!]$.
– Recall $[\![\psi_1 \Rightarrow \psi_2]\!] = [\![\texttt{true}]\!]$. Then, by Defn. 6, $[\![\neg(\psi_1 \wedge \neg\psi_2)]\!] = [\![\texttt{true}]\!]$. Then, by Defn. 13, $[\![\texttt{true}]\!] \setminus [\![\psi_1 \wedge \neg\psi_2]\!] = [\![\texttt{true}]\!]$. Then, by Defn. 13, $[\![\texttt{true}]\!] \setminus ([\![\psi_1]\!] \cap [\![\neg\psi_2]\!]) = [\![\texttt{true}]\!]$. Then, by Defn. 13, $[\![\texttt{true}]\!] \setminus ([\![\psi_1]\!] \cap ([\![\texttt{true}]\!] \setminus [\![\psi_2]\!])) = [\![\texttt{true}]\!]$. Then, $[\![\psi_1]\!] \cap ([\![\texttt{true}]\!] \setminus [\![\psi_2]\!]) = \emptyset$. Then, $[\![\psi_1]\!] \cap (([\![\psi_1]\!] \cup [\![\texttt{true}]\!]) \setminus [\![\psi_2]\!]) = \emptyset$. Then, $[\![\psi_1]\!] \cap (([\![\psi_1]\!] \setminus [\![\psi_2]\!]) \cup ([\![\texttt{true}]\!] \setminus [\![\psi_2]\!])) = \emptyset$. Then, $([\![\psi_1]\!] \cap ([\![\psi_1]\!] \setminus [\![\psi_2]\!])) \cup ([\![\psi_1]\!] \cap ([\![\texttt{true}]\!] \setminus [\![\psi_2]\!])) = \emptyset$. Then, $[\![\psi_1]\!] \cap ([\![\psi_1]\!] \setminus [\![\psi_2]\!]) = \emptyset$. Then, $[\![\psi_1]\!] \subseteq [\![\psi_2]\!]$. □

Proof (of Lem. 24).

1. By Defn. 13:
   – **Case:** $[\![\forall\psi]\!] = [\![\texttt{true}]\!]$ and $[\![\psi]\!] = [\![\texttt{true}]\!]$. Then, $[\![\forall\psi]\!] = [\![\psi]\!]$. Then, $[\![\forall\psi]\!] \subseteq [\![\psi]\!]$.
   – **Case:** $[\![\forall\psi]\!] = [\![\texttt{false}]\!]$.
   Recall $\emptyset \subseteq [\![\psi]\!]$. Then, by Lem. 15, $[\![\texttt{false}]\!] \subseteq [\![\psi]\!]$. Then, $[\![\forall\psi]\!] \subseteq [\![\psi]\!]$. □
2. – By 1, $[\![\forall\psi]\!] \subseteq [\![\psi]\!]$.
   – Recall $[\![\psi]\!] \cap \Sigma_1 \subseteq \Sigma_2$ and $[\![\forall\psi]\!] \subseteq [\![\psi]\!]$. Then, $[\![\forall\psi]\!] \cap \Sigma_1 \subseteq \Sigma_2$. □
3. – By 1, $[\![\forall\psi]\!] \subseteq [\![\psi]\!]$.
   – Recall $\Sigma_1 \subseteq [\![\forall\psi]\!] \cap \Sigma_2$ and $[\![\forall\psi]\!] \subseteq [\![\psi]\!]$. Then, $\Sigma_1 \subseteq [\![\psi]\!] \cap \Sigma_2$. □

Proof (of Lem. 25).

1. – By Lem. 15, $[\![\psi_1]\!], [\![\neg\psi_2]\!] \subseteq [\![\texttt{true}]\!]$. Then, $[\![\texttt{true}]\!] \setminus ([\![\psi_1]\!] \cap [\![\neg\psi_2]\!]) = ([\![\texttt{true}]\!] \setminus [\![\psi_1]\!]) \cup ([\![\texttt{true}]\!] \setminus [\![\neg\psi_2]\!])$.
   – Recall $[\![\psi_1 \Rightarrow \psi_2]\!] \subseteq [\![\psi_1 \Rightarrow \psi_2]\!]$. Then, by Defn. 6, $[\![\neg(\psi_1 \wedge \neg\psi_2)]\!] \subseteq [\![\psi_1 \Rightarrow \psi_2]\!]$. Then, by Defn. 13, $[\![\texttt{true}]\!] \setminus [\![\psi_1 \wedge \neg\psi_2]\!] \subseteq [\![\psi_1 \Rightarrow \psi_2]\!]$. Then, by Defn. 13, $[\![\texttt{true}]\!] \setminus ([\![\psi_1]\!] \cap [\![\neg\psi_2]\!]) \subseteq [\![\psi_1 \Rightarrow \psi_2]\!]$. Then, $([\![\texttt{true}]\!] \setminus [\![\psi_1]\!]) \cup ([\![\texttt{true}]\!] \setminus [\![\neg\psi_2]\!]) \subseteq [\![\psi_1 \Rightarrow \psi_2]\!]$. Then, $[\![\texttt{true}]\!] \setminus [\![\neg\psi_2]\!] \subseteq [\![\psi_1 \Rightarrow \psi_2]\!]$. Then, by Defn. 13, $[\![\texttt{true}]\!] \setminus ([\![\texttt{true}]\!] \setminus [\![\psi_2]\!]) \subseteq [\![\psi_1 \Rightarrow \psi_2]\!]$. Then, $[\![\psi_2]\!] \subseteq [\![\psi_1 \Rightarrow \psi_2]\!]$. □
2. – By 1, $[\![\psi_2]\!] \subseteq [\![\psi_1 \Rightarrow \psi_2]\!]$.
   – Recall $[\![\psi_1 \Rightarrow \psi_2]\!] \cap \Sigma_1 \subseteq \Sigma_2$ and $[\![\psi_2]\!] \subseteq [\![\psi_1 \Rightarrow \psi_2]\!]$. Then, $[\![\psi_2]\!] \cap \Sigma_1 \subseteq \Sigma_2$. □
3. – By 1, $[\![\psi_2]\!] \subseteq [\![\psi_1 \Rightarrow \psi_2]\!]$.
   – Recall $\Sigma_1 \subseteq [\![\psi_2]\!] \cap \Sigma_2$ and $[\![\psi_2]\!] \subseteq [\![\psi_1 \Rightarrow \psi_2]\!]$. Then, $\Sigma_1 \subseteq [\![\psi_1 \Rightarrow \psi_2]\!] \cap \Sigma_2$. □

Proof (of Lem. 26).

1. By Defn. 5:
   - **Base:** $\xi = \texttt{true}$.
     Recall $[\![\texttt{true}]\!] = \bigcap\{[\![\texttt{true}]\!]\}_{r \in \mathsf{subj}(\xi)}$. Then, by Defn. 11, $[\![\texttt{true}]\!] = \bigcap\{[\![\texttt{true} \restriction r]\!]\}_{r \in \mathsf{subj}(\xi)}$. Then, by Defn. 13, $[\![\texttt{true}]\!] = [\![\bigwedge\{\texttt{true} \restriction r\}_{r \in \mathsf{subj}(\xi)}]\!]$. Then, $[\![\xi]\!] = [\![\bigwedge\{\xi \restriction r\}_{r \in \mathsf{subj}(\xi)}]\!]$.
   - **Base:** $\xi = \bigwedge\{e_{\hat{r}}\}_{\hat{r} \in \hat{R}}$, for some $\{e_{\hat{r}}\}_{\hat{r} \in \hat{R}}$.
     - Recall $\xi = \bigwedge\{e_{\hat{r}}\}_{\hat{r} \in \hat{R}}$. Then, by Rem. 1, $\{\hat{\hat{r}} \mid \hat{\hat{r}}.\hat{\hat{x}} \in \mathsf{rx}(e_{\hat{r}})\} = \{\hat{r}\}$ for every $\hat{r} \in \hat{R}$.
     - By Lem. 40, $\mathsf{subj}(\bigwedge\{e_{\hat{r}}\}_{\hat{r} \in \hat{R}}) = \bigcup\{\mathsf{subj}(e_{\hat{r}})\}_{\hat{r} \in \hat{R}}$. Then, $\mathsf{subj}(\xi) = \bigcup\{\mathsf{subj}(e_{\hat{r}})\}_{\hat{r} \in \hat{R}}$. Then, by Defn. 10, $\mathsf{subj}(\xi) = \bigcup\{\{\hat{\hat{r}} \mid \hat{\hat{r}}.\hat{\hat{x}} \in \mathsf{rx}(e_{\hat{r}})\}\}_{\hat{r} \in \hat{R}}$. Then, $\mathsf{subj}(\xi) = \bigcup\{\{\hat{r}\}\}_{\hat{r} \in \hat{R}}$. Then, $\mathsf{subj}(\xi) = \{\hat{r} \mid \hat{r} \in \hat{R}\}$. Then, $\mathsf{subj}(\xi) = \hat{R}$.
     - By Defn. 11, $\bigwedge\{e_{\hat{r}}\}_{\hat{r} \in \hat{R}} \restriction r = e_r$ for every $r \in \hat{R}$. Then, $\xi \restriction r = e_r$ for every $r \in \mathsf{subj}(\xi)$. Then, $\bigwedge\{\xi \restriction r\}_{r \in \mathsf{subj}(\xi)} = \bigwedge\{e_r\}_{r \in \mathsf{subj}(\xi)}$. Then, $\bigwedge\{\xi \restriction r\}_{r \in \mathsf{subj}(\xi)} = \xi$. Then, by Defn. 13, $[\![\bigwedge\{\xi \restriction r\}_{r \in \mathsf{subj}(\xi)}]\!] = [\![\xi]\!]$.
   - **Base:** $\xi = \bigwedge\{\neg e_{\hat{r}}\}_{\hat{r} \in R}$, for some $\{e_{\hat{r}}\}_{\hat{r} \in R}$. Similar to case "$\xi = \bigwedge\{e_{\hat{r}}\}_{\hat{r} \in R}$, for some $\{e_{\hat{r}}\}_{\hat{r} \in R}$". □
2. By Defn. 5:
   - **Base:** $\xi = \texttt{true}$.
     Recall $[\![\texttt{true}]\!] = \bigcap\{[\![\texttt{true}]\!]\}_{r \in R}$. Then, by Defn. 11, $[\![\texttt{true}]\!] = \bigcap\{[\![\texttt{true} \restriction r]\!]\}_{r \in R}$. Then, by Defn. 13, $[\![\texttt{true}]\!] = [\![\bigwedge\{\texttt{true} \restriction r\}_{r \in R}]\!]$. Then, $[\![\xi]\!] = [\![\bigwedge\{\xi \restriction r\}_{r \in R}]\!]$.
   - **Base:** $\xi = \bigwedge\{e_{\hat{r}}\}_{\hat{r} \in \hat{R}}$, for some $\{e_{\hat{r}}\}_{\hat{r} \in \hat{R}}$.
     - Recall $\xi = \bigwedge\{e_{\hat{r}}\}_{\hat{r} \in \hat{R}}$. Then, by Rem. 1, $\{\hat{\hat{r}} \mid \hat{\hat{r}}.\hat{\hat{x}} \in \mathsf{rx}(e_{\hat{r}})\} = \{\hat{r}\}$ for every $\hat{r} \in \hat{R}$.
     - By Lem. 40, $\mathsf{subj}(\bigwedge\{e_{\hat{r}}\}_{\hat{r} \in \hat{R}}) = \bigcup\{\mathsf{subj}(e_{\hat{r}})\}_{\hat{r} \in \hat{R}}$. Then, $\mathsf{subj}(\xi) = \bigcup\{\mathsf{subj}(e_{\hat{r}})\}_{\hat{r} \in \hat{R}}$. Then, by Defn. 10, $\mathsf{subj}(\xi) = \bigcup\{\{\hat{\hat{r}} \mid \hat{\hat{r}}.\hat{\hat{x}} \in \mathsf{rx}(e_{\hat{r}})\}\}_{\hat{r} \in \hat{R}}$. Then, $\mathsf{subj}(\xi) = \bigcup\{\{\hat{r}\}\}_{\hat{r} \in \hat{R}}$. Then, $\mathsf{subj}(\xi) = \{\hat{r} \mid \hat{r} \in \hat{R}\}$. Then, $\mathsf{subj}(\xi) = \hat{R}$.
     - By Defn. 11, $\bigwedge\{e_{\hat{r}}\}_{\hat{r} \in \hat{R}} \restriction r = \texttt{true}$ for every $r \notin \hat{R}$. Then, $\xi \restriction r = \texttt{true}$ for every $r \notin \mathsf{subj}(\xi)$. Then, $\xi \restriction r = \texttt{true}$ for every $r \in R \setminus \mathsf{subj}(\xi)$.
     - Recall $\mathsf{subj}(\xi) \subseteq R$. Then, $R = (R \setminus \mathsf{subj}(\xi)) \cup \mathsf{subj}(\xi)$.
     - By Lem. 15, $[\![\bigwedge\{\xi \restriction r\}_{r \in \mathsf{subj}(\xi)}]\!] \subseteq [\![\texttt{true}]\!]$. Then, $[\![\bigwedge\{\xi \restriction r\}_{r \in \mathsf{subj}(\xi)}]\!] \subseteq \bigcap\{[\![\texttt{true}]\!]\}_{r \in R \setminus \mathsf{subj}(\xi)}$. Then, $[\![\bigwedge\{\xi \restriction r\}_{r \in \mathsf{subj}(\xi)}]\!] \subseteq \bigcap\{[\![\xi \restriction r]\!]\}_{r \in R \setminus \mathsf{subj}(\xi)}$. Then, by Defn. 13, $\bigcap\{[\![\xi \restriction r]\!]\}_{r \in \mathsf{subj}(\xi)} \subseteq \bigcap\{[\![\xi \restriction r]\!]\}_{r \in R \setminus \mathsf{subj}(\xi)}$. Then, $\bigcap\{[\![\xi \restriction r]\!]\}_{r \in \mathsf{subj}(\xi)} = \bigcap\{[\![\xi \restriction r]\!]\}_{r \in \mathsf{subj}(\xi)} \cap \bigcap\{[\![\xi \restriction r]\!]\}_{r \in R \setminus \mathsf{subj}(\xi)}$. Then, $\bigcap\{[\![\xi \restriction r]\!]\}_{r \in \mathsf{subj}(\xi)} = \bigcap\{[\![\xi \restriction r]\!]\}_{r \in (R \setminus \mathsf{subj}(\xi)) \cup \mathsf{subj}(\xi)}$. Then, $\bigcap\{[\![\xi \restriction r]\!]\}_{r \in \mathsf{subj}(\xi)} = \bigcap\{[\![\xi \restriction r]\!]\}_{r \in R}$. Then, by Defn. 13, $[\![\bigwedge\{\xi \restriction r\}_{r \in \mathsf{subj}(\xi)}]\!] = [\![\bigwedge\{\xi \restriction r\}_{r \in R}]\!]$. Then, by 1, $[\![\xi]\!] = [\![\bigwedge\{\xi \restriction r\}_{r \in R}]\!]$.
   - **Base:** $\xi = \bigwedge\{\neg e_{\hat{r}}\}_{\hat{r} \in R}$, for some $\{e_{\hat{r}}\}_{\hat{r} \in R}$. Similar to case "$\xi = \bigwedge\{e_{\hat{r}}\}_{\hat{r} \in R}$, for some $\{e_{\hat{r}}\}_{\hat{r} \in R}$". □

Proof (of Lem. 27).

1. By Defn. 5:
   - **Base:** $\xi = \texttt{true}$.
     Recall $\mathcal{S} \in [\![\xi^{\equiv}]\!]$. Then, $\mathcal{S} \in [\![\texttt{true}^{\equiv}]\!]$. Then, by Defn. 12, $\mathcal{S} \in [\![\texttt{true}]\!]$. Then, by Defn. 12, $\mathcal{S} \in [\![\texttt{true}^{+}]\!]$. Then, $\mathcal{S} \in [\![\xi^{+}]\!]$.
   - **Base:** $\xi = \bigwedge\{e_r\}_{r \in R}$, for some $\{e_r\}_{r \in R}$.
     - Recall $\xi = \bigwedge\{e_r\}_{r \in R}$. Then, by Rem. 1, $R \neq \emptyset$. Then, $\hat{r} \in R$, for some $\hat{r}$.
     - By Lem. 15, $[\![e_{\hat{r}}]\!] \subseteq [\![\texttt{true}]\!]$. Then, $[\![\texttt{true}]\!] = ([\![\texttt{true}]\!] \setminus [\![e_{\hat{r}}]\!]) \cup [\![e_{\hat{r}}]\!]$.
     - By Defn. 4, $\mathsf{eval}_{\mathcal{S}}(\texttt{true}) = \texttt{true}$. Then, $\mathcal{S} \in \{\hat{\mathcal{S}} \mid \mathsf{eval}_{\hat{\mathcal{S}}}(\texttt{true}) = \texttt{true}\}$. Then, by Defn. 13, $\mathcal{S} \in [\![\texttt{true}]\!]$. Then, $\mathcal{S} \in ([\![\texttt{true}]\!] \setminus [\![e_{\hat{r}}]\!]) \cup [\![e_{\hat{r}}]\!]$. Then, by Defn. 13, $\mathcal{S} \in [\![\neg e_{\hat{r}}]\!] \cup [\![e_{\hat{r}}]\!]$. Then:
       - **Case:** $\mathcal{S} \in [\![e_{\hat{r}}]\!]$.
         - Recall $\hat{r} \in R$. Then, $R = (R \setminus \{\hat{r}\}) \cup \{\hat{r}\}$.

· Recall $\mathcal{S} \in [\![\xi^{\equiv}]\!]$. Then, $\mathcal{S} \in [\![\bigwedge\{e_r\}^{\equiv}_{r\in R}]\!]$. Then, by Defn. 12, $\mathcal{S} \in [\![\bigwedge\{e_{r_1} \Rightarrow e_{r_2}\}_{r_1,r_2\in R}]\!]$. Then, by Defn. 13, $\mathcal{S} \in \bigcap\{[\![e_{r_1} \Rightarrow e_{r_2}]\!]\}_{r_1,r_2\in R}$. Then, $\mathcal{S} \in \bigcap\{[\![e_{r_1} \Rightarrow e_{r_2}]\!]\}_{r_1\in(R\setminus\{\hat{r}\})\cup\{\hat{r}\},r_2\in R}$. Then, $\mathcal{S} \in \bigcap\{[\![e_{r_1} \Rightarrow e_{r_2}]\!]\}_{r_1\in R\setminus\{\hat{r}\},r_2\in R} \cap \bigcap\{[\![e_{\hat{r}} \Rightarrow e_{r_2}]\!]\}_{r_2\in R}$. Then, $\mathcal{S} \in \bigcap\{[\![e_{\hat{r}} \Rightarrow e_{r_2}]\!]\}_{r_2\in R}$.
· Recall $\mathcal{S} \in [\![e_{\hat{r}}]\!]$ and $\mathcal{S} \in \bigcap\{[\![e_{\hat{r}} \Rightarrow e_{r_2}]\!]\}_{r_2\in R}$. Then, $\mathcal{S} \in [\![e_{\hat{r}}]\!] \cap \bigcap\{[\![e_{\hat{r}} \Rightarrow e_{r_2}]\!]\}_{r_2\in R}$. Then, by Lem. 19, $\mathcal{S} \in [\![e_{\hat{r}}]\!] \cap \bigcap\{[\![e_{r_2}]\!]\}_{r_2\in R}$. Then, $\mathcal{S} \in \bigcap\{[\![e_{r_2}]\!]\}_{r_2\in R}$. Then, by Defn. 13, $\mathcal{S} \in [\![\bigwedge\{e_{r_2}\}_{r_2\in R}]\!]$. Then, by Defn. 12, $\mathcal{S} \in [\![\bigwedge\{e_{r_2}\}^{+}_{r_2\in R}]\!]$. Then, $\mathcal{S} \in [\![\xi^{+}]\!]$.

∗ **Case:** $\mathcal{S} \in [\![\neg e_{\hat{r}}]\!]$.
· Recall $\hat{r} \in R$. Then, $R = (R \setminus \{\hat{r}\}) \cup \{\hat{r}\}$.
· Recall $\mathcal{S} \in [\![\xi^{\equiv}]\!]$. Then, $\mathcal{S} \in [\![\bigwedge\{e_r\}^{\equiv}_{r\in R}]\!]$. Then, by Defn. 12, $\mathcal{S} \in [\![\bigwedge\{e_{r_1} \Rightarrow e_{r_2}\}_{r_1,r_2\in R}]\!]$. Then, by Defn. 13, $\mathcal{S} \in \bigcap\{[\![e_{r_1} \Rightarrow e_{r_2}]\!]\}_{r_1,r_2\in R}$. Then, by Lem. 17, $\mathcal{S} \in \bigcap\{[\![\neg e_{r_2} \Rightarrow \neg e_{r_1}]\!]\}_{r_1,r_2\in R}$. Then, $\mathcal{S} \in \bigcap\{[\![\neg e_{r_2} \Rightarrow \neg e_{r_1}]\!]\}_{r_1\in R,r_2\in(R\setminus\{\hat{r}\})\cup\{\hat{r}\}}$. Then, $\mathcal{S} \in \bigcap\{[\![\neg e_{r_2} \Rightarrow \neg e_{r_1}]\!]\}_{r_1\in R,r_2\in R\setminus\{\hat{r}\}} \cap \bigcap\{[\![\neg e_{\hat{r}} \Rightarrow \neg e_{r_1}]\!]\}_{r_1\in R}$. Then, $\mathcal{S} \in \bigcap\{[\![\neg e_{\hat{r}} \Rightarrow \neg e_{r_1}]\!]\}_{r_1\in R}$.
· Recall $\mathcal{S} \in [\![\neg e_{\hat{r}}]\!]$ and $\mathcal{S} \in \bigcap\{[\![\neg e_{\hat{r}} \Rightarrow \neg e_{r_1}]\!]\}_{r_1\in R}$. Then, $\mathcal{S} \in [\![\neg e_{\hat{r}}]\!] \cap \bigcap\{[\![\neg e_{\hat{r}} \Rightarrow \neg e_{r_1}]\!]\}_{r_1\in R}$. Then, by Lem. 19, $\mathcal{S} \in [\![\neg e_{\hat{r}}]\!] \cap \bigcap\{[\![\neg e_{r_1}]\!]\}_{r_1\in R}$. Then, $\mathcal{S} \in \bigcap\{[\![\neg e_{r_1}]\!]\}_{r_1\in R}$. Then, by Defn. 13, $\mathcal{S} \in [\![\bigwedge\{\neg e_{r_1}\}_{r_1\in R}]\!]$. Then, by Defn. 12, $\mathcal{S} \in [\![\bigwedge\{e_{r_1}\}^{-}_{r_1\in R}]\!]$. Then, $\mathcal{S} \in [\![\xi^{-}]\!]$.

– **Base:** $\xi = \bigwedge\{\neg e_r\}_{r\in R}$, for some $\{e_r\}_{r\in R}$. Similar to case "$\xi = \bigwedge\{e_r\}_{r\in R}$, for some $\{e_r\}_{r\in R}$". □

2. Recall $\mathcal{S} \in [\![\xi^{\equiv}]\!]$. Then, by 1:
   – **Case:** $\mathcal{S} \in [\![\xi^{+}]\!]$.
   By Defn. 5:
   • **Base:** $\xi = \texttt{true}$.
   Recall $\mathcal{S} \in [\![\xi^{+}]\!]$. Then, $\mathcal{S} \in [\![\texttt{true}^{+}]\!]$. Then, by Defn. 11, $\mathcal{S} \in [\![(\texttt{true} \restriction r)^{+}]\!]$. Then, by Lem. 11, $\mathcal{S} \in [\![\texttt{true}^{+} \restriction r]\!]$. Then, $\mathcal{S} \in [\![\xi^{+} \restriction r]\!]$.
   • **Base:** $\xi = \bigwedge\{e_{\hat{r}}\}_{\hat{r}\in R}$, for some $\{e_{\hat{r}}\}_{\hat{r}\in R}$.
   By Defn. 11:
   ∗ **Case:** $\bigwedge\{e_{\hat{r}}\}_{\hat{r}\in R} \restriction r = e_r$ and $r \in R$.
   · Recall $r \in R$. Then, $R = (R \setminus \{r\}) \cup \{r\}$.
   · Recall $\mathcal{S} \in [\![\xi^{+}]\!]$. Then, $\mathcal{S} \in [\![\bigwedge\{e_{\hat{r}}\}^{+}_{\hat{r}\in R}]\!]$. Then, by Defn. 12, $\mathcal{S} \in [\![\bigwedge\{e_{\hat{r}}\}_{\hat{r}\in R}]\!]$. Then, by Defn. 13, $\mathcal{S} \in \bigcap\{[\![e_{\hat{r}}]\!]\}_{\hat{r}\in R}$. Then, $\mathcal{S} \in \bigcap\{[\![e_{\hat{r}}]\!]\}_{\hat{r}\in(R\setminus\{r\})\cup\{r\}}$. Then, $\mathcal{S} \in \bigcap\{[\![e_{\hat{r}}]\!]\}_{\hat{r}\in R\setminus\{r\}} \cap [\![e_r]\!]$. Then, $\mathcal{S} \in [\![e_r]\!]$. Then, $\mathcal{S} \in [\![\bigwedge\{e_{\hat{r}}\}_{\hat{r}\in R} \restriction r]\!]$. Then, by Defn. 12, $\mathcal{S} \in [\![\bigwedge\{e_{\hat{r}}\}^{+}_{\hat{r}\in R} \restriction r]\!]$. Then, $\mathcal{S} \in [\![\xi^{+} \restriction r]\!]$.
   ∗ **Case:** $\bigwedge\{e_{\hat{r}}\}_{\hat{r}\in R} \restriction r = \texttt{true}$.
   By Defn. 4, $\mathsf{eval}_{\mathcal{S}}(\texttt{true}) = \texttt{true}$ for every $\mathcal{S}$. Then, $\mathcal{S} \in \{\hat{\mathcal{S}} \mid \mathsf{eval}_{\hat{\mathcal{S}}}(\texttt{true}) = \texttt{true}\}$ for every $\mathcal{S}$. Then, by Defn. 13, $\mathcal{S} \in [\![\texttt{true}]\!]$. Then, by Defn. 12, $\mathcal{S} \in [\![\texttt{true}^{+}]\!]$. Then, $\mathcal{S} \in [\![(\bigwedge\{e_{\hat{r}}\}_{\hat{r}\in R} \restriction r)^{+}]\!]$. Then, by Lem. 11, $\mathcal{S} \in [\![\bigwedge\{e_{\hat{r}}\}^{+}_{\hat{r}\in R} \restriction r]\!]$. Then, $\mathcal{S} \in [\![\xi^{+} \restriction r]\!]$.
   • **Base:** $\xi = \bigwedge\{\neg e_r\}_{r\in R}$, for some $\{e_r\}_{r\in R}$. Similar to case "$\xi = \bigwedge\{e_r\}_{r\in R}$, for some $\{e_r\}_{r\in R}$". □
   – **Case:** $\mathcal{S} \in [\![\xi^{-}]\!]$. Similar to case $\mathcal{S} \in [\![\xi^{+}]\!]$. □

Proof (of Lem. 28).

1. By Defn. 5:
   – **Base:** $\xi = \texttt{true}$.
   Recall $r \notin \emptyset$. Then, by Lem. 4, $r \notin \mathsf{subj}(\texttt{true})$. Then, $r \notin \mathsf{subj}(\xi)$. Then, **false**.
   – **Base:** $\xi = \bigwedge\{e_{\hat{r}}\}_{\hat{r}\in R}$, for some $\{e_{\hat{r}}\}_{\hat{r}\in R}$.
   Recall:
   • **Case:** $r \in R$.

    * Recall $r \in R$. Then, by Defn. 11, $\bigwedge\{e_{\hat{r}}\}_{\hat{r}\in R} \restriction r = e_r$ and $\bigwedge\{\neg e_{\hat{r}}\}_{\hat{r}\in R} \restriction r = \neg e_r$.
    * Recall $[\![\xi^+ \restriction r]\!] \cap [\![\xi^- \restriction r]\!] = [\![\xi^+ \restriction r]\!] \cap [\![\xi^- \restriction r]\!]$. Then, $[\![\xi^+ \restriction r]\!] \cap [\![\xi^- \restriction r]\!] = [\![\bigwedge\{e_{\hat{r}}\}^+_{\hat{r}\in R} \restriction r]\!] \cap [\![\bigwedge\{e_{\hat{r}}\}^-_{\hat{r}\in R} \restriction r]\!]$. Then, by Defn. 12, $[\![\xi^+ \restriction r]\!] \cap [\![\xi^- \restriction r]\!] = [\![\bigwedge\{e_{\hat{r}}\}_{\hat{r}\in R} \restriction r]\!] \cap [\![\bigwedge\{\neg e_{\hat{r}}\}_{\hat{r}\in R} \restriction r]\!]$. Then, $[\![\xi^+ \restriction r]\!] \cap [\![\xi^- \restriction r]\!] = [\![e_r]\!] \cap [\![\neg e_r]\!]$. Then, by Defn. 13, $[\![\xi^+ \restriction r]\!] \cap [\![\xi^- \restriction r]\!] = [\![e_r]\!] \cap ([\![\texttt{true}]\!] \setminus [\![e_r]\!])$. Then, $[\![\xi^+ \restriction r]\!] \cap [\![\xi^- \restriction r]\!] = \emptyset$. Then, by Lem. 15, $[\![\xi^+ \restriction r]\!] \cap [\![\xi^- \restriction r]\!] = [\![\texttt{false}]\!]$.
  - **Case:** $r \notin R$.
    * Recall $\xi = \bigwedge\{e_{\hat{r}}\}_{\hat{r}\in R}$. Then, by Rem. 1, $\{\hat{\hat{r}} \mid \hat{\hat{r}}.\hat{\hat{x}} \in \mathsf{rx}(e_{\hat{r}})\} = \{\hat{r}\}$ for every $\hat{r} \in R$.
    * By Lem. 40, $\mathsf{subj}(\bigwedge\{e_{\hat{r}}\}_{\hat{r}\in R}) = \bigcup\{\mathsf{subj}(e_{\hat{r}})\}_{\hat{r}\in R}$. Then, $\mathsf{subj}(\xi) = \bigcup\{\mathsf{subj}(e_{\hat{r}})\}_{\hat{r}\in R}$. Then, by Defn. 10, $\mathsf{subj}(\xi) = \bigcup\{\{\hat{\hat{r}} \mid \hat{\hat{r}}.\hat{\hat{x}} \in \mathsf{rx}(e_{\hat{r}})\}\}_{\hat{r}\in R}$. Then, $\mathsf{subj}(\xi) = \bigcup\{\{\hat{r}\}\}_{\hat{r}\in R}$. Then, $\mathsf{subj}(\xi) = \{\hat{r} \mid \hat{r} \in R\}$. Then, $\mathsf{subj}(\xi) = R$.
    * Recall $r \notin R$. Then, $r \notin \mathsf{subj}(\xi)$. Then, **false**.
- **Base:** $\xi = \bigwedge\{\neg e_{\hat{r}}\}_{\hat{r}\in R}$, for some $\{e_{\hat{r}}\}_{\hat{r}\in R}$. Similar to case "$\xi = \bigwedge\{e_{\hat{r}}\}_{\hat{r}\in R}$, for some $\{e_{\hat{r}}\}_{\hat{r}\in R}$". □

2. By Defn. 5:
- **Base:** $\xi = \texttt{true}$.
Recall $r \notin \emptyset$. Then, by Lem. 4, $r \notin \mathsf{subj}(\texttt{true})$. Then, $r \notin \mathsf{subj}(\xi)$. Then, **false**.
- **Base:** $\xi = \bigwedge\{e_{\hat{r}}\}_{\hat{r}\in R}$, for some $\{e_{\hat{r}}\}_{\hat{r}\in R}$.
By Defn. 11:
  - **Case:** $\bigwedge\{e_{\hat{r}}\}_{\hat{r}\in R} \restriction r = e_r$ and $r \in R$.
    * Recall $r \in R$. Then, $R = (R \setminus \{r\}) \cup \{r\}$.
    * Recall $[\![\xi^\equiv]\!] \cap [\![\xi^+ \restriction r]\!] \subseteq [\![\xi^\equiv]\!] \cap [\![\xi^+ \restriction r]\!]$. Then, $[\![\xi^\equiv]\!] \cap [\![\xi^+ \restriction r]\!] \subseteq [\![\bigwedge\{e_{\hat{r}}\}^\equiv_{\hat{r}\in R}]\!] \cap [\![\bigwedge\{e_{\hat{r}}\}^+_{\hat{r}\in R} \restriction r]\!]$. Then, by Defn. 12, $[\![\xi^\equiv]\!] \cap [\![\xi^+ \restriction r]\!] \subseteq [\![\bigwedge\{e_{r_1} \Rightarrow e_{r_2}\}_{r_1,r_2\in R}]\!] \cap [\![\bigwedge\{e_{\hat{r}}\}_{\hat{r}\in R} \restriction r]\!]$. Then, $[\![\xi^\equiv]\!] \cap [\![\xi^+ \restriction r]\!] \subseteq [\![\bigwedge\{e_{r_1} \Rightarrow e_{r_2}\}_{r_1,r_2\in R}]\!] \cap [\![e_r]\!]$. Then, by Defn. 13, $[\![\xi^\equiv]\!] \cap [\![\xi^+ \restriction r]\!] \subseteq \bigcap\{[\![e_{r_1} \Rightarrow e_{r_2}]\!]\}_{r_1,r_2\in R} \cap [\![e_r]\!]$. Then, $[\![\xi^\equiv]\!] \cap [\![\xi^+ \restriction r]\!] \subseteq \bigcap\{[\![e_{r_1} \Rightarrow e_{r_2}]\!]\}_{r_1\in(R\setminus\{r\})\cup\{r\},r_2\in R} \cap [\![e_r]\!]$. Then, $[\![\xi^\equiv]\!] \cap [\![\xi^+ \restriction r]\!] \subseteq \bigcap\{[\![e_{r_1} \Rightarrow e_{r_2}]\!]\}_{r_1\in R\setminus\{r\},r_2\in R} \cap \bigcap\{[\![e_r \Rightarrow e_{r_2}]\!]\}_{r_2\in R} \cap [\![e_r]\!]$. Then, by Lem. 19, $[\![\xi^\equiv]\!] \cap [\![\xi^+ \restriction r]\!] \subseteq \bigcap\{[\![e_{r_1} \Rightarrow e_{r_2}]\!]\}_{r_1\in R\setminus\{r\},r_2\in R} \cap \bigcap\{[\![e_{r_2}]\!]\}_{r_2\in R} \cap [\![e_r]\!]$. Then, $[\![\xi^\equiv]\!] \cap [\![\xi^+ \restriction r]\!] \subseteq \bigcap\{[\![e_{r_2}]\!]\}_{r_2\in R}$. Then, by Defn. 13, $[\![\xi^\equiv]\!] \cap [\![\xi^+ \restriction r]\!] \subseteq [\![\bigwedge\{e_{r_2}\}_{r_2\in R}]\!]$. Then, by Defn. 12, $[\![\xi^\equiv]\!] \cap [\![\xi^+ \restriction r]\!] \subseteq [\![\bigwedge\{e_{r_2}\}^+_{r_2\in R}]\!]$. Then, $[\![\xi^\equiv]\!] \cap [\![\xi^+ \restriction r]\!] \subseteq [\![\xi^+]\!]$.
  - **Case:** $r \notin R$.
    * Recall $\xi = \bigwedge\{e_{\hat{r}}\}_{\hat{r}\in R}$. Then, by Rem. 1, $\{\hat{\hat{r}} \mid \hat{\hat{r}}.\hat{\hat{x}} \in \mathsf{rx}(e_{\hat{r}})\} = \{\hat{r}\}$ for every $\hat{r} \in R$.
    * By Lem. 40, $\mathsf{subj}(\bigwedge\{e_{\hat{r}}\}_{\hat{r}\in R}) = \bigcup\{\mathsf{subj}(e_{\hat{r}})\}_{\hat{r}\in R}$. Then, $\mathsf{subj}(\xi) = \bigcup\{\mathsf{subj}(e_{\hat{r}})\}_{\hat{r}\in R}$. Then, by Defn. 10, $\mathsf{subj}(\xi) = \bigcup\{\{\hat{\hat{r}} \mid \hat{\hat{r}}.\hat{\hat{x}} \in \mathsf{rx}(e_{\hat{r}})\}\}_{\hat{r}\in R}$. Then, $\mathsf{subj}(\xi) = \bigcup\{\{\hat{r}\}\}_{\hat{r}\in R}$. Then, $\mathsf{subj}(\xi) = \{\hat{r} \mid \hat{r} \in R\}$. Then, $\mathsf{subj}(\xi) = R$.
    * Recall $r \notin R$. Then, $r \notin \mathsf{subj}(\xi)$. Then, **false**.
- **Base:** $\xi = \bigwedge\{\neg e_{\hat{r}}\}_{\hat{r}\in R}$, for some $\{e_{\hat{r}}\}_{\hat{r}\in R}$. Similar to case "$\xi = \bigwedge\{e_{\hat{r}}\}_{\hat{r}\in R}$, for some $\{e_{\hat{r}}\}_{\hat{r}\in R}$". □

3. By Defn. 5:
- **Base:** $\xi = \texttt{true}$.
Recall $r \notin \emptyset$. Then, by Lem. 4, $r \notin \mathsf{subj}(\texttt{true})$. Then, $r \notin \mathsf{subj}(\xi)$. Then, **false**.
- **Base:** $\xi = \bigwedge\{e_{\hat{r}}\}_{\hat{r}\in R}$, for some $\{e_{\hat{r}}\}_{\hat{r}\in R}$.
  - Recall $\xi = \bigwedge\{e_{\hat{r}}\}_{\hat{r}\in R}$. Then, by Rem. 1, $\{\hat{\hat{r}} \mid \hat{\hat{r}}.\hat{\hat{x}} \in \mathsf{rx}(e_{\hat{r}})\} = \{\hat{r}\}$ for every $\hat{r} \in R$.
  - By Lem. 40, $\mathsf{subj}(\bigwedge\{e_{\hat{r}}\}_{\hat{r}\in R}) = \bigcup\{\mathsf{subj}(e_{\hat{r}})\}_{\hat{r}\in R}$. Then, $\mathsf{subj}(\xi) = \bigcup\{\mathsf{subj}(e_{\hat{r}})\}_{\hat{r}\in R}$. Then, by Defn. 10, $\mathsf{subj}(\xi) = \bigcup\{\{\hat{\hat{r}} \mid \hat{\hat{r}}.\hat{\hat{x}} \in \mathsf{rx}(e_{\hat{r}})\}\}_{\hat{r}\in R}$. Then, $\mathsf{subj}(\xi) = \bigcup\{\{\hat{r}\}\}_{\hat{r}\in R}$. Then, $\mathsf{subj}(\xi) = \{\hat{r} \mid \hat{r} \in R\}$. Then, $\mathsf{subj}(\xi) = R$.
  - Recall $r \in \mathsf{subj}(\xi)$. Then, by 1, $[\![\xi^\equiv]\!] \cap [\![\xi^+ \restriction r]\!] \subseteq [\![\xi^+]\!]$. Then, $[\![\xi^\equiv]\!] \cap [\![\xi^+ \restriction r]\!] \subseteq [\![\bigwedge\{e_{\hat{r}}\}^+_{\hat{r}\in R}]\!]$. Then, by Defn. 12, $[\![\xi^\equiv]\!] \cap [\![\xi^+ \restriction r]\!] \subseteq [\![\bigwedge\{e_{\hat{r}}\}_{\hat{r}\in R}]\!]$. Then, by Defn. 13, $[\![\xi^\equiv]\!] \cap [\![\xi^+ \restriction r]\!] \subseteq \bigcup\{[\![e_{\hat{r}}]\!]\}_{\hat{r}\in R}$. Then, by Defn. 11, $[\![\xi^\equiv]\!] \cap [\![\xi^+ \restriction r]\!] \subseteq \bigcup\{[\![\bigwedge\{e_{\hat{\hat{r}}}\}_{\hat{\hat{r}}\in R} \restriction \hat{r}]\!]\}_{\hat{r}\in R}$. Then,

$[\![\xi^{\equiv}]\!] \cap [\![\xi^{+} \restriction r]\!] \subseteq \bigcup\{[\![\bigwedge\{e_{\hat{r}}\}_{\hat{r}\in R} \restriction \hat{r}]\!]\}_{\hat{r}\in R\setminus\{r\}}$. Then, by Defn. 13,
$[\![\xi^{\equiv}]\!] \cap [\![\xi^{+} \restriction r]\!] \subseteq [\![\bigwedge\{\bigwedge\{e_{\hat{r}}\}_{\hat{r}\in R} \restriction \hat{r}\}_{\hat{r}\in R\setminus\{r\}}]\!]$. Then, by Defn. 12,
$[\![\xi^{\equiv}]\!] \cap [\![\xi^{+} \restriction r]\!] \subseteq [\![\bigwedge\{\bigwedge\{e_{\hat{r}}\}^{+}_{\hat{r}\in R} \restriction \hat{r}\}_{\hat{r}\in R\setminus\{r\}}]\!]$. Then,
$[\![\xi^{\equiv}]\!] \cap [\![\xi^{+} \restriction r]\!] \subseteq [\![\bigwedge\{\xi^{+} \restriction \hat{r}\}_{\hat{r}\in \mathsf{subj}(\xi)\setminus\{r\}}]\!]$.

– **Base:** $\xi = \bigwedge\{\neg e_{\hat{r}}\}_{\hat{r}\in R}$, for some $\{e_{\hat{r}}\}_{\hat{r}\in R}$. Similar to case "$\xi = \bigwedge\{e_{\hat{r}}\}_{\hat{r}\in R}$, for some $\{e_{\hat{r}}\}_{\hat{r}\in R}$". □

4. By Defn. 5:
   – **Base:** $\xi = \mathtt{true}$.
   Recall $r \notin \emptyset$. Then, by Lem. 4, $r \notin \mathsf{subj}(\mathtt{true})$. Then, $r \notin \mathsf{subj}(\xi)$. Then, **false**.
   – **Base:** $\xi = \bigwedge\{e_{\hat{r}}\}_{\hat{r}\in R}$, for some $\{e_{\hat{r}}\}_{\hat{r}\in R}$.
   By Defn. 11:
   - **Case:** $\bigwedge\{\neg e_{\hat{r}}\}_{\hat{r}\in R} \restriction r = \neg e_r$ and $r \in R$.
     * Recall $r \in R$. Then, $R = (R \setminus \{r\}) \cup \{r\}$.
     * Recall $[\![\xi^{\equiv}]\!] \cap [\![\xi^{-} \restriction r]\!] \subseteq [\![\xi^{\equiv}]\!] \cap [\![\xi^{-} \restriction r]\!]$. Then,
     $[\![\xi^{\equiv}]\!] \cap [\![\xi^{-} \restriction r]\!] \subseteq [\![\bigwedge\{e_{\hat{r}}\}^{\equiv}_{\hat{r}\in R}]\!] \cap [\![\bigwedge\{e_{\hat{r}}\}^{-}_{\hat{r}\in R} \restriction r]\!]$. Then, by Defn. 12,
     $[\![\xi^{\equiv}]\!] \cap [\![\neg\xi \restriction r]\!] \subseteq [\![\bigwedge\{e_{r_1} \Rightarrow e_{r_2}\}_{r_1,r_2\in R}]\!] \cap [\![\bigwedge\{\neg e_{\hat{r}}\}_{\hat{r}\in R} \restriction r]\!]$. Then,
     $[\![\xi^{\equiv}]\!] \cap [\![\neg\xi \restriction r]\!] \subseteq [\![\bigwedge\{e_{r_1} \Rightarrow e_{r_2}\}_{r_1,r_2\in R}]\!] \cap [\![\neg e_r]\!]$. Then, by Defn. 13,
     $[\![\xi^{\equiv}]\!] \cap [\![\xi^{-} \restriction r]\!] \subseteq \bigcap\{[\![e_{r_1} \Rightarrow e_{r_2}]\!]\}_{r_1,r_2\in R} \cap [\![\neg e_r]\!]$. Then, by Lem. 17,
     $[\![\xi^{\equiv}]\!] \cap [\![\xi^{-} \restriction r]\!] \subseteq \bigcap\{[\![\neg e_{r_2} \Rightarrow \neg e_{r_1}]\!]\}_{r_1,r_2\in R} \cap [\![\neg e_r]\!]$. Then,
     $[\![\xi^{\equiv}]\!] \cap [\![\xi^{-} \restriction r]\!] \subseteq \bigcap\{[\![\neg e_{r_1} \Rightarrow \neg e_{r_2}]\!]\}_{r_1\in R, r_2\in (R\setminus\{r\})\cup\{r\}} \cap [\![\neg e_r]\!]$. Then,
     $[\![\xi^{\equiv}]\!] \cap [\![\xi^{-} \restriction r]\!] \subseteq \bigcap\{[\![\neg e_{r_2} \Rightarrow \neg e_{r_1}]\!]\}_{r_1\in R, r_2\in R\setminus\{r\}} \cap \bigcap\{[\![\neg e_r \Rightarrow \neg e_{r_1}]\!]\}_{r_1\in R} \cap [\![\neg e_r]\!]$.
     Then, by Lem. 19,
     $[\![\xi^{\equiv}]\!] \cap [\![\xi^{-} \restriction r]\!] \subseteq \bigcap\{[\![\neg e_{r_2} \Rightarrow \neg e_{r_1}]\!]\}_{r_1\in R, r_2\in R\setminus\{r\}} \cap \bigcap\{[\![\neg e_{r_1}]\!]\}_{r_1\in R} \cap [\![\neg e_r]\!]$. Then,
     $[\![\xi^{\equiv}]\!] \cap [\![\xi^{-} \restriction r]\!] \subseteq \bigcap\{[\![\neg e_{r_1}]\!]\}_{r_1\in R}$. Then, by Defn. 13,
     $[\![\xi^{\equiv}]\!] \cap [\![\xi^{-} \restriction r]\!] \subseteq [\![\bigwedge\{\neg e_{r_1}\}_{r_1\in R}]\!]$. Then, by Defn. 12,
     $[\![\xi^{\equiv}]\!] \cap [\![\xi^{-} \restriction r]\!] \subseteq [\![\bigwedge\{e_{r_1}\}^{-}_{r_1\in R}]\!]$. Then, $[\![\xi^{\equiv}]\!] \cap [\![\xi^{-} \restriction r]\!] \subseteq [\![\xi^{-}]\!]$.
   - **Case:** $r \notin R$.
     * Recall $\xi = \bigwedge\{e_{\hat{r}}\}_{\hat{r}\in R}$. Then, by Rem. 1, $\{\hat{r} \mid \hat{r}.\hat{x} \in \mathsf{rx}(e_{\hat{r}})\} = \{\hat{r}\}$ for every $\hat{r} \in R$.
     * By Lem. 40, $\mathsf{subj}(\bigwedge\{e_{\hat{r}}\}_{\hat{r}\in R}) = \bigcup\{\mathsf{subj}(e_{\hat{r}})\}_{\hat{r}\in R}$. Then, $\mathsf{subj}(\xi) = \bigcup\{\mathsf{subj}(e_{\hat{r}})\}_{\hat{r}\in R}$.
     Then, by Defn. 10, $\mathsf{subj}(\xi) = \bigcup\{\{\hat{r} \mid \hat{r}.\hat{x} \in \mathsf{rx}(e_{\hat{r}})\}\}_{\hat{r}\in R}$. Then,
     $\mathsf{subj}(\xi) = \bigcup\{\{\hat{r}\}\}_{\hat{r}\in R}$. Then, $\mathsf{subj}(\xi) = \{\hat{r} \mid \hat{r} \in R\}$. Then, $\mathsf{subj}(\xi) = R$.
     * Recall $r \notin R$. Then, $r \notin \mathsf{subj}(\xi)$. Then, **false**.
   – **Base:** $\xi = \bigwedge\{\neg e_{\hat{r}}\}_{\hat{r}\in R}$, for some $\{e_{\hat{r}}\}_{\hat{r}\in R}$. Similar to case "$\xi = \bigwedge\{e_{\hat{r}}\}_{\hat{r}\in R}$, for some $\{e_{\hat{r}}\}_{\hat{r}\in R}$". □
5. Similar to 4. □

Proof (of Lem. 29).

1. – Recall $r \in R$. Then, $R = (R \setminus \{r\}) \cup \{r\}$.
   – Recall $r \in R \subseteq \mathsf{subj}(\xi)$. Then, $r \in \mathsf{subj}(\xi)$. Then, by Lem. 28, $[\![\xi^{+} \restriction r]\!] \cap [\![\xi^{-} \restriction r]\!] = [\![\mathtt{false}]\!]$.
   – Recall $[\![\bigwedge\{\xi^{-} \restriction \hat{r}\}_{\hat{r}\in R}]\!] \cap [\![\xi^{+} \restriction r]\!] \subseteq [\![\bigwedge\{\xi^{-} \restriction \hat{r}\}_{\hat{r}\in R}]\!] \cap [\![\xi^{+} \restriction r]\!]$. Then, by Defn. 13,
   $[\![\bigwedge\{\xi^{-} \restriction \hat{r}\}_{\hat{r}\in R}]\!] \cap [\![\xi^{+} \restriction r]\!] \subseteq \bigcap\{[\![\xi^{-} \restriction \hat{r}]\!]\}_{\hat{r}\in R} \cap [\![\xi^{+} \restriction r]\!]$. Then,
   $[\![\bigwedge\{\xi^{-} \restriction \hat{r}\}_{\hat{r}\in R}]\!] \cap [\![\xi^{+} \restriction r]\!] \subseteq \bigcap\{[\![\xi^{-} \restriction \hat{r}]\!]\}_{\hat{r}\in (R\setminus\{r\})\cup\{r\}} \cap [\![\xi^{+} \restriction r]\!]$. Then,
   $[\![\bigwedge\{\xi^{-} \restriction \hat{r}\}_{\hat{r}\in R}]\!] \cap [\![\xi^{+} \restriction r]\!] \subseteq \bigcap\{[\![\xi^{-} \restriction \hat{r}]\!]\}_{\hat{r}\in R\setminus\{r\}} \cap [\![\xi^{-} \restriction r]\!] \cap [\![\xi^{+} \restriction r]\!]$. Then,
   $[\![\bigwedge\{\xi^{-} \restriction \hat{r}\}_{\hat{r}\in R}]\!] \cap [\![\xi^{+} \restriction r]\!] \subseteq [\![\xi^{-} \restriction r]\!] \cap [\![\xi^{+} \restriction r]\!]$. Then,
   $[\![\bigwedge\{\xi^{-} \restriction \hat{r}\}_{\hat{r}\in R}]\!] \cap [\![\xi^{+} \restriction r]\!] \subseteq [\![\mathtt{false}]\!]$.
2. Similar to 2. □

C Actions

Proof (of Lem. 30). By Defn. 14:

- **Base:** $\alpha = \tau$. Then, **false**.
- **Base:** $\alpha = q.y := e$, for some q, y, e.
  Recall $\{q\} \neq \emptyset$. Then, by Defn. 16, $\mathsf{subj}(q.y := e) \neq \emptyset$. Then, $\mathsf{subj}(\alpha) \neq \emptyset$.
- **Base:** $\alpha = p.e \twoheadrightarrow q.y$, for some p, q, y, e. Similar to case "$\alpha = q.y := e$, for some q, y, e".
- **Base:** $\alpha = i^R_R$, for some R.
  Recall $\alpha = i^R_R$. Then, by Rem. 2, $R \neq \emptyset$. Then, by Defn. 16, $\mathsf{subj}(i^R_R) \neq \emptyset$. Then, $\mathsf{subj}(\alpha) \neq \emptyset$.
- **Base:** $\alpha = pq!e$, for some p, q, e. Similar to case "$\alpha = q.y := e$, for some q, y, e".
- **Base:** $\alpha = pq?y$, for some p, q, y. Similar to case "$\alpha = q.y := e$, for some q, y, e".
- **Base:** $\alpha = i^R_{\{r\}}$, for some p, q, y. Similar to case "$\alpha = q.y := e$, for some q, y, e". □

Proof (of Lem. 31).

1. By Defn. 14:
   - **Base:** $\alpha = \tau$.
     Recall $\emptyset \subseteq \{r.x \mid r \in \mathsf{subj}(\alpha)\}$. Then, by Defn. 15, $\mathsf{read}(\tau) \subseteq \{r.x \mid r \in \mathsf{subj}(\alpha)\}$. Then, $\mathsf{read}(\alpha) \subseteq \{r.x \mid r \in \mathsf{subj}(\alpha)\}$.
   - **Base:** $\alpha = q.y := e$, for some q, y, e.
     Recall $\alpha = q.y := e$. Then, by Rem. 2, $\{r \mid r.x \in \mathsf{rx}(e)\} \subseteq \{q\}$. Then, $\hat{r} \in \{q\}$ for every $\hat{r} \in \{r \mid r.x \in \mathsf{rx}(e)\}$. Then, $\hat{r} \in \{q\}$ for every $\hat{r}.x \in \mathsf{rx}(e)$. Then, $\mathsf{rx}(e) \subseteq \{r.x \mid r \in \{q\}\}$. Then, by Defn. 15, $\mathsf{read}(q.y := e) \subseteq \{r.x \mid r \in \{q\}\}$. Then, by Defn. 16, $\mathsf{read}(q.y := e) \subseteq \{r.x \mid r \in \mathsf{subj}(q.y := e)\}$. Then, $\mathsf{read}(\alpha) \subseteq \{r.x \mid r \in \mathsf{subj}(\alpha)\}$.
   - **Base:** $\alpha = p.e \twoheadrightarrow q.y$, for some p, q, y, e.
     Recall $\alpha = p.e \twoheadrightarrow q.y$. Then, by Rem. 2, $\{r \mid r.x \in \mathsf{rx}(e)\} \subseteq \{p\}$. Then, $\{r \mid r.x \in \mathsf{rx}(e)\} \subseteq \{p, q\}$. Then, $\hat{r} \in \{p, q\}$ for every $\hat{r} \in \{r \mid r.x \in \mathsf{rx}(e)\}$. Then, $\hat{r} \in \{p, q\}$ for every $\hat{r}.x \in \mathsf{rx}(e)$. Then, $\mathsf{rx}(e) \subseteq \{r.x \mid r \in \{p, q\}\}$. Then, by Defn. 15, $\mathsf{read}(q.y := e) \subseteq \{r.x \mid r \in \{p, q\}\}$. Then, by Defn. 16, $\mathsf{read}(q.y := e) \subseteq \{r.x \mid r \in \mathsf{subj}(q.y := e)\}$. Then, $\mathsf{read}(\alpha) \subseteq \{r.x \mid r \in \mathsf{subj}(\alpha)\}$.
   - **Base:** $\alpha = i^R_R$, for some R. Similar to case $\alpha = \tau$.
   - **Base:** $\alpha = pq!e$, for some p, q, e. Similar to case "$\alpha = q.y := e$, for some q, y, e".
   - **Base:** $\alpha = pq?y$, for some p, q, y. Similar to case $\alpha = \tau$.
   - **Base:** $\alpha = i^R_R r$, for some R, r. Similar to case $\alpha = \tau$. □
2. By Defn. 14:
   - **Base:** $\alpha = \tau$.
     Recall $\emptyset \subseteq \{r.x \mid r \in \mathsf{subj}(\alpha)\}$. Then, by Defn. 15, $\mathsf{write}(\tau) \subseteq \{r.x \mid r \in \mathsf{subj}(\alpha)\}$. Then, $\mathsf{write}(\alpha) \subseteq \{r.x \mid r \in \mathsf{subj}(\alpha)\}$.
   - **Base:** $\alpha = q.y := e$, for some q, y, e.
     Recall $\{q.y\} \subseteq \{q.y\}$. Then, $\{q.y\} \subseteq \{r.x \mid r \in \{q\}\}$. Then, by Defn. 15, $\mathsf{write}(q.y := e) \subseteq \{r.x \mid r \in \{q\}\}$. Then, by Defn. 16, $\mathsf{write}(q.y := e) \subseteq \{r.x \mid r \in \mathsf{subj}(q.y := e)\}$. Then, $\mathsf{write}(\alpha) \subseteq \{r.x \mid r \in \mathsf{subj}(\alpha)\}$.
   - **Base:** $\alpha = i^R_R$, for some R. Similar to case $\alpha = \tau$.
   - **Base:** $\alpha = p.e \twoheadrightarrow q.y$, for some p, q, y, e. Similar to case "$\alpha = q.y := e$, for some q, y, e".
   - **Base:** $\alpha = pq!e$, for some p, q, e. Similar to case $\alpha = \tau$.
   - **Base:** $\alpha = pq?y$, for some p, q, y. Similar to case "$\alpha = q.y := e$, for some q, y, e".
   - **Base:** $\alpha = i^R_{\{r\}}$, for some R, r. Similar to case $\alpha = \tau$. □

Proof (of Lem. 32).

1. By Defn. 14:
   - **Base:** $\gamma = \tau$.
     Recall $r \notin \emptyset$. Then, by Defn. 16, $r \notin \mathsf{subj}(\tau)$. Then, $r \notin \mathsf{subj}(\gamma)$. Then, **false**.

- **Base:** $\gamma = q.y{:=}e$, for some q, y, e.
  By Defn. 18:
  - **Case:** $q.y{:=}e \restriction r = q.y{:=}e$ and $r = q$.
    Recall $q.y{:=}e \restriction r = q.y{:=}e$. Then, $\mathsf{subj}(q.y{:=}e \restriction r) = \mathsf{subj}(q.y{:=}e)$. Then, by Defn. 16, $\mathsf{subj}(q.y{:=}e \restriction r) = \{q\}$. Then, $\mathsf{subj}(\gamma \restriction r) = \{r\}$.
  - **Case:** $r \neq q$. Then, $r \notin \{q\}$. Then, by Defn. 16, $r \notin \mathsf{subj}(q.y{:=}e)$. Then, $r \notin \mathsf{subj}(\gamma)$. Then, **false**.
- **Base:** $\gamma = p.e \twoheadrightarrow q.y$, for some p, q, y, e.
  By Defn. 18:
  - **Case:** $p.e \twoheadrightarrow q.y \restriction r = pq!e$ and $p = r \neq q$.
    Recall $p.e \twoheadrightarrow q.y \restriction r = pq!e$. Then, $\mathsf{subj}(p.e \twoheadrightarrow q.y \restriction r) = \mathsf{subj}(pq!e)$. Then, by Defn. 16, $\mathsf{subj}(p.e \twoheadrightarrow q.y \restriction r) = \{p\}$. Then, $\mathsf{subj}(\gamma \restriction r) = \{r\}$.
  - **Case:** $p.e \twoheadrightarrow q.y \restriction r = pq?y$ and $p \neq r = q$. Similar to case "$p.e \twoheadrightarrow q.y \restriction r = pq!e$ and $p = r \neq q$".
  - **Case:** $p.e \twoheadrightarrow q.y \restriction r = \tau$ and $p \neq r \neq q$.
    Recall $p \neq r \neq q$. Then, $r \notin \{p, q\}$. Then, by Defn. 16, $r \notin \mathsf{subj}(p.e \twoheadrightarrow q.y)$. Then, $r \notin \mathsf{subj}(\gamma)$. Then, **false**.
- **Base:** $\gamma = i^R_R$, for some R.
  By Defn. 18:
  - **Case:** $i^R_R \restriction r = i^R_{\{r\}}$. Then, $\mathsf{subj}(i^R_R \restriction r) = \mathsf{subj}(i^R_{\{r\}})$. Then, by Defn. 16, $\mathsf{subj}(i^R_R \restriction r) = \{r\}$. Then, $\mathsf{subj}(\gamma \restriction r) = \{r\}$.
  - **Case:** $r \notin R$. Then, by Defn. 16, $r \notin \mathsf{subj}(i^R_R)$. Then, $r \notin \mathsf{subj}(\gamma)$. Then, **false**. □

2. By Defn. 14:
   - **Base:** $\gamma = \tau$.
     By Defn. 16, $\mathsf{subj}(\tau) = \emptyset$. Then, by Defn. 18, $\mathsf{subj}(\tau \restriction r) = \emptyset$. Then, $\mathsf{subj}(\gamma \restriction r) = \emptyset$.
   - **Base:** $\gamma = q.y{:=}e$, for some q, y, e.
     Recall $r \notin \mathsf{subj}(\gamma)$. Then, $r \notin \mathsf{subj}(q.y{:=}e)$. Then, by Defn. 16, $r \notin \{q\}$. Then, $r \neq q$. Then, by Defn. 18, $q.y{:=}e \restriction r = \tau$. Then, $\mathsf{subj}(q.y{:=}e \restriction r) = \mathsf{subj}(\tau)$. Then, by Defn. 16, $\mathsf{subj}(q.y{:=}e \restriction r) = \emptyset$. Then, $\mathsf{subj}(\gamma \restriction r) = \emptyset$.
   - **Base:** $\gamma = p.e \twoheadrightarrow q.y$, for some p, q, y, e. Similar to case "$\gamma = q.y{:=}e$, for some q, y, e".
   - **Base:** $\gamma = i^R_R$, for some R. Similar to case "$\gamma = q.y{:=}e$, for some q, y, e". □

Proof (of Lem. 33). Recall:

- **Case:** $r \in \mathsf{subj}(\gamma)$.
  - Recall $r \in \mathsf{subj}(\gamma)$. Then, by Lem. 32, $\mathsf{subj}(\gamma \restriction r) = \{r\}$.
  - Recall $r \in \mathsf{subj}(\gamma)$. Then, $\{r\} \subseteq \mathsf{subj}(\gamma)$. Then, $\mathsf{subj}(\gamma \restriction r) \subseteq \mathsf{subj}(\gamma)$.
- **Case:** $r \notin \mathsf{subj}(\gamma)$.
  - Recall $r \notin \mathsf{subj}(\gamma)$. Then, by Lem. 32, $\mathsf{subj}(\gamma \restriction r) = \emptyset$.
  - Recall $\emptyset \subseteq \mathsf{subj}(\gamma)$. Then, $\mathsf{subj}(\gamma \restriction r) \subseteq \mathsf{subj}(\gamma)$. □

Proof (of Lem. 34). By Defn. 14:

- **Base:** $\gamma = \tau$.
  By Defn. 16, $\tau \restriction r = \tau$. Then, $\gamma \restriction r = \tau$.
- **Base:** $\gamma = q.y{:=}e$, for some q, y, e.
  Recall $r \notin \mathsf{subj}(\gamma)$. Then, $r \notin \mathsf{subj}(q.y{:=}e)$. Then, by Defn. 16, $r \notin \{q\}$. Then, $r \neq q$. Then, by Defn. 18, $q.y{:=}e \restriction r = \tau$. Then, $\alpha \restriction r = \tau$.
- **Base:** $\gamma = p.e \twoheadrightarrow q.y$, for some p, q, y, e. Similar to case "$\gamma = q.y{:=}e$, for some q, y, e".
- **Base:** $\gamma = i^R_R$, for some R. Similar to case "$\gamma = q.y{:=}e$, for some q, y, e". □

Proof (of Lem. 35). By Defn. 14:

- **Base:** $\gamma = \tau$.
  Recall $\emptyset \subseteq \mathsf{chan}(\gamma)$. Then, by Defn. 17, $\mathsf{chan}(\tau) \subseteq \mathsf{chan}(\gamma)$. Then, by Defn. 18, $\mathsf{chan}(\tau \upharpoonright r) \subseteq \mathsf{chan}(\gamma)$. Then, $\mathsf{chan}(\gamma \upharpoonright r) \subseteq \mathsf{chan}(\gamma)$.
- **Base:** $\gamma = q.y := e$, for some q, y, e.
  By Defn. 18:
  - **Case:** $q.y := e \upharpoonright r = q.y := e$.
    Recall $\emptyset \subseteq \mathsf{chan}(\gamma)$. Then, by Defn. 17, $\mathsf{chan}(q.y := e) \subseteq \mathsf{chan}(\gamma)$. Then, $\mathsf{chan}(q.y := e \upharpoonright r) \subseteq \mathsf{chan}(\gamma)$. Then, $\mathsf{chan}(\gamma \upharpoonright r) \subseteq \mathsf{chan}(\gamma)$.
  - **Case:** $q.y := e \upharpoonright r = \tau$. Similar to case $q.y := e \upharpoonright r = q.y := e$.
- **Base:** $\gamma = p.e \twoheadrightarrow q.y$, for some p, q, y, e.
  By Defn. 18:
  - **Case:** $p.e \twoheadrightarrow q.y \upharpoonright r = pq!e$.
    Recall $\{pq\} \subseteq \{pq\}$. Then, by Defn. 17, $\mathsf{chan}(pq!e) \subseteq \mathsf{chan}(p.e \twoheadrightarrow q.y)$. Then, $\mathsf{chan}(p.e \twoheadrightarrow q.y \upharpoonright r) \subseteq \mathsf{chan}(p.e \twoheadrightarrow q.y)$. Then, $\mathsf{chan}(\gamma \upharpoonright r) \subseteq \mathsf{chan}(\gamma)$.
  - **Case:** $p.e \twoheadrightarrow q.y \upharpoonright r = pq?y$. Similar to case $p.e \twoheadrightarrow q.y \upharpoonright r = pq!e$.
  - **Case:** $p.e \twoheadrightarrow q.y \upharpoonright r = \tau$.
    Recall $\emptyset \subseteq \mathsf{chan}(\gamma)$. Then, by Defn. 17, $\mathsf{chan}(\tau) \subseteq \mathsf{chan}(\gamma)$. Then, $\mathsf{chan}(p.e \twoheadrightarrow q.y \upharpoonright r) \subseteq \mathsf{chan}(\gamma)$. Then, $\mathsf{chan}(\gamma \upharpoonright r) \subseteq \mathsf{chan}(\gamma)$.
- **Base:** $\gamma = i^R_R$, for some R.
  By Defn. 18:
  - **Case:** $i^R_R \upharpoonright r = i^R_{\{r\}}$.
    Recall $\{R\} \subseteq \{R\}$. Then, by Defn. 17, $\mathsf{chan}(i^R_{\{r\}}) \subseteq \mathsf{chan}(i^R_R)$. Then, $\mathsf{chan}(i^R_R \upharpoonright r) \subseteq \mathsf{chan}(i^R_R)$. Then, $\mathsf{chan}(\gamma \upharpoonright r) \subseteq \mathsf{chan}(\gamma)$.
  - **Case:** $i^R_R \upharpoonright r = \tau$.
    Recall $\emptyset \subseteq \mathsf{chan}(\gamma)$. Then, by Defn. 17, $\mathsf{chan}(\tau) \subseteq \mathsf{chan}(\gamma)$. Then, $\mathsf{chan}(i^R_R \upharpoonright r) \subseteq \mathsf{chan}(\gamma)$. Then, $\mathsf{chan}(\gamma \upharpoonright r) \subseteq \mathsf{chan}(\gamma)$. □

D Programs

Proof (of Lem. 36).

1. By Defn. 22, $\mathsf{read}(\mathbf{if}\ \xi|_n\ A_1|_{R_1}\ A_2|_{R_2}) = (\mathsf{read}(\xi) \setminus \{r.x \mid r \in R_1 \cup R_2\}) \cup (\mathsf{read}(A_1) \setminus \{r.x \mid r \in R_2\}) \cup (\mathsf{read}(A_2) \setminus \{r.x \mid r \in R_1\})$. Then, $\mathsf{read}(\mathbf{if}\ \xi|_n\ A_1|_{R_1}\ A_2|_{R_2}) = (\mathsf{read}(\xi) \setminus \{r.x \mid r \in \emptyset \cup \emptyset\}) \cup (\mathsf{read}(A_1) \setminus \{r.x \mid r \in \emptyset\}) \cup (\mathsf{read}(A_2) \setminus \{r.x \mid r \in \emptyset\})$. Then, $\mathsf{read}(\mathbf{if}\ \xi|_n\ A_1|_{R_1}\ A_2|_{R_2}) = (\mathsf{read}(\xi) \setminus \{r.x \mid r \in \emptyset\}) \cup (\mathsf{read}(A_1) \setminus \{r.x \mid r \in \emptyset\}) \cup (\mathsf{read}(A_2) \setminus \{r.x \mid r \in \emptyset\})$. Then, $\mathsf{read}(\mathbf{if}\ \xi|_n\ A_1|_{R_1}\ A_2|_{R_2}) = (\mathsf{read}(\xi) \setminus \emptyset) \cup (\mathsf{read}(A_1) \setminus \emptyset) \cup (\mathsf{read}(A_2) \setminus \emptyset)$. Then, $\mathsf{read}(\mathbf{if}\ \xi|_n\ A_1|_{R_1}\ A_2|_{R_2}) = \mathsf{read}(\xi) \cup \mathsf{read}(A_1) \cup \mathsf{read}(A_2)$. □
2. By Defn. 22, $\mathsf{read}(\mathbf{if}\ \xi|_n\ A_1|_{R_1}\ A_2|_{R_2}) = (\mathsf{read}(\xi) \setminus \{r.x \mid r \in R_1 \cup R_2\}) \cup (\mathsf{read}(A_1) \setminus \{r.x \mid r \in R_2\}) \cup (\mathsf{read}(A_2) \setminus \{r.x \mid r \in R_1\})$. Then, $\mathsf{read}(\mathbf{if}\ \xi|_n\ A_1|_{R_1}\ A_2|_{R_2}) = (\mathsf{read}(\xi) \setminus \{r.x \mid r \in \emptyset \cup R_2\}) \cup (\mathsf{read}(A_1) \setminus \{r.x \mid r \in R_2\}) \cup (\mathsf{read}(A_2) \setminus \{r.x \mid r \in \emptyset\})$. Then, $\mathsf{read}(\mathbf{if}\ \xi|_n\ A_1|_{R_1}\ A_2|_{R_2}) = (\mathsf{read}(\xi) \setminus \{r.x \mid r \in R_2\}) \cup (\mathsf{read}(A_1) \setminus \{r.x \mid r \in R_2\}) \cup (\mathsf{read}(A_2) \setminus \{r.x \mid r \in \emptyset\})$. Then, $\mathsf{read}(\mathbf{if}\ \xi|_n\ A_1|_{R_1}\ A_2|_{R_2}) = (\mathsf{read}(\xi) \setminus \{r.x \mid r \in R_2\}) \cup (\mathsf{read}(A_1) \setminus \{r.x \mid r \in R_2\}) \cup (\mathsf{read}(A_2) \setminus \emptyset)$. Then, $\mathsf{read}(\mathbf{if}\ \xi|_n\ A_1|_{R_1}\ A_2|_{R_2}) = (\mathsf{read}(\xi) \setminus \{r.x \mid r \in R_2\}) \cup (\mathsf{read}(A_1) \setminus \{r.x \mid r \in R_2\}) \cup \mathsf{read}(A_2)$. □
3. Similar to 2. □
4. Similar to 1. □
5. Similar to 2. □
6. Similar to 2. □

Proof (of Lem. 37). Recall $A_1 \# A_2$. Then, by Defn. 23, $\mathsf{read}(A_1) \cap \mathsf{read}(A_2) = \emptyset$ and $\mathsf{read}(A_1) \cap \mathsf{write}(A_2) = \emptyset$ and $\mathsf{write}(A_1) \cap \mathsf{read}(A_2) = \emptyset$ and $\mathsf{write}(A_1) \cap \mathsf{write}(A_2) = \emptyset$. Then, $\mathsf{read}(A_2) \cap \mathsf{read}(A_1) = \emptyset$ and $\mathsf{write}(A_2) \cap \mathsf{read}(A_1) = \emptyset$ and $\mathsf{read}(A_2) \cap \mathsf{write}(A_1) = \emptyset$ and $\mathsf{write}(A_2) \cap \mathsf{write}(A_1) = \emptyset$. Then, by [#], $A_2 \# A_1$. □

Proof (of Lem. 38).

1. Recall $A_1 \circ A_2 \# A$. Then, by Defn. 23, $\mathsf{read}(A_1 \circ A_2) \cap \mathsf{read}(A) = \emptyset$ and $\mathsf{read}(A_1 \circ A_2) \cap \mathsf{write}(A) = \emptyset$ and $\mathsf{write}(A_1 \circ A_2) \cap \mathsf{read}(A) = \emptyset$ and $\mathsf{write}(A_1 \circ A_2) \cap \mathsf{write}(A) = \emptyset$. Then, by Defn. 22, $(\mathsf{read}(A_1) \cup \mathsf{read}(A_2)) \cap \mathsf{read}(A) = \emptyset$ and $(\mathsf{read}(A_1) \cup \mathsf{read}(A_2)) \cap \mathsf{write}(A) = \emptyset$ and $(\mathsf{write}(A_1) \cup \mathsf{write}(A_2)) \cap \mathsf{read}(A) = \emptyset$ and $(\mathsf{write}(A_1) \cup \mathsf{write}(A_2)) \cap \mathsf{write}(A) = \emptyset$. Then, $\mathsf{read}(A_1) \cap \mathsf{read}(A) = \emptyset$ and $\mathsf{read}(A_1) \cap \mathsf{write}(A) = \emptyset$ and $\mathsf{write}(A_1) \cap \mathsf{read}(A) = \emptyset$ and $\mathsf{write}(A_1) \cap \mathsf{write}(A) = \emptyset$ and $\mathsf{read}(A_2) \cap \mathsf{read}(A) = \emptyset$ and $\mathsf{read}(A_2) \cap \mathsf{write}(A) = \emptyset$ and $\mathsf{write}(A_2) \cap \mathsf{read}(A) = \emptyset$ and $\mathsf{write}(A_2) \cap \mathsf{write}(A) = \emptyset$. Then, by [#], $A_1 \# A$ and $A_2 \# A$. □
2. Similar to 1. □
3. Similar to 1. □
4.
   - Recall $R_1 = \emptyset = R_2$. Then, by Lem. 36 $\mathsf{read}(\mathbf{if}\ \xi|_n\ A_1|_{R_1}\ A_2|_{R_2}) = \mathsf{read}(\xi) \cup \mathsf{read}(A_1) \cup \mathsf{read}(A_2)$ and $\mathsf{write}(\mathbf{if}\ \xi|_n\ A_1|_{R_1}\ A_2|_{R_2}) = \mathsf{write}(\xi) \cup \mathsf{write}(A_1) \cup \mathsf{write}(A_2)$.
   - Recall $\mathbf{if}\ \xi|_n\ A_1|_{R_1}\ A_2|_{R_2} \# A$. Then, by Defn. 23, $\mathsf{read}(\mathbf{if}\ \xi|_n\ A_1|_{R_1}\ A_2|_{R_2}) \cap \mathsf{read}(A) = \emptyset$ and $\mathsf{read}(\mathbf{if}\ \xi|_n\ A_1|_{R_1}\ A_2|_{R_2}) \cap \mathsf{write}(A) = \emptyset$ and $\mathsf{write}(\mathbf{if}\ \xi|_n\ A_1|_{R_1}\ A_2|_{R_2}) \cap \mathsf{read}(A) = \emptyset$ and $\mathsf{write}(\mathbf{if}\ \xi|_n\ A_1|_{R_1}\ A_2|_{R_2}) \cap \mathsf{write}(A) = \emptyset$. Then, $(\mathsf{read}(\xi) \cup \mathsf{read}(A_1) \cup \mathsf{read}(A_2)) \cap \mathsf{read}(A) = \emptyset$ and $(\mathsf{read}(\xi) \cup \mathsf{read}(A_1) \cup \mathsf{read}(A_2)) \cap \mathsf{write}(A) = \emptyset$ and $(\mathsf{write}(\xi) \cup \mathsf{write}(A_1) \cup \mathsf{write}(A_2)) \cap \mathsf{read}(A) = \emptyset$ and $(\mathsf{write}(\xi) \cup \mathsf{write}(A_1) \cup \mathsf{write}(A_2)) \cap \mathsf{write}(A) = \emptyset$. Then, $\mathsf{read}(\xi) \cap \mathsf{read}(A) = \emptyset$ and $\mathsf{read}(\xi) \cap \mathsf{write}(A) = \emptyset$ and $\mathsf{write}(\xi) \cap \mathsf{read}(A) = \emptyset$ and $\mathsf{write}(\xi) \cap \mathsf{write}(A) = \emptyset$ and $\mathsf{read}(A_1) \cap \mathsf{read}(A) = \emptyset$ and $\mathsf{read}(A_1) \cap \mathsf{write}(A) = \emptyset$ and $\mathsf{write}(A_1) \cap \mathsf{read}(A) = \emptyset$ and $\mathsf{write}(A_1) \cap \mathsf{write}(A) = \emptyset$ and $\mathsf{read}(A_2) \cap \mathsf{read}(A) = \emptyset$ and $\mathsf{read}(A_2) \cap \mathsf{write}(A) = \emptyset$ and

$\mathsf{write}(A_2) \cap \mathsf{read}(A) = \emptyset$ and $\mathsf{write}(A_2) \cap \mathsf{write}(A) = \emptyset$. Then, by [#], $\xi \# A$ and $A_1 \# A$ and $A_2 \# A$. $\square$

5. – Recall $R_1 = \emptyset \neq R_2$. Then, by Lem. 36,
$\mathsf{read}(\mathbf{if}\ \xi|_n\ A_1|_{R_1}\ A_2|_{R_2}) = (\mathsf{read}(\xi) \setminus \{r.x \mid r \in R_2\}) \cup (\mathsf{read}(A_1) \setminus \{r.x \mid r \in R_2\}) \cup \mathsf{read}(A_2)$ and $\mathsf{write}(\mathbf{if}\ \xi|_n\ A_1|_{R_1}\ A_2|_{R_2}) = (\mathsf{write}(\xi) \setminus \{r.x \mid r \in R_2\}) \cup (\mathsf{write}(A_1) \setminus \{r.x \mid r \in R_2\}) \cup \mathsf{write}(A_2)$.
   – Recall $\mathbf{if}\ \xi|_n\ A_1|_{R_1}\ A_2|_{R_2} \# A$. Then, by Defn. 23, $\mathsf{read}(\mathbf{if}\ \xi|_n\ A_1|_{R_1}\ A_2|_{R_2}) \cap \mathsf{read}(A) = \emptyset$ and $\mathsf{read}(\mathbf{if}\ \xi|_n\ A_1|_{R_1}\ A_2|_{R_2}) \cap \mathsf{write}(A) = \emptyset$ and $\mathsf{write}(\mathbf{if}\ \xi|_n\ A_1|_{R_1}\ A_2|_{R_2}) \cap \mathsf{read}(A) = \emptyset$ and $\mathsf{write}(\mathbf{if}\ \xi|_n\ A_1|_{R_1}\ A_2|_{R_2}) \cap \mathsf{write}(A) = \emptyset$. Then,
$((\mathsf{read}(\xi) \setminus \{r.x \mid r \in R_2\}) \cup (\mathsf{read}(A_1) \setminus \{r.x \mid r \in R_2\}) \cup \mathsf{read}(A_2)) \cap \mathsf{read}(A) = \emptyset$ and
$((\mathsf{read}(\xi) \setminus \{r.x \mid r \in R_2\}) \cup (\mathsf{read}(A_1) \setminus \{r.x \mid r \in R_2\}) \cup \mathsf{read}(A_2)) \cap \mathsf{write}(A) = \emptyset$ and
$((\mathsf{write}(\xi) \setminus \{r.x \mid r \in R_2\}) \cup (\mathsf{write}(A_1) \setminus \{r.x \mid r \in R_2\}) \cup \mathsf{write}(A_2)) \cap \mathsf{read}(A) = \emptyset$ and
$((\mathsf{write}(\xi) \setminus \{r.x \mid r \in R_2\}) \cup (\mathsf{write}(A_1) \setminus \{r.x \mid r \in R_2\}) \cup \mathsf{write}(A_2)) \cap \mathsf{write}(A) = \emptyset$.
Then, $(\mathsf{read}(\xi) \setminus \{r.x \mid r \in R_2\}) \cap \mathsf{read}(A) = \emptyset$ and
$(\mathsf{read}(\xi) \setminus \{r.x \mid r \in R_2\}) \cap \mathsf{write}(A) = \emptyset$ and $(\mathsf{write}(\xi) \setminus \{r.x \mid r \in R_2\}) \cap \mathsf{read}(A) = \emptyset$ and
$(\mathsf{write}(\xi) \setminus \{r.x \mid r \in R_2\}) \cap \mathsf{write}(A) = \emptyset$ and $\mathsf{read}(A_2) \cap \mathsf{read}(A) = \emptyset$ and
$\mathsf{read}(A_2) \cap \mathsf{write}(A) = \emptyset$ and $\mathsf{write}(A_2) \cap \mathsf{read}(A) = \emptyset$ and $\mathsf{write}(A_2) \cap \mathsf{write}(A) = \emptyset$.
   – Recall $\hat{r} \notin R_2$ for every $\hat{r} \in \mathsf{subj}(\xi) \setminus R_2$. Then, $\{\hat{r}\} \cap R_2 = \emptyset$ for every $\hat{r} \in \mathsf{subj}(\xi) \setminus R_2$. Then, $\{r.x \mid r \in \{\hat{r}\}\} \cap \{r.x \mid r \in R_2\} = \emptyset$ for every $\hat{r} \in \mathsf{subj}(\xi) \setminus R_2$.
   – By Lem. 5, $\mathsf{read}(\xi \upharpoonright \hat{r}) \subseteq \{r.x \mid r \in \mathsf{subj}(\xi \upharpoonright \hat{r})\}$ and $\mathsf{write}(\xi \upharpoonright \hat{r}) \subseteq \{r.x \mid r \in \mathsf{subj}(\xi \upharpoonright \hat{r})\}$, for every $\hat{r} \in \mathsf{subj}(\xi)$. Then, by Lem. 9, $\mathsf{read}(\xi \upharpoonright \hat{r}) \subseteq \{r.x \mid r \in \{\hat{r}\}\}$ and $\mathsf{write}(\xi \upharpoonright \hat{r}) \subseteq \{r.x \mid r \in \{\hat{r}\}\}$, for every $\hat{r} \in \mathsf{subj}(\xi)$. Then, $\mathsf{read}(\xi \upharpoonright \hat{r}) \subseteq \{r.x \mid r \in \{\hat{r}\}\}$ and $\mathsf{write}(\xi \upharpoonright \hat{r}) \subseteq \{r.x \mid r \in \{\hat{r}\}\}$, for every $\hat{r} \in \mathsf{subj}(\xi) \setminus R_2$.
   – Recall $\{r.x \mid r \in \{\hat{r}\}\} \cap \{r.x \mid r \in R_2\} = \emptyset$ and $\mathsf{read}(\xi \upharpoonright \hat{r}) \subseteq \{r.x \mid r \in \{\hat{r}\}\}$ and $\mathsf{write}(\xi \upharpoonright \hat{r}) \subseteq \{r.x \mid r \in \{\hat{r}\}\}$, for every $\hat{r} \in \mathsf{subj}(\xi) \setminus R_2$. Then, $\mathsf{read}(\xi \upharpoonright \hat{r}) \cap \{r.x \mid r \in R_2\} = \emptyset$ and $\mathsf{write}(\xi \upharpoonright \hat{r}) \cap \{r.x \mid r \in R_2\} = \emptyset$, for every $\hat{r} \in \mathsf{subj}(\xi) \setminus R_2$. Then, $\mathsf{read}(\xi \upharpoonright \hat{r}) \setminus \{r.x \mid r \in R_2\} = \mathsf{read}(\xi \upharpoonright \hat{r})$ and $\mathsf{write}(\xi \upharpoonright \hat{r}) \setminus \{r.x \mid r \in R_2\} = \mathsf{write}(\xi \upharpoonright \hat{r})$, for every $\hat{r} \in \mathsf{subj}(\xi) \setminus R_2$.
   – By Lem. 8, $\mathsf{read}(\xi \upharpoonright \hat{r}) \subseteq \mathsf{read}(\xi)$ and $\mathsf{write}(\xi \upharpoonright \hat{r}) \subseteq \mathsf{write}(\xi)$, for every $\hat{r} \in \mathsf{subj}(\xi)$. Then, $\mathsf{read}(\xi \upharpoonright \hat{r}) \setminus \{r.x \mid r \in R_2\} \subseteq \mathsf{read}(\xi) \setminus \{r.x \mid r \in R_2\}$, and $\mathsf{write}(\xi \upharpoonright \hat{r}) \setminus \{r.x \mid r \in R_2\} \subseteq \mathsf{write}(\xi) \setminus \{r.x \mid r \in R_2\}$, for every $\hat{r} \in \mathsf{subj}(\xi)$. Then, $\mathsf{read}(\xi \upharpoonright \hat{r}) \setminus \{r.x \mid r \in R_2\} \subseteq \mathsf{read}(\xi) \setminus \{r.x \mid r \in R_2\}$ and $\mathsf{write}(\xi \upharpoonright \hat{r}) \setminus \{r.x \mid r \in R_2\} \subseteq \mathsf{write}(\xi) \setminus \{r.x \mid r \in R_2\}$, for every $\hat{r} \in \mathsf{subj}(\xi) \setminus R_2$. Then, $\mathsf{read}(\xi \upharpoonright \hat{r}) \subseteq \mathsf{read}(\xi) \setminus \{r.x \mid r \in R_2\}$ and $\mathsf{write}(\xi \upharpoonright \hat{r}) \subseteq \mathsf{write}(\xi) \setminus \{r.x \mid r \in R_2\}$, for every $\hat{r} \in \mathsf{subj}(\xi) \setminus R_2$.
   – Recall $(\mathsf{read}(\xi) \setminus \{r.x \mid r \in R_2\}) \cap \mathsf{read}(A) = \emptyset$ and $(\mathsf{read}(\xi) \setminus \{r.x \mid r \in R_2\}) \cap \mathsf{write}(A) = \emptyset$ and $(\mathsf{write}(\xi) \setminus \{r.x \mid r \in R_2\}) \cap \mathsf{read}(A) = \emptyset$ and $(\mathsf{write}(\xi) \setminus \{r.x \mid r \in R_2\}) \cap \mathsf{write}(A) = \emptyset$ and $\mathsf{read}(\xi \upharpoonright \hat{r}) \subseteq \mathsf{read}(\xi) \setminus \{r.x \mid r \in R_2\}$ and $\mathsf{write}(\xi \upharpoonright \hat{r}) \subseteq \mathsf{write}(\xi) \setminus \{r.x \mid r \in R_2\}$, for every $\hat{r} \in \mathsf{subj}(\xi) \setminus R_2$. Then, $\mathsf{read}(\xi \upharpoonright \hat{r}) \cap \mathsf{read}(A) = \emptyset$ and $\mathsf{read}(\xi \upharpoonright \hat{r}) \cap \mathsf{write}(A) = \emptyset$ and $\mathsf{write}(\xi \upharpoonright \hat{r}) \cap \mathsf{read}(A) = \emptyset$ and $\mathsf{write}(\xi \upharpoonright \hat{r}) \cap \mathsf{write}(A) = \emptyset$, for every $\hat{r} \in \mathsf{subj}(\xi) \setminus R_2$. Then, by [#], $\xi \upharpoonright \hat{r} \# A$ for every $\hat{r} \in \mathsf{subj}(\xi) \setminus R_2$.
   – Recall $\mathsf{read}(A_2) \cap \mathsf{read}(A) = \emptyset$ and $\mathsf{read}(A_2) \cap \mathsf{write}(A) = \emptyset$ and $\mathsf{write}(A_2) \cap \mathsf{read}(A) = \emptyset$ and $\mathsf{write}(A_2) \cap \mathsf{write}(A) = \emptyset$. Then, by [#], and $A_2 \# A$. $\square$
6. Similar to 5.
7. Similar to 1. $\square$

Proof (of Lem. 39).

1. Recall $\xi_r \# A$ for every $r \in R$. Then, by Defn. 23, $\mathsf{read}(\xi_r) \cap \mathsf{read}(A) = \emptyset$ and $\mathsf{read}(\xi_r) \cap \mathsf{write}(A) = \emptyset$ and $\mathsf{write}(\xi_r) \cap \mathsf{read}(A) = \emptyset$ and $\mathsf{write}(\xi_r) \cap \mathsf{write}(A) = \emptyset$ and $\mathsf{read}(\xi_2) \cap \mathsf{read}(A) = \emptyset$, for every $r \in R$. Then, $\bigcup\{\mathsf{read}(\xi_r)\}_{r \in R} \cap \mathsf{read}(A) = \emptyset$ and $\bigcup\{\mathsf{read}(\xi_r)\}_{r \in R} \cap \mathsf{write}(A) = \emptyset$ and $\bigcup\{\mathsf{write}(\xi_r)\}_{r \in R} \cap \mathsf{read}(A) = \emptyset$ and $\bigcup\{\mathsf{write}(\xi_r)\}_{r \in R} \cap \mathsf{write}(A) = \emptyset$. Then, by Lem. 12, $\bigcup\{\mathsf{read}(\xi_r^+)\}_{r \in R} \cap \mathsf{read}(A) = \emptyset$ and $\bigcup\{\mathsf{read}(\xi_r^+)\}_{r \in R} \cap \mathsf{write}(A) = \emptyset$ and $\bigcup\{\mathsf{write}(\xi_r^+)\}_{r \in R} \cap \mathsf{read}(A) = \emptyset$ and

$\bigcup\{\mathsf{write}(\xi_r^+)\}_{r\in R} \cap \mathsf{write}(A) = \emptyset$. Then, by Lem. 3, $\mathsf{read}(\bigwedge\{\xi_r^+\}_{r\in R}) \cap \mathsf{read}(A) = \emptyset$ and $\mathsf{read}(\bigwedge\{\xi_r^+\}_{r\in R}) \cap \mathsf{write}(A) = \emptyset$ and $\mathsf{write}(\bigwedge\{\xi_r^+\}_{r\in R}) \cap \mathsf{read}(A) = \emptyset$ and $\mathsf{write}(\bigwedge\{\xi_r^+\}_{r\in R}) \cap \mathsf{write}(A) = \emptyset$. Then, by [#], $\bigwedge\{\xi_r^+\}_{r\in R} \# A$. □

2. Similar to 1. □

Proof (of Lem. 40).

1. By Defn. 21:
   - **Base:** $A = \xi$, for some ξ.
     By Lem. 5, $\mathsf{read}(\psi) \subseteq \{r.x \mid r \in \mathsf{subj}(\psi)\}$. Then, $\mathsf{read}(A) \subseteq \{r.x \mid r \in \mathsf{subj}(A)\}$.
   - **Base:** $A = \alpha$, for some α.
     By Lem. 31, $\mathsf{read}(\alpha) \subseteq \{r.x \mid r \in \mathsf{subj}(\alpha)\}$. Then, $\mathsf{read}(A) \subseteq \{r.x \mid r \in \mathsf{subj}(A)\}$.
   - **Base:** $A = \mathbf{skip}$.
     Recall $\emptyset \subseteq \{r.x \mid r \in \mathsf{subj}(A)\}$. Then, by Defn. 22, $\mathsf{read}(\mathbf{skip}) \subseteq \{r.x \mid r \in \mathsf{subj}(A)\}$. Then, $\mathsf{read}(A) \subseteq \{r.x \mid r \in \mathsf{subj}(A)\}$.
   - **Step:** $A = A_1 \circ A_2$, for some A_1, A_2.
     By induction, $\mathsf{read}(A_1) \subseteq \{r.x \mid r \in \mathsf{subj}(A_1)\}$ and $\mathsf{read}(A_2) \subseteq \{r.x \mid r \in \mathsf{subj}(A_2)\}$. Then, $\mathsf{read}(A_1) \cup \mathsf{read}(A_2) \subseteq \{r.x \mid r \in \mathsf{subj}(A_1)\} \cup \{r.x \mid r \in \mathsf{subj}(A_2)\}$. Then, $\mathsf{read}(A_1) \cup \mathsf{read}(A_2) \subseteq \{r.x \mid r \in \mathsf{subj}(A_1) \cup \mathsf{subj}(A_2)\}$. Then, by Defn. 22, $\mathsf{read}(A_1 \circ A_2) \subseteq \{r.x \mid r \in \mathsf{subj}(A_1) \cup \mathsf{subj}(A_2)\}$. Then, by Defn. 24, $\mathsf{read}(A_1 \circ A_2) \subseteq \{r.x \mid r \in \mathsf{subj}(A_1 \circ A_2)\}$. Then, $\mathsf{read}(A) \subseteq \{r.x \mid r \in \mathsf{subj}(A)\}$.
   - **Step:** $A = R.\mathbf{if}\ \xi\ A_1\ A_2$, for some R, A_1, A_2, ξ. Similar to case "$A = A_1 \circ A_2$, for some A_1, A_2".
   - **Step:** $A = R.\mathbf{while}\ \xi\ \{\psi\}\ \hat{A}$, for some $R, \hat{A}, \psi, \xi$. Similar to case "$A = A_1 \circ A_2$, for some A_1, A_2".
   - **Step:** $A = \mathbf{if}\ \xi|_n\ A_1|_{R_1}\ A_2|_{R_2}$, for some $R_1, R_2, A_1, A_2, \xi, n$.
     - By induction, $\mathsf{read}(\xi) \subseteq \{r.x \mid r \in \mathsf{subj}(\xi)\}$. Then, $\mathsf{read}(\xi) \setminus \{r.x \mid r \in R_1 \cup R_2\} \subseteq \{r.x \mid r \in \mathsf{subj}(\xi)\} \setminus \{r.x \mid r \in R_1 \cup R_2\}$. Then, $\mathsf{read}(\xi) \setminus \{r.x \mid r \in R_1 \cup R_2\} \subseteq \{r.x \mid r \in \mathsf{subj}(\xi) \setminus (R_1 \cup R_2)\}$.
     - By induction, $\mathsf{read}(A_1) \subseteq \{r.x \mid r \in \mathsf{subj}(A_1)\}$. Then, $\mathsf{read}(A_1) \setminus \{r.x \mid r \in R_2\} \subseteq \{r.x \mid r \in \mathsf{subj}(A_1)\} \setminus \{r.x \mid r \in R_2\}$. Then, $\mathsf{read}(A_1) \setminus \{r.x \mid r \in R_2\} \subseteq \{r.x \mid r \in \mathsf{subj}(A_1) \setminus R_2\}$.
     - By induction, $\mathsf{read}(A_2) \subseteq \{r.x \mid r \in \mathsf{subj}(A_2)\}$. Then, $\mathsf{read}(A_2) \setminus \{r.x \mid r \in R_1\} \subseteq \{r.x \mid r \in \mathsf{subj}(A_2)\} \setminus \{r.x \mid r \in R_1\}$. Then, $\mathsf{read}(A_2) \setminus \{r.x \mid r \in R_1\} \subseteq \{r.x \mid r \in \mathsf{subj}(A_2) \setminus R_1\}$.
     - Recall $\mathsf{read}(\xi) \setminus \{r.x \mid r \in R_1 \cup R_2\} \subseteq \{r.x \mid r \in \mathsf{subj}(\xi) \setminus (R_1 \cup R_2)\}$ and $\mathsf{read}(A_1) \setminus \{r.x \mid r \in R_2\} \subseteq \{r.x \mid r \in \mathsf{subj}(A_1) \setminus R_2\}$ and $\mathsf{read}(A_2) \setminus \{r.x \mid r \in R_1\} \subseteq \{r.x \mid r \in \mathsf{subj}(A_2) \setminus R_1\}$. Then, $(\mathsf{read}(\xi) \setminus \{r.x \mid r \in R_1 \cup R_2\}) \cup (\mathsf{read}(A_1) \setminus \{r.x \mid r \in R_2\}) \cup (\mathsf{read}(A_2) \setminus \{r.x \mid r \in R_1\}) \subseteq \{r.x \mid r \in \mathsf{subj}(\xi) \setminus (R_1 \cup R_2)\} \cup \{r.x \mid r \in \mathsf{subj}(A_1) \setminus R_2\} \cup \{r.x \mid r \in \mathsf{subj}(A_2) \setminus R_1\}$. Then, $(\mathsf{read}(\xi) \setminus \{r.x \mid r \in R_1 \cup R_2\}) \cup (\mathsf{read}(A_1) \setminus \{r.x \mid r \in R_2\}) \cup (\mathsf{read}(A_2) \setminus \{r.x \mid r \in R_1\}) \subseteq \{r.x \mid r \in (\mathsf{subj}(\xi) \setminus (R_1 \cup R_2)) \cup (\mathsf{subj}(A_1) \setminus R_2) \cup (\mathsf{subj}(A_2) \setminus R_1)\}$. Then, by Defn. 22, $\mathsf{read}(\mathbf{if}\ \xi|_n\ A_1|_{R_1}\ A_2|_{R_2}) \subseteq \{r.x \mid r \in (\mathsf{subj}(\xi) \setminus (R_1 \cup R_2)) \cup (\mathsf{subj}(A_1) \setminus R_2) \cup (\mathsf{subj}(A_2) \setminus R_1)\}$. Then, by Defn. 24, $\mathsf{read}(\mathbf{if}\ \xi|_n\ A_1|_{R_1}\ A_2|_{R_2}) \subseteq \{r.x \mid r \in \mathsf{subj}(\mathbf{if}\ \xi|_n\ A_1|_{R_1}\ A_2|_{R_2})\}$. $\mathsf{read}(A) \subseteq \{r.x \mid r \in \mathsf{subj}(A)\}$.
   - **Step:** $A = \mathbf{while}\ \xi|_n\ \{\psi\}\ \hat{A}|_\emptyset$, for some $\hat{A}, \psi, \xi$. Similar to case "$A = A_1 \circ A_2$, for some A_1, A_2". □
2. Similar to 1. □

Proof (of Lem. 41).

– Recall $\mathsf{subj}(A_1) \cap \mathsf{subj}(A_2) = \emptyset$. Then, $\{r.x \mid r \in \mathsf{subj}(A_1)\} \cap \{r.x \mid r \in \mathsf{subj}(A_2)\} = \emptyset$.

- By Lem. 40, $\mathsf{read}(A_1) \subseteq \{r.x \mid r \in \mathsf{subj}(A_1)\}$ and $\mathsf{write}(A_1) \subseteq \{r.x \mid r \in \mathsf{subj}(A_1)\}$ and $\mathsf{read}(A_2) \subseteq \{r.x \mid r \in \mathsf{subj}(A_2)\}$ and $\mathsf{write}(A_2) \subseteq \{r.x \mid r \in \mathsf{subj}(A_2)\}$.
- Recall $\{r.x \mid r \in \mathsf{subj}(A_1)\} \cap \{r.x \mid r \in \mathsf{subj}(A_2)\} = \emptyset$ and $\mathsf{read}(A_1) \subseteq \{r.x \mid r \in \mathsf{subj}(A_1)\}$ and $\mathsf{read}(A_2) \subseteq \{r.x \mid r \in \mathsf{subj}(A_2)\}$. Then, $\mathsf{read}(A_1) \cap \mathsf{read}(A_2) = \emptyset$.
- Recall $\{r.x \mid r \in \mathsf{subj}(A_1)\} \cap \{r.x \mid r \in \mathsf{subj}(A_2)\} = \emptyset$ and $\mathsf{read}(A_1) \subseteq \{r.x \mid r \in \mathsf{subj}(A_1)\}$ and $\mathsf{write}(A_2) \subseteq \{r.x \mid r \in \mathsf{subj}(A_2)\}$. Then, $\mathsf{read}(A_1) \cap \mathsf{write}(A_2) = \emptyset$.
- Recall $\{r.x \mid r \in \mathsf{subj}(A_1)\} \cap \{r.x \mid r \in \mathsf{subj}(A_2)\} = \emptyset$ and $\mathsf{write}(A_1) \subseteq \{r.x \mid r \in \mathsf{subj}(A_1)\}$ and $\mathsf{read}(A_2) \subseteq \{r.x \mid r \in \mathsf{subj}(A_2)\}$. Then, $\mathsf{write}(A_1) \cap \mathsf{read}(A_2) = \emptyset$.
- Recall $\{r.x \mid r \in \mathsf{subj}(A_1)\} \cap \{r.x \mid r \in \mathsf{subj}(A_2)\} = \emptyset$ and $\mathsf{write}(A_1) \subseteq \{r.x \mid r \in \mathsf{subj}(A_1)\}$ and $\mathsf{write}(A_2) \subseteq \{r.x \mid r \in \mathsf{subj}(A_2)\}$. Then, $\mathsf{write}(A_1) \cap \mathsf{write}(A_2) = \emptyset$.
- Recall $\mathsf{read}(A_1) \cap \mathsf{read}(A_2) = \emptyset$ and $\mathsf{read}(A_1) \cap \mathsf{write}(A_2) = \emptyset$ and $\mathsf{write}(A_1) \cap \mathsf{read}(A_2) = \emptyset$ and $\mathsf{write}(A_1) \cap \mathsf{write}(A_2) = \emptyset$. Then, by [#], $A_1 \# A_2$. □

Proof (of Lem. 42). By Defn. 21:

- **Base:** $G = \xi$, for some ξ.
  1. Recall $r \in \mathsf{subj}(G)$. Then, $r \in \mathsf{subj}(\xi)$. Then, by Lem. 9, $\mathsf{subj}(\xi \upharpoonright r) = \{r\}$. Then, $\mathsf{subj}(G \upharpoonright r) = \{r\}$.
  2. Recall $r \notin \mathsf{subj}(G)$. Then, $r \notin \mathsf{subj}(\xi)$. Then, by Lem. 9, $\mathsf{subj}(\xi \upharpoonright r) = \emptyset$. Then, $\mathsf{subj}(G \upharpoonright r) = \emptyset$.
- **Base:** $G = \gamma$, for some γ.
  1. Recall $r \in \mathsf{subj}(G)$. Then, $r \in \mathsf{subj}(\gamma)$. Then, by Lem. 32, $\mathsf{subj}(\gamma \upharpoonright r) = \{r\}$. Then, $\mathsf{subj}(G \upharpoonright r) = \{r\}$.
  2. Recall $r \notin \mathsf{subj}(G)$. Then, $r \notin \mathsf{subj}(\gamma)$. Then, by Lem. 32, $\mathsf{subj}(\gamma \upharpoonright r) = \emptyset$. Then, $\mathsf{subj}(G \upharpoonright r) = \emptyset$.
- **Base:** $G = \mathbf{skip}$.
  1. Recall $r \in \mathsf{subj}(G)$. Then, $r \in \mathsf{subj}(\mathbf{skip})$. Then, by Defn. 24, $r \in \emptyset$. Then, **false**.
  2. By Defn. 24, $\mathsf{subj}(\mathbf{skip}) = \emptyset$. Then, by Defn. 26, $\mathsf{subj}(\mathbf{skip} \upharpoonright r) = \emptyset$. Then, $\mathsf{subj}(G \upharpoonright r) = \emptyset$.
- **Step:** $G = G_1 \circ G_2$, for some G_1, G_2.
  1. Recall:
     - **Case:** $r \in \mathsf{subj}(G_1)$ and $r \in \mathsf{subj}(G_2)$. Then, by induction, $\mathsf{subj}(G_1 \upharpoonright r) = \{r\}$ and $\mathsf{subj}(G_2 \upharpoonright r) = \{r\}$. Then, $\mathsf{subj}(G_1 \upharpoonright r) \cup \mathsf{subj}(G_2 \upharpoonright r) = \{r\} \cup \{r\}$. Then, $\mathsf{subj}(G_1 \upharpoonright r) \cup \mathsf{subj}(G_2 \upharpoonright r) = \{r\}$. Then, by Defn. 24, $\mathsf{subj}((G_1 \upharpoonright r) \circ (G_2 \upharpoonright r)) = \{r\}$. Then, by Defn. 26, $\mathsf{subj}(G_1 \circ G_2 \upharpoonright r) = \{r\}$. Then, $\mathsf{subj}(G \upharpoonright r) = \{r\}$.
     - **Case:** $r \in \mathsf{subj}(G_1)$ and $r \notin \mathsf{subj}(G_2)$. Similar to case "$r \in \mathsf{subj}(G_1)$ and $r \in \mathsf{subj}(G_2)$".
     - **Case:** $r \notin \mathsf{subj}(G_1)$ and $r \in \mathsf{subj}(G_2)$. Similar to case "$r \in \mathsf{subj}(G_1)$ and $r \in \mathsf{subj}(G_2)$".
     - **Case:** $r \notin \mathsf{subj}(G_1)$ and $r \notin \mathsf{subj}(G_2)$. Recall $r \in \mathsf{subj}(G)$. Then, $r \in \mathsf{subj}(G_1 \circ G_2)$. Then, by Defn. 24, $r \in \mathsf{subj}(G_1) \cup \mathsf{subj}(G_2)$. Then, $r \in \mathsf{subj}(G_1)$ or $r \in \mathsf{subj}(G_2)$. Then, **false**.
  2. Recall $r \notin \mathsf{subj}(G)$. Then, $r \notin \mathsf{subj}(G_1 \circ G_2)$. Then, by Defn. 24, $r \notin \mathsf{subj}(G_1) \cup \mathsf{subj}(G_2)$. Then, $r \notin \mathsf{subj}(G_1)$ and $r \notin \mathsf{subj}(G_2)$. Then, by induction, $\mathsf{subj}(G_1 \upharpoonright r) = \emptyset$ and $\mathsf{subj}(G_2 \upharpoonright r) = \emptyset$. Then, $\mathsf{subj}(G_1 \upharpoonright r) \cup \mathsf{subj}(G_2 \upharpoonright r) = \emptyset \cup \emptyset$. Then, $\mathsf{subj}(G_1 \upharpoonright r) \cup \mathsf{subj}(G_2 \upharpoonright r) = \emptyset$. Then, by Defn. 24, $\mathsf{subj}((G_1 \upharpoonright r) \circ (G_2 \upharpoonright r)) = \emptyset$. Then, by Defn. 26, $\mathsf{subj}(G_1 \circ G_2 \upharpoonright r) = \emptyset$. Then, $\mathsf{subj}(G \upharpoonright r) = \emptyset$.
- **Step:** $G = R.\mathbf{if}\ \xi\ G_1\ G_2$, for some R, G_1, G_2, ξ.
  1. Recall:
     - **Case:** $r \in \mathsf{subj}(\xi)$ and $r \in \mathsf{subj}(G_1)$ and $r \in \mathsf{subj}(G_2)$. Then, by induction, $\mathsf{subj}(\xi \upharpoonright r) = \{r\}$ and $\mathsf{subj}(G_1 \upharpoonright r) = \{r\}$ and $\mathsf{subj}(G_2 \upharpoonright r) = \{r\}$. Then, $\mathsf{subj}(\xi \upharpoonright r) \cup \mathsf{subj}(G_1 \upharpoonright r) \cup \mathsf{subj}(G_2 \upharpoonright r) = \{r\} \cup \{r\} \cup \{r\}$. Then, $\mathsf{subj}(\xi \upharpoonright r) \cup \mathsf{subj}(G_1 \upharpoonright r) \cup \mathsf{subj}(G_2 \upharpoonright r) = \{r\}$. Then, by Defn. 24, $\mathsf{subj}(R.\mathbf{if}\ (\xi \upharpoonright r)\ (G_1 \upharpoonright r)\ (G_2 \upharpoonright r)) = \{r\}$. Then, by Defn. 26, $\mathsf{subj}(R.\mathbf{if}\ \xi\ G_1\ G_2 \upharpoonright r) = \{r\}$. Then, $\mathsf{subj}(G \upharpoonright r) = \{r\}$.

- **Case:** $r \in \mathsf{subj}(\xi)$ and $r \in \mathsf{subj}(G_1)$ and $r \notin \mathsf{subj}(G_2)$. Similar to case "$r \in \mathsf{subj}(\xi)$ and $r \in \mathsf{subj}(G_1)$ and $r \in \mathsf{subj}(G_2)$".
- **Case:** $r \in \mathsf{subj}(\xi)$ and $r \notin \mathsf{subj}(G_1)$ and $r \in \mathsf{subj}(G_2)$. Similar to case "$r \in \mathsf{subj}(\xi)$ and $r \in \mathsf{subj}(G_1)$ and $r \in \mathsf{subj}(G_2)$".
- **Case:** $r \in \mathsf{subj}(\xi)$ and $r \notin \mathsf{subj}(G_1)$ and $r \notin \mathsf{subj}(G_2)$. Similar to case "$r \in \mathsf{subj}(\xi)$ and $r \in \mathsf{subj}(G_1)$ and $r \in \mathsf{subj}(G_2)$".
- **Case:** $r \notin \mathsf{subj}(\xi)$ and $r \in \mathsf{subj}(G_1)$ and $r \in \mathsf{subj}(G_2)$. Similar to case "$r \in \mathsf{subj}(\xi)$ and $r \in \mathsf{subj}(G_1)$ and $r \in \mathsf{subj}(G_2)$".
- **Case:** $r \notin \mathsf{subj}(\xi)$ and $r \in \mathsf{subj}(G_1)$ and $r \notin \mathsf{subj}(G_2)$. Similar to case "$r \in \mathsf{subj}(\xi)$ and $r \in \mathsf{subj}(G_1)$ and $r \in \mathsf{subj}(G_2)$".
- **Case:** $r \notin \mathsf{subj}(\xi)$ and $r \notin \mathsf{subj}(G_1)$ and $r \in \mathsf{subj}(G_2)$. Similar to case "$r \in \mathsf{subj}(\xi)$ and $r \in \mathsf{subj}(G_1)$ and $r \in \mathsf{subj}(G_2)$".
- **Case:** $r \notin \mathsf{subj}(\xi)$ and $r \notin \mathsf{subj}(G_1)$ and $r \notin \mathsf{subj}(G_2)$.
  Recall $r \in \mathsf{subj}(G)$. Then, $r \in \mathsf{subj}(R.\mathbf{if}\ \xi\ G_1\ G_2)$. Then, by Defn. 24,
  $r \in \mathsf{subj}(\xi) \cup \mathsf{subj}(G_1) \cup \mathsf{subj}(G_2)$. Then, $r \in \mathsf{subj}(\xi)$ or $r \in \mathsf{subj}(G_1)$ or $r \in \mathsf{subj}(G_2)$.
  Then, **false**.

2. Recall $r \notin \mathsf{subj}(G)$. Then, $r \notin \mathsf{subj}(R.\mathbf{if}\ \xi\ G_1\ G_2)$. Then, by Defn. 24,
   $r \notin \mathsf{subj}(\xi) \cup \mathsf{subj}(G_1) \cup \mathsf{subj}(G_2)$. Then, $r \notin \mathsf{subj}(\xi)$ and $r \notin \mathsf{subj}(G_1)$ and $r \notin \mathsf{subj}(G_2)$.
   Then, by induction, $\mathsf{subj}(\xi \restriction r) = \emptyset$ and $\mathsf{subj}(G_1 \restriction r) = \emptyset$ and $\mathsf{subj}(G_2 \restriction r) = \emptyset$. Then,
   $\mathsf{subj}(\xi \restriction r) \cup \mathsf{subj}(G_1 \restriction r) \cup \mathsf{subj}(G_2 \restriction r) = \emptyset \cup \emptyset \cup \emptyset$. Then,
   $\mathsf{subj}(\xi \restriction r) \cup \mathsf{subj}(G_1 \restriction r) \cup \mathsf{subj}(G_2 \restriction r) = \emptyset$. Then, by Defn. 24,
   $\mathsf{subj}(R.\mathbf{if}\ (\xi \restriction r)\ (G_1 \restriction r)\ (G_2 \restriction r)) = \emptyset$. Then, by Defn. 26, $\mathsf{subj}(R.\mathbf{if}\ \xi\ G_1\ G_2 \restriction r) = \emptyset$.
   Then, $\mathsf{subj}(G \restriction r) = \emptyset$.

– **Step:** $G = R.\mathbf{while}\ \xi\ \{\psi\}\ \hat{G}$, for some $R, \hat{G}, \psi, \xi$. Similar to case "$G = G_1 \circ G_2$, for some G_1, G_2".

– **Step:** $G = \mathbf{if}\ \xi|_0\ G_1|_{R_1}\ G_2|_{R_2}$, for some R_1, R_2, G_1, G_2, ξ.

1. Recall:
   - **Case:** $r \in \mathsf{subj}(\xi) \setminus (R_1 \cup R_2)$ and $r \in \mathsf{subj}(G_1) \setminus R_2$ and $r \in \mathsf{subj}(G_2) \setminus R_1$.
     * Recall $r \in \mathsf{subj}(\xi) \setminus (R_1 \cup R_2)$. Then, $r \in \mathsf{subj}(\xi)$ and $r \notin R_1 \cup R_2$.
     * Recall $r \notin R_1 \cup R_2$. Then, $\{r\} \cap (R_1 \cup R_2) = \emptyset$.
     * Recall $r \in \mathsf{subj}(\xi)$. Then, by induction, $\mathsf{subj}(\xi \restriction r) = \{r\}$. Then, $\mathsf{subj}(\xi \restriction r) \setminus \emptyset = \{r\}$. Then, $\mathsf{subj}(\xi \restriction r) \setminus ((R_1 \cup R_2) \cap \{r\}) = \{r\}$. Then, $\mathsf{subj}(\xi \restriction r) \setminus ((R_1 \cap \{r\}) \cup (R_2 \cap \{r\})) = \{r\}$.
     * Recall $r \in \mathsf{subj}(G_1) \setminus R_2$. Then, $r \in \mathsf{subj}(G_1)$ and $r \notin R_2$.
     * Recall $r \notin R_2$. Then, $\{r\} \cap R_2 = \emptyset$.
     * Recall $r \in \mathsf{subj}(G_1)$. Then, by induction, $\mathsf{subj}(G_1 \restriction r) = \{r\}$. Then, $\mathsf{subj}(G_1 \restriction r) \setminus \emptyset = \{r\}$. Then, $\mathsf{subj}(G_1 \restriction r) \setminus (R_2 \cap \{r\}) = \{r\}$.
     * Recall $r \in \mathsf{subj}(G_2) \setminus R_1$. Then, $r \in \mathsf{subj}(G_2)$ and $r \notin R_1$.
     * Recall $r \notin R_1$. Then, $\{r\} \cap R_1 = \emptyset$.
     * Recall $r \in \mathsf{subj}(G_2)$. Then, by induction, $\mathsf{subj}(G_2 \restriction r) = \{r\}$. Then, $\mathsf{subj}(G_2 \restriction r) \setminus \emptyset = \{r\}$. Then, $\mathsf{subj}(G_2 \restriction r) \setminus (R_1 \cap \{r\}) = \{r\}$.
     * Recall $\{r\} \cup \{r\} \cup \{r\} = \{r\}$. Then, $(\mathsf{subj}(\xi \restriction r) \setminus ((R_1 \cap \{r\}) \cup (R_2 \cap \{r\}))) \cup (\mathsf{subj}(G_1 \restriction r) \setminus (R_2 \cap \{r\})) \cup (\mathsf{subj}(G_2 \restriction r) \setminus (R_1 \cap \{r\})) = \{r\}$. Then, by Defn. 24, $\mathsf{subj}(\mathbf{if}\ (\xi \restriction r)|_{|\mathsf{subj}(\xi) \setminus (R_1 \cup R_2 \cup \{r\})|}\ (G_1 \restriction r)|_{R_1 \cap \{r\}}\ (G_2 \restriction r)|_{R_2 \cap \{r\}}) = \{r\}$. Then, by Defn. 26, $\mathsf{subj}(\mathbf{if}\ \xi|_0\ G_1|_{R_1}\ G_2|_{R_2} \restriction r) = \{r\}$. Then, $\mathsf{subj}(G \restriction r) = \{r\}$.
   - **Case:** $r \in \mathsf{subj}(\xi) \setminus (R_1 \cup R_2)$ and $r \in \mathsf{subj}(G_1) \setminus R_2$ and $r \notin \mathsf{subj}(G_2) \setminus R_1$.
     * Recall $r \in \mathsf{subj}(\xi) \setminus (R_1 \cup R_2)$. Then, $r \in \mathsf{subj}(\xi)$ and $r \notin R_1 \cup R_2$.
     * Recall $r \notin R_1 \cup R_2$. Then, $\{r\} \cap (R_1 \cup R_2) = \emptyset$.
     * Recall $r \in \mathsf{subj}(\xi)$. Then, by induction, $\mathsf{subj}(\xi \restriction r) = \{r\}$. Then, $\mathsf{subj}(\xi \restriction r) \setminus \emptyset = \{r\}$. Then, $\mathsf{subj}(\xi \restriction r) \setminus ((R_1 \cup R_2) \cap \{r\}) = \{r\}$. Then, $\mathsf{subj}(\xi \restriction r) \setminus ((R_1 \cap \{r\}) \cup (R_2 \cap \{r\})) = \{r\}$.
     * Recall $r \in \mathsf{subj}(G_1) \setminus R_2$. Then, $r \in \mathsf{subj}(G_1)$ and $r \notin R_2$.
     * Recall $r \notin R_2$. Then, $\{r\} \cap R_2 = \emptyset$.

  * Recall $r \in \mathsf{subj}(G_1)$. Then, by induction, $\mathsf{subj}(G_1 \upharpoonright r) = \{r\}$. Then, $\mathsf{subj}(G_1 \upharpoonright r) \setminus \emptyset = \{r\}$. Then, $\mathsf{subj}(G_1 \upharpoonright r) \setminus (R_2 \cap \{r\}) = \{r\}$.
  * Recall $r \notin R_1 \cup R_2$. Then, $r \notin R_1$.
  * Recall $r \notin \mathsf{subj}(G_2) \setminus R_1$ and $r \notin R_1$. Then, $r \notin \mathsf{subj}(G_2)$. Then, by induction, $\mathsf{subj}(G_2 \upharpoonright r) = \emptyset$. Then, $\mathsf{subj}(G_2 \upharpoonright r) \setminus (R_1 \cap \{r\}) = \emptyset \setminus (R_1 \cap \{r\})$. Then, $\mathsf{subj}(G_2 \upharpoonright r) \setminus (R_1 \cap \{r\}) = \emptyset$.
  * Recall $\{r\} \cup \{r\} \cup \emptyset = \{r\}$. Then, $(\mathsf{subj}(\xi \upharpoonright r) \setminus ((R_1 \cap \{r\}) \cup (R_2 \cap \{r\}))) \cup (\mathsf{subj}(G_1 \upharpoonright r) \setminus (R_2 \cap \{r\})) \cup (\mathsf{subj}(G_2 \upharpoonright r) \setminus (R_1 \cap \{r\})) = \{r\}$. Then, by Defn. 24, $\mathsf{subj}(\textbf{if } (\xi \upharpoonright r)|_{|\mathsf{subj}(\xi)\setminus(R_1 \cup R_2 \cup \{r\})|}\ (G_1 \upharpoonright r)|_{R_1 \cap \{r\}}\ (G_2 \upharpoonright r)|_{R_2 \cap \{r\}}) = \{r\}$. Then, by Defn. 26, $\mathsf{subj}(\textbf{if } \xi|_0\ G_1|_{R_1}\ G_2|_{R_2} \upharpoonright r) = \{r\}$. Then, $\mathsf{subj}(G \upharpoonright r) = \{r\}$.
- **Case:** $r \in \mathsf{subj}(\xi) \setminus (R_1 \cup R_2)$ and $r \notin \mathsf{subj}(G_1) \setminus R_2$ and $r \in \mathsf{subj}(G_2) \setminus R_1$. Similar to case "$r \in \mathsf{subj}(\xi) \setminus (R_1 \cup R_2)$ and $r \in \mathsf{subj}(G_1) \setminus R_2$ and $r \notin \mathsf{subj}(G_2) \setminus R_1$".
- **Case:** $r \in \mathsf{subj}(\xi) \setminus (R_1 \cup R_2)$ and $r \notin \mathsf{subj}(G_1) \setminus R_2$ and $r \notin \mathsf{subj}(G_2) \setminus R_1$.
  * Recall $r \in \mathsf{subj}(\xi) \setminus (R_1 \cup R_2)$. Then, $r \in \mathsf{subj}(\xi)$ and $r \notin R_1 \cup R_2$.
  * Recall $r \notin R_1 \cup R_2$. Then, $\{r\} \cap (R_1 \cup R_2) = \emptyset$.
  * Recall $r \in \mathsf{subj}(\xi)$. Then, by induction, $\mathsf{subj}(\xi \upharpoonright r) = \{r\}$. Then, $\mathsf{subj}(\xi \upharpoonright r) \setminus \emptyset = \{r\}$. Then, $\mathsf{subj}(\xi \upharpoonright r) \setminus ((R_1 \cup R_2) \cap \{r\}) = \{r\}$. Then, $\mathsf{subj}(\xi \upharpoonright r) \setminus ((R_1 \cap \{r\}) \cup (R_2 \cap \{r\})) = \{r\}$.
  * Recall $r \notin R_1 \cup R_2$. Then, $r \notin R_2$.
  * Recall $r \in \mathsf{subj}(G_1) \setminus R_2$ and $r \notin R_2$. Then, $r \notin \mathsf{subj}(G_1)$. Then, by induction, $\mathsf{subj}(G_1 \upharpoonright r) = \emptyset$. Then, $\mathsf{subj}(G_1 \upharpoonright r) \setminus (R_2 \cap \{r\}) = \emptyset \setminus (R_2 \cap \{r\})$. Then, $\mathsf{subj}(G_1 \upharpoonright r) \setminus (R_2 \cap \{r\}) = \emptyset$.
  * Recall $r \notin R_1 \cup R_2$. Then, $r \notin R_1$.
  * Recall $r \notin \mathsf{subj}(G_2) \setminus R_1$ and $r \notin R_1$. Then, $r \notin \mathsf{subj}(G_2)$. Then, by induction, $\mathsf{subj}(G_2 \upharpoonright r) = \emptyset$. Then, $\mathsf{subj}(G_2 \upharpoonright r) \setminus (R_1 \cap \{r\}) = \emptyset \setminus (R_1 \cap \{r\})$. Then, $\mathsf{subj}(G_2 \upharpoonright r) \setminus (R_1 \cap \{r\}) = \emptyset$.
  * Recall $\{r\} \cup \emptyset \cup \emptyset = \{r\}$. Then, $(\mathsf{subj}(\xi \upharpoonright r) \setminus ((R_1 \cap \{r\}) \cup (R_2 \cap \{r\}))) \cup (\mathsf{subj}(G_1 \upharpoonright r) \setminus (R_2 \cap \{r\})) \cup (\mathsf{subj}(G_2 \upharpoonright r) \setminus (R_1 \cap \{r\})) = \{r\}$. Then, by Defn. 24, $\mathsf{subj}(\textbf{if } (\xi \upharpoonright r)|_{|\mathsf{subj}(\xi)\setminus(R_1 \cup R_2 \cup \{r\})|}\ (G_1 \upharpoonright r)|_{R_1 \cap \{r\}}\ (G_2 \upharpoonright r)|_{R_2 \cap \{r\}}) = \{r\}$. Then, by Defn. 26, $\mathsf{subj}(\textbf{if } \xi|_0\ G_1|_{R_1}\ G_2|_{R_2} \upharpoonright r) = \{r\}$. Then, $\mathsf{subj}(G \upharpoonright r) = \{r\}$.
- **Case:** $r \notin \mathsf{subj}(\xi) \setminus (R_1 \cup R_2)$ and $r \in \mathsf{subj}(G_1) \setminus R_2$ and $r \in \mathsf{subj}(G_2) \setminus R_1$.
  * Recall $r \in \mathsf{subj}(G_1) \setminus R_2$. Then, $r \in \mathsf{subj}(G_1)$ and $r \notin R_2$.
  * Recall $r \notin R_2$. Then, $\{r\} \cap R_2 = \emptyset$.
  * Recall $r \in \mathsf{subj}(G_1)$. Then, by induction, $\mathsf{subj}(G_1 \upharpoonright r) = \{r\}$. Then, $\mathsf{subj}(G_1 \upharpoonright r) \setminus \emptyset = \{r\}$. Then, $\mathsf{subj}(G_1 \upharpoonright r) \setminus (R_2 \cap \{r\}) = \{r\}$.
  * Recall $r \in \mathsf{subj}(G_2) \setminus R_1$. Then, $r \in \mathsf{subj}(G_2)$ and $r \notin R_1$.
  * Recall $r \notin R_1$. Then, $\{r\} \cap R_1 = \emptyset$.
  * Recall $r \in \mathsf{subj}(G_2)$. Then, by induction, $\mathsf{subj}(G_2 \upharpoonright r) = \{r\}$. Then, $\mathsf{subj}(G_2 \upharpoonright r) \setminus \emptyset = \{r\}$. Then, $\mathsf{subj}(G_2 \upharpoonright r) \setminus (R_1 \cap \{r\}) = \{r\}$.
  * Recall $r \notin R_1$ and $r \notin R_2$. Then, $r \notin R_1 \cup R_2$.
  * Recall $r \notin \mathsf{subj}(\xi) \setminus (R_1 \cup R_2)$ and $r \notin R_1 \cup R_2$. Then, $r \notin \mathsf{subj}(\xi)$. Then, by induction, $\mathsf{subj}(\xi \upharpoonright r) = \emptyset$. Then, $\mathsf{subj}(\xi \upharpoonright r) \setminus ((R_1 \cap \{r\}) \cup (R_2 \cap \{r\})) = \emptyset \setminus ((R_1 \cap \{r\}) \cup (R_2 \cap \{r\}))$. Then, $\mathsf{subj}(\xi \upharpoonright r) \setminus ((R_1 \cap \{r\}) \cup (R_2 \cap \{r\})) = \emptyset$.
  * Recall $\emptyset \cup \{r\} \cup \{r\} = \{r\}$. Then, $(\mathsf{subj}(\xi \upharpoonright r) \setminus ((R_1 \cap \{r\}) \cup (R_2 \cap \{r\}))) \cup (\mathsf{subj}(G_1 \upharpoonright r) \setminus (R_2 \cap \{r\})) \cup (\mathsf{subj}(G_2 \upharpoonright r) \setminus (R_1 \cap \{r\})) = \{r\}$. Then, by Defn. 24, $\mathsf{subj}(\textbf{if } (\xi \upharpoonright r)|_{|\mathsf{subj}(\xi)\setminus(R_1 \cup R_2 \cup \{r\})|}\ (G_1 \upharpoonright r)|_{R_1 \cap \{r\}}\ (G_2 \upharpoonright r)|_{R_2 \cap \{r\}}) = \{r\}$. Then, by Defn. 26, $\mathsf{subj}(\textbf{if } \xi|_0\ G_1|_{R_1}\ G_2|_{R_2} \upharpoonright r) = \{r\}$. Then, $\mathsf{subj}(G \upharpoonright r) = \{r\}$.
- **Case:** $r \notin \mathsf{subj}(\xi) \setminus (R_1 \cup R_2)$ and $r \in \mathsf{subj}(G_1) \setminus R_2$ and $r \notin \mathsf{subj}(G_2) \setminus R_1$.
  * Recall $r \in \mathsf{subj}(G_1) \setminus R_2$. Then, $r \in \mathsf{subj}(G_1)$ and $r \notin R_2$.
  * Recall $r \notin R_2$. Then, $\{r\} \cap R_2 = \emptyset$.
  * Recall $r \in \mathsf{subj}(G_1)$. Then, by induction, $\mathsf{subj}(G_1 \upharpoonright r) = \{r\}$. Then, $\mathsf{subj}(G_1 \upharpoonright r) \setminus \emptyset = \{r\}$. Then, $\mathsf{subj}(G_1 \upharpoonright r) \setminus (R_2 \cap \{r\}) = \{r\}$.
  * Recall:

  - **Case:** $r \in \mathsf{subj}(\xi)$ and $r \in \mathsf{subj}(G_2)$.
    - Recall $r \notin \mathsf{subj}(\xi) \setminus (R_1 \cup R_2)$ and $r \in \mathsf{subj}(\xi)$. Then, $r \in R_1 \cup R_2$. Then, $\{r\} \cap (R_1 \cup R_2) = \{r\}$.
    - Recall $r \in \mathsf{subj}(\xi)$. Then, by induction, $\mathsf{subj}(\xi \restriction r) = \{r\}$. Then, $\mathsf{subj}(\xi \restriction r) \setminus \{r\} = \{r\} \setminus \{r\}$. Then, $\mathsf{subj}(\xi \restriction r) \setminus ((R_1 \cup R_2) \cap \{r\}) = \{r\} \setminus \{r\}$. Then, $\mathsf{subj}(\xi \restriction r) \setminus ((R_1 \cap \{r\}) \cup (R_2 \cap \{r\})) = \{r\} \setminus \{r\}$. Then, $\mathsf{subj}(\xi \restriction r) \setminus ((R_1 \cap \{r\}) \cup (R_2 \cap \{r\})) = \emptyset$.
    - Recall $r \notin \mathsf{subj}(G_2) \setminus R_1$ and $r \in \mathsf{subj}(G_2)$. Then $r \in R_1$. Then, $\{r\} \cap R_1 = \{r\}$.
    - Recall $r \in \mathsf{subj}(G_2)$. Then, by induction, $\mathsf{subj}(G_2 \restriction r) = \{r\}$. Then, $\mathsf{subj}(G_2 \restriction r) \setminus \{r\} = \{r\} \setminus \{r\}$. Then, $\mathsf{subj}(G_2 \restriction r) \setminus (R_1 \cap \{r\}) = \{r\} \setminus \{r\}$. Then, $\mathsf{subj}(G_2 \restriction r) \setminus (R_1 \cap \{r\}) = \emptyset$.
    - Recall $\emptyset \cup \{r\} \cup \emptyset = \{r\}$. Then, $(\mathsf{subj}(\xi \restriction r) \setminus ((R_1 \cap \{r\}) \cup (R_2 \cap \{r\}))) \cup (\mathsf{subj}(G_1 \restriction r) \setminus (R_2 \cap \{r\})) \cup (\mathsf{subj}(G_2 \restriction r) \setminus (R_1 \cap \{r\})) = \{r\}$. Then, by Defn. 24, $\mathsf{subj}(\textbf{if } (\xi \restriction r)|_{|\mathsf{subj}(\xi) \setminus (R_1 \cup R_2 \cup \{r\})|} \, (G_1 \restriction r)|_{R_1 \cap \{r\}} \, (G_2 \restriction r)|_{R_2 \cap \{r\}}) = \{r\}$. Then, by Defn. 26, $\mathsf{subj}(\textbf{if } \xi|_0 \, G_1|_{R_1} \, G_2|_{R_2} \restriction r) = \{r\}$. Then, $\mathsf{subj}(G \restriction r) = \{r\}$.
  - **Case:** $r \in \mathsf{subj}(\xi)$ and $r \notin \mathsf{subj}(G_2)$.
    - Recall $r \notin \mathsf{subj}(\xi) \setminus (R_1 \cup R_2)$ and $r \in \mathsf{subj}(\xi)$. Then, $r \in R_1 \cup R_2$. Then, $\{r\} \cap (R_1 \cup R_2) = \{r\}$.
    - Recall $r \in \mathsf{subj}(\xi)$. Then, by induction, $\mathsf{subj}(\xi \restriction r) = \{r\}$. Then, $\mathsf{subj}(\xi \restriction r) \setminus \{r\} = \{r\} \setminus \{r\}$. Then, $\mathsf{subj}(\xi \restriction r) \setminus ((R_1 \cup R_2) \cap \{r\}) = \{r\} \setminus \{r\}$. Then, $\mathsf{subj}(\xi \restriction r) \setminus ((R_1 \cap \{r\}) \cup (R_2 \cap \{r\})) = \{r\} \setminus \{r\}$. Then, $\mathsf{subj}(\xi \restriction r) \setminus ((R_1 \cap \{r\}) \cup (R_2 \cap \{r\})) = \emptyset$.
    - Recall $r \notin \mathsf{subj}(G_2)$. Then, by induction, $\mathsf{subj}(G_2 \restriction r) = \emptyset$. Then, $\mathsf{subj}(G_2 \restriction r) \setminus (R_1 \cap \{r\}) = \emptyset \setminus (R_1 \cap \{r\})$. Then, $\mathsf{subj}(G_2 \restriction r) \setminus (R_1 \cap \{r\}) = \emptyset$.
    - Recall $\emptyset \cup \{r\} \cup \emptyset = \{r\}$. Then, $(\mathsf{subj}(\xi \restriction r) \setminus ((R_1 \cap \{r\}) \cup (R_2 \cap \{r\}))) \cup (\mathsf{subj}(G_1 \restriction r) \setminus (R_2 \cap \{r\})) \cup (\mathsf{subj}(G_2 \restriction r) \setminus (R_1 \cap \{r\})) = \{r\}$. Then, by Defn. 24, $\mathsf{subj}(\textbf{if } (\xi \restriction r)|_{|\mathsf{subj}(\xi) \setminus (R_1 \cup R_2 \cup \{r\})|} \, (G_1 \restriction r)|_{R_1 \cap \{r\}} \, (G_2 \restriction r)|_{R_2 \cap \{r\}}) = \{r\}$. Then, by Defn. 26, $\mathsf{subj}(\textbf{if } \xi|_0 \, G_1|_{R_1} \, G_2|_{R_2} \restriction r) = \{r\}$. Then, $\mathsf{subj}(G \restriction r) = \{r\}$.
  - **Case:** $r \notin \mathsf{subj}(\xi)$ and $r \in \mathsf{subj}(G_2)$. Similar to case "$r \in \mathsf{subj}(\xi)$ and $r \notin \mathsf{subj}(G_2)$".
  - **Case:** $r \notin \mathsf{subj}(\xi)$ and $r \notin \mathsf{subj}(G_2)$.
    - Recall $r \notin \mathsf{subj}(\xi)$. Then, by induction, $\mathsf{subj}(\xi \restriction r) = \emptyset$. Then, $\mathsf{subj}(\xi \restriction r) \setminus ((R_1 \cap \{r\}) \cup (R_2 \cap \{r\})) = \emptyset \setminus ((R_1 \cap \{r\}) \cup (R_2 \cap \{r\}))$. Then, $\mathsf{subj}(\xi \restriction r) \setminus ((R_1 \cap \{r\}) \cup (R_2 \cap \{r\})) = \emptyset$.
    - Recall $r \notin \mathsf{subj}(G_2)$. Then, by induction, $\mathsf{subj}(G_2 \restriction r) = \emptyset$. Then, $\mathsf{subj}(G_2 \restriction r) \setminus (R_1 \cap \{r\}) = \emptyset \setminus (R_1 \cap \{r\})$. Then, $\mathsf{subj}(G_2 \restriction r) \setminus (R_1 \cap \{r\}) = \emptyset$.
    - Recall $\emptyset \cup \{r\} \cup \emptyset = \{r\}$. Then, $(\mathsf{subj}(\xi \restriction r) \setminus ((R_1 \cap \{r\}) \cup (R_2 \cap \{r\}))) \cup (\mathsf{subj}(G_1 \restriction r) \setminus (R_2 \cap \{r\})) \cup (\mathsf{subj}(G_2 \restriction r) \setminus (R_1 \cap \{r\})) = \{r\}$. Then, by Defn. 24, $\mathsf{subj}(\textbf{if } (\xi \restriction r)|_{|\mathsf{subj}(\xi) \setminus (R_1 \cup R_2 \cup \{r\})|} \, (G_1 \restriction r)|_{R_1 \cap \{r\}} \, (G_2 \restriction r)|_{R_2 \cap \{r\}}) = \{r\}$. Then, by Defn. 26, $\mathsf{subj}(\textbf{if } \xi|_0 \, G_1|_{R_1} \, G_2|_{R_2} \restriction r) = \{r\}$. Then, $\mathsf{subj}(G \restriction r) = \{r\}$.
- **Case:** $r \notin \mathsf{subj}(\xi) \setminus (R_1 \cup R_2)$ and $r \notin \mathsf{subj}(G_1) \setminus R_2$ and $r \in \mathsf{subj}(G_2) \setminus R_1$. Similar to case "$r \notin \mathsf{subj}(\xi) \setminus (R_1 \cup R_2)$ and $r \in \mathsf{subj}(G_1) \setminus R_2$ and $r \notin \mathsf{subj}(G_2) \setminus R_1$".
- **Case:** $r \notin \mathsf{subj}(\xi) \setminus (R_1 \cup R_2)$ and $r \notin \mathsf{subj}(G_1) \setminus R_2$ and $r \notin \mathsf{subj}(G_2) \setminus R_1$. Recall $r \in \mathsf{subj}(G)$. Then, $r \in \mathsf{subj}(\textbf{if } \xi|_0 \, G_1|_{R_1} \, G_2|_{R_2})$. Then, by Defn. 24, $r \in (\mathsf{subj}(\xi) \setminus (R_1 \cup R_2)) \cup (\mathsf{subj}(G_1) \setminus R_2) \cup (\mathsf{subj}(G_2) \setminus R_1)$. Then, $r \in \mathsf{subj}(\xi) \setminus (R_1 \cup R_2)$ or $r \in \mathsf{subj}(G_1) \setminus R_2$ or $r \in \mathsf{subj}(G_2) \setminus R_1$. Then, **false**.

2. Recall $r \notin \mathsf{subj}(G)$. Then, $r \notin \mathsf{subj}(\textbf{if } \xi|_0 \, G_1|_{R_1} \, G_2|_{R_2})$. Then, by Defn. 24, $r \notin (\mathsf{subj}(\xi) \setminus (R_1 \cup R_2)) \cup (\mathsf{subj}(G_1) \setminus R_2) \cup (\mathsf{subj}(G_2) \setminus R_1)$. Then, $r \notin \mathsf{subj}(\xi) \setminus (R_1 \cup R_2)$ and $r \notin \mathsf{subj}(G_1) \setminus R_2$ and $r \notin \mathsf{subj}(G_2) \setminus R_1$. Then:
   - **Case:** $r \notin \mathsf{subj}(\xi)$ and $r \notin \mathsf{subj}(G_1)$ and $r \notin \mathsf{subj}(G_2)$.

- $*$ Recall $r \notin \mathsf{subj}(\xi)$. Then, by induction, $\mathsf{subj}(\xi \upharpoonright r) = \emptyset$. Then, $\mathsf{subj}(\xi \upharpoonright r) \setminus ((R_1 \cap \{r\}) \cup (R_2 \cap \{r\})) = \emptyset \setminus ((R_1 \cap \{r\}) \cup (R_2 \cap \{r\}))$. Then, $\mathsf{subj}(\xi \upharpoonright r) \setminus ((R_1 \cap \{r\}) \cup (R_2 \cap \{r\})) = \emptyset$.
- $*$ Recall $r \notin \mathsf{subj}(G_1)$. Then, by induction, $\mathsf{subj}(G_1 \upharpoonright r) = \emptyset$. Then, $\mathsf{subj}(G_1 \upharpoonright r) \setminus (R_2 \cap \{r\}) = \emptyset \setminus (R_2 \cap \{r\})$. Then, $\mathsf{subj}(G_1 \upharpoonright r) \setminus (R_2 \cap \{r\}) = \emptyset$.
- $*$ Recall $r \notin \mathsf{subj}(G_2)$. Then, by induction, $\mathsf{subj}(G_2 \upharpoonright r) = \emptyset$. Then, $\mathsf{subj}(G_2 \upharpoonright r) \setminus (R_1 \cap \{r\}) = \emptyset \setminus (R_1 \cap \{r\})$. Then, $\mathsf{subj}(G_2 \upharpoonright r) \setminus (R_1 \cap \{r\}) = \emptyset$.
- $*$ Recall $\emptyset \cup \emptyset \cup \emptyset = \emptyset$. Then, $(\mathsf{subj}(\xi \upharpoonright r) \setminus ((R_1 \cap \{r\}) \cup (R_2 \cap \{r\}))) \cup (\mathsf{subj}(G_1 \upharpoonright r) \setminus (R_2 \cap \{r\})) \cup (\mathsf{subj}(G_2 \upharpoonright r) \setminus (R_1 \cap \{r\})) = \emptyset$. Then, by Defn. 24, $\mathsf{subj}(\mathbf{if}\ (\xi \upharpoonright r)|_{|\mathsf{subj}(\xi) \setminus (R_1 \cup R_2 \cup \{r\})|}\ (G_1 \upharpoonright r)|_{R_1 \cap \{r\}}\ (G_2 \upharpoonright r)|_{R_2 \cap \{r\}}) = \emptyset$. Then, by Defn. 26, $\mathsf{subj}(\mathbf{if}\ \xi|_0\ G_1|_{R_1}\ G_2|_{R_2} \upharpoonright r) = \emptyset$ Then, $\mathsf{subj}(G \upharpoonright r) = \emptyset$.

- **Case:** $r \notin \mathsf{subj}(\xi)$ and $r \notin \mathsf{subj}(G_1)$ and $r \in \mathsf{subj}(G_2)$ and $r \in R_1$.
  - $*$ Recall $r \notin \mathsf{subj}(\xi)$. Then, by induction, $\mathsf{subj}(\xi \upharpoonright r) = \emptyset$. Then, $\mathsf{subj}(\xi \upharpoonright r) \setminus ((R_1 \cap \{r\}) \cup (R_2 \cap \{r\})) = \emptyset \setminus ((R_1 \cap \{r\}) \cup (R_2 \cap \{r\}))$. Then, $\mathsf{subj}(\xi \upharpoonright r) \setminus ((R_1 \cap \{r\}) \cup (R_2 \cap \{r\})) = \emptyset$.
  - $*$ Recall $r \notin \mathsf{subj}(G_1)$. Then, by induction, $\mathsf{subj}(G_1 \upharpoonright r) = \emptyset$. Then, $\mathsf{subj}(G_1 \upharpoonright r) \setminus (R_2 \cap \{r\}) = \emptyset \setminus (R_2 \cap \{r\})$. Then, $\mathsf{subj}(G_1 \upharpoonright r) \setminus (R_2 \cap \{r\}) = \emptyset$.
  - $*$ Recall $r \in R_1$. Then, $\{r\} \cap R_1 = \{r\}$.
  - $*$ Recall $r \in \mathsf{subj}(G_2)$. Then, by induction, $\mathsf{subj}(G_2 \upharpoonright r) = \{r\}$. Then, $\mathsf{subj}(G_2 \upharpoonright r) \setminus \{r\} = \{r\} \setminus \{r\}$. Then, $\mathsf{subj}(G_2 \upharpoonright r) \setminus (R_1 \cap \{r\}) = \{r\} \setminus \{r\}$. Then, $\mathsf{subj}(G_2 \upharpoonright r) \setminus (R_1 \cap \{r\}) = \emptyset$.
  - $*$ Recall $\emptyset \cup \emptyset \cup \emptyset = \emptyset$. Then, $(\mathsf{subj}(\xi \upharpoonright r) \setminus ((R_1 \cap \{r\}) \cup (R_2 \cap \{r\}))) \cup (\mathsf{subj}(G_1 \upharpoonright r) \setminus (R_2 \cap \{r\})) \cup (\mathsf{subj}(G_2 \upharpoonright r) \setminus (R_1 \cap \{r\})) = \emptyset$. Then, by Defn. 24, $\mathsf{subj}(\mathbf{if}\ (\xi \upharpoonright r)|_{|\mathsf{subj}(\xi) \setminus (R_1 \cup R_2 \cup \{r\})|}\ (G_1 \upharpoonright r)|_{R_1 \cap \{r\}}\ (G_2 \upharpoonright r)|_{R_2 \cap \{r\}}) = \emptyset$. Then, by Defn. 26, $\mathsf{subj}(\mathbf{if}\ \xi|_0\ G_1|_{R_1}\ G_2|_{R_2} \upharpoonright r) = \emptyset$ Then, $\mathsf{subj}(G \upharpoonright r) = \emptyset$.
- **Case:** $r \notin \mathsf{subj}(\xi)$ and $r \in \mathsf{subj}(G_1)$ and $r \in R_2$ and $r \notin \mathsf{subj}(G_2)$.
  - $*$ Recall $r \notin \mathsf{subj}(\xi)$. Then, by induction, $\mathsf{subj}(\xi \upharpoonright r) = \emptyset$. Then, $\mathsf{subj}(\xi \upharpoonright r) \setminus ((R_1 \cap \{r\}) \cup (R_2 \cap \{r\})) = \emptyset \setminus ((R_1 \cap \{r\}) \cup (R_2 \cap \{r\}))$. Then, $\mathsf{subj}(\xi \upharpoonright r) \setminus ((R_1 \cap \{r\}) \cup (R_2 \cap \{r\})) = \emptyset$.
  - $*$ Recall $r \in R_2$. Then, $\{r\} \cap R_2 = \{r\}$.
  - $*$ Recall $r \in \mathsf{subj}(G_1)$. Then, by induction, $\mathsf{subj}(G_1 \upharpoonright r) = \{r\}$. Then, $\mathsf{subj}(G_1 \upharpoonright r) \setminus \{r\} = \{r\} \setminus \{r\}$. Then, $\mathsf{subj}(G_1 \upharpoonright r) \setminus (R_2 \cap \{r\}) = \{r\} \setminus \{r\}$. Then, $\mathsf{subj}(G_1 \upharpoonright r) \setminus (R_2 \cap \{r\}) = \emptyset$.
  - $*$ Recall $r \notin \mathsf{subj}(G_2)$. Then, by induction, $\mathsf{subj}(G_2 \upharpoonright r) = \emptyset$. Then, $\mathsf{subj}(G_2 \upharpoonright r) \setminus (R_1 \cap \{r\}) = \emptyset \setminus (R_1 \cap \{r\})$. Then, $\mathsf{subj}(G_2 \upharpoonright r) \setminus (R_1 \cap \{r\}) = \emptyset$.
  - $*$ Recall $\emptyset \cup \emptyset \cup \emptyset = \emptyset$. Then, $(\mathsf{subj}(\xi \upharpoonright r) \setminus ((R_1 \cap \{r\}) \cup (R_2 \cap \{r\}))) \cup (\mathsf{subj}(G_1 \upharpoonright r) \setminus (R_2 \cap \{r\})) \cup (\mathsf{subj}(G_2 \upharpoonright r) \setminus (R_1 \cap \{r\})) = \emptyset$. Then, by Defn. 24, $\mathsf{subj}(\mathbf{if}\ (\xi \upharpoonright r)|_{|\mathsf{subj}(\xi) \setminus (R_1 \cup R_2 \cup \{r\})|}\ (G_1 \upharpoonright r)|_{R_1 \cap \{r\}}\ (G_2 \upharpoonright r)|_{R_2 \cap \{r\}}) = \emptyset$. Then, by Defn. 26, $\mathsf{subj}(\mathbf{if}\ \xi|_0\ G_1|_{R_1}\ G_2|_{R_2} \upharpoonright r) = \emptyset$ Then, $\mathsf{subj}(G \upharpoonright r) = \emptyset$.
- **Case:** $r \notin \mathsf{subj}(\xi)$ and $r \in \mathsf{subj}(G_1)$ and $r \in R_2$ and $r \in \mathsf{subj}(G_2)$ and $r \in R_1$.
  - $*$ Recall $r \notin \mathsf{subj}(\xi)$. Then, by induction, $\mathsf{subj}(\xi \upharpoonright r) = \emptyset$. Then, $\mathsf{subj}(\xi \upharpoonright r) \setminus ((R_1 \cap \{r\}) \cup (R_2 \cap \{r\})) = \emptyset \setminus ((R_1 \cap \{r\}) \cup (R_2 \cap \{r\}))$. Then, $\mathsf{subj}(\xi \upharpoonright r) \setminus ((R_1 \cap \{r\}) \cup (R_2 \cap \{r\})) = \emptyset$.
  - $*$ Recall $r \in R_2$. Then, $\{r\} \cap R_2 = \{r\}$.
  - $*$ Recall $r \in \mathsf{subj}(G_1)$. Then, by induction, $\mathsf{subj}(G_1 \upharpoonright r) = \{r\}$. Then, $\mathsf{subj}(G_1 \upharpoonright r) \setminus \{r\} = \{r\} \setminus \{r\}$. Then, $\mathsf{subj}(G_1 \upharpoonright r) \setminus (R_2 \cap \{r\}) = \{r\} \setminus \{r\}$. Then, $\mathsf{subj}(G_1 \upharpoonright r) \setminus (R_2 \cap \{r\}) = \emptyset$.
  - $*$ Recall $r \in R_1$. Then, $\{r\} \cap R_1 = \{r\}$.
  - $*$ Recall $r \in \mathsf{subj}(G_2)$. Then, by induction, $\mathsf{subj}(G_2 \upharpoonright r) = \{r\}$. Then, $\mathsf{subj}(G_2 \upharpoonright r) \setminus \{r\} = \{r\} \setminus \{r\}$. Then, $\mathsf{subj}(G_2 \upharpoonright r) \setminus (R_1 \cap \{r\}) = \{r\} \setminus \{r\}$. Then, $\mathsf{subj}(G_2 \upharpoonright r) \setminus (R_1 \cap \{r\}) = \emptyset$.
  - $*$ Recall $\emptyset \cup \emptyset \cup \emptyset = \emptyset$. Then, $(\mathsf{subj}(\xi \upharpoonright r) \setminus ((R_1 \cap \{r\}) \cup (R_2 \cap \{r\}))) \cup (\mathsf{subj}(G_1 \upharpoonright r) \setminus (R_2 \cap \{r\})) \cup (\mathsf{subj}(G_2 \upharpoonright r) \setminus (R_1 \cap \{r\})) = \emptyset$. Then, by Defn. 24,

$\mathsf{subj}(\textbf{if } (\xi \restriction r)|_{|\mathsf{subj}(\xi)\setminus(R_1\cup R_2\cup\{r\})|}\ (G_1 \restriction r)|_{R_1\cap\{r\}}\ (G_2 \restriction r)|_{R_2\cap\{r\}}) = \emptyset$. Then, by Defn. 26, $\mathsf{subj}(\textbf{if } \xi|_0\ G_1|_{R_1}\ G_2|_{R_2} \restriction r) = \emptyset$ Then, $\mathsf{subj}(G \restriction r) = \emptyset$.

- **Case:** $r \in \mathsf{subj}(\xi)$ and $r \in R_1 \cup R_2$ and $r \notin \mathsf{subj}(G_1)$ and $r \notin \mathsf{subj}(G_2)$.
  - ∗ Recall $r \in R_1 \cup R_2$. Then, $\{r\} \cap (R_1 \cup R_2) = \{r\}$.
  - ∗ Recall $r \in \mathsf{subj}(\xi)$. Then, by induction, $\mathsf{subj}(\xi \restriction r) = \{r\}$. Then, $\mathsf{subj}(\xi \restriction r) \setminus \{r\} = \{r\} \setminus \{r\}$. Then, $\mathsf{subj}(\xi \restriction r) \setminus ((R_1 \cup R_2) \cap \{r\}) = \{r\} \setminus \{r\}$. Then, $\mathsf{subj}(\xi \restriction r) \setminus ((R_1 \cap \{r\}) \cup (R_2 \cap \{r\})) = \{r\} \setminus \{r\}$. Then, $\mathsf{subj}(\xi \restriction r) \setminus ((R_1 \cap \{r\}) \cup (R_2 \cap \{r\})) = \emptyset$.
  - ∗ Recall $r \notin \mathsf{subj}(G_1)$. Then, by induction, $\mathsf{subj}(G_1 \restriction r) = \emptyset$. Then, $\mathsf{subj}(G_1 \restriction r) \setminus (R_2 \cap \{r\}) = \emptyset \setminus (R_2 \cap \{r\})$. Then, $\mathsf{subj}(G_1 \restriction r) \setminus (R_2 \cap \{r\}) = \emptyset$.
  - ∗ Recall $r \notin \mathsf{subj}(G_2)$. Then, by induction, $\mathsf{subj}(G_2 \restriction r) = \emptyset$. Then, $\mathsf{subj}(G_2 \restriction r) \setminus (R_1 \cap \{r\}) = \emptyset \setminus (R_1 \cap \{r\})$. Then, $\mathsf{subj}(G_2 \restriction r) \setminus (R_1 \cap \{r\}) = \emptyset$.
  - ∗ Recall $\emptyset \cup \emptyset \cup \emptyset = \emptyset$. Then, $(\mathsf{subj}(\xi \restriction r) \setminus ((R_1 \cap \{r\}) \cup (R_2 \cap \{r\}))) \cup (\mathsf{subj}(G_1 \restriction r) \setminus (R_2 \cap \{r\})) \cup (\mathsf{subj}(G_2 \restriction r) \setminus (R_1 \cap \{r\})) = \emptyset$. Then, by Defn. 24, $\mathsf{subj}(\textbf{if } (\xi \restriction r)|_{|\mathsf{subj}(\xi)\setminus(R_1\cup R_2\cup\{r\})|}\ (G_1 \restriction r)|_{R_1\cap\{r\}}\ (G_2 \restriction r)|_{R_2\cap\{r\}}) = \emptyset$. Then, by Defn. 26, $\mathsf{subj}(\textbf{if } \xi|_0\ G_1|_{R_1}\ G_2|_{R_2} \restriction r) = \emptyset$ Then, $\mathsf{subj}(G \restriction r) = \emptyset$.
- **Case:** $r \in \mathsf{subj}(\xi)$ and $r \in R_1 \cup R_2$ and $r \notin \mathsf{subj}(G_1)$ and $r \in \mathsf{subj}(G_2)$ and $r \in R_1$.
  - ∗ Recall $r \in R_1 \cup R_2$. Then, $\{r\} \cap (R_1 \cup R_2) = \{r\}$.
  - ∗ Recall $r \in \mathsf{subj}(\xi)$. Then, by induction, $\mathsf{subj}(\xi \restriction r) = \{r\}$. Then, $\mathsf{subj}(\xi \restriction r) \setminus \{r\} = \{r\} \setminus \{r\}$. Then, $\mathsf{subj}(\xi \restriction r) \setminus ((R_1 \cup R_2) \cap \{r\}) = \{r\} \setminus \{r\}$. Then, $\mathsf{subj}(\xi \restriction r) \setminus ((R_1 \cap \{r\}) \cup (R_2 \cap \{r\})) = \{r\} \setminus \{r\}$. Then, $\mathsf{subj}(\xi \restriction r) \setminus ((R_1 \cap \{r\}) \cup (R_2 \cap \{r\})) = \emptyset$.
  - ∗ Recall $r \notin \mathsf{subj}(G_1)$. Then, by induction, $\mathsf{subj}(G_1 \restriction r) = \emptyset$. Then, $\mathsf{subj}(G_1 \restriction r) \setminus (R_2 \cap \{r\}) = \emptyset \setminus (R_2 \cap \{r\})$. Then, $\mathsf{subj}(G_1 \restriction r) \setminus (R_2 \cap \{r\}) = \emptyset$.
  - ∗ Recall $r \in R_1$. Then, $\{r\} \cap R_1 = \{r\}$.
  - ∗ Recall $r \in \mathsf{subj}(G_2)$. Then, by induction, $\mathsf{subj}(G_2 \restriction r) = \{r\}$. Then, $\mathsf{subj}(G_2 \restriction r) \setminus \{r\} = \{r\} \setminus \{r\}$. Then, $\mathsf{subj}(G_2 \restriction r) \setminus (R_1 \cap \{r\}) = \{r\} \setminus \{r\}$. Then, $\mathsf{subj}(G_2 \restriction r) \setminus (R_1 \cap \{r\}) = \emptyset$.
  - ∗ Recall $\emptyset \cup \emptyset \cup \emptyset = \emptyset$. Then, $(\mathsf{subj}(\xi \restriction r) \setminus ((R_1 \cap \{r\}) \cup (R_2 \cap \{r\}))) \cup (\mathsf{subj}(G_1 \restriction r) \setminus (R_2 \cap \{r\})) \cup (\mathsf{subj}(G_2 \restriction r) \setminus (R_1 \cap \{r\})) = \emptyset$. Then, by Defn. 24, $\mathsf{subj}(\textbf{if } (\xi \restriction r)|_{|\mathsf{subj}(\xi)\setminus(R_1\cup R_2\cup\{r\})|}\ (G_1 \restriction r)|_{R_1\cap\{r\}}\ (G_2 \restriction r)|_{R_2\cap\{r\}}) = \emptyset$. Then, by Defn. 26, $\mathsf{subj}(\textbf{if } \xi|_0\ G_1|_{R_1}\ G_2|_{R_2} \restriction r) = \emptyset$ Then, $\mathsf{subj}(G \restriction r) = \emptyset$.
- **Case:** $r \in \mathsf{subj}(\xi)$ and $r \in R_1 \cup R_2$ and $r \in \mathsf{subj}(G_1)$ and $r \in R_2$ and $r \notin \mathsf{subj}(G_2)$.
  - ∗ Recall $r \in R_1 \cup R_2$. Then, $\{r\} \cap (R_1 \cup R_2) = \{r\}$.
  - ∗ Recall $r \in \mathsf{subj}(\xi)$. Then, by induction, $\mathsf{subj}(\xi \restriction r) = \{r\}$. Then, $\mathsf{subj}(\xi \restriction r) \setminus \{r\} = \{r\} \setminus \{r\}$. Then, $\mathsf{subj}(\xi \restriction r) \setminus ((R_1 \cup R_2) \cap \{r\}) = \{r\} \setminus \{r\}$. Then, $\mathsf{subj}(\xi \restriction r) \setminus ((R_1 \cap \{r\}) \cup (R_2 \cap \{r\})) = \{r\} \setminus \{r\}$. Then, $\mathsf{subj}(\xi \restriction r) \setminus ((R_1 \cap \{r\}) \cup (R_2 \cap \{r\})) = \emptyset$.
  - ∗ Recall $r \in R_2$. Then, $\{r\} \cap R_2 = \{r\}$.
  - ∗ Recall $r \in \mathsf{subj}(G_1)$. Then, by induction, $\mathsf{subj}(G_1 \restriction r) = \{r\}$. Then, $\mathsf{subj}(G_1 \restriction r) \setminus \{r\} = \{r\} \setminus \{r\}$. Then, $\mathsf{subj}(G_1 \restriction r) \setminus (R_2 \cap \{r\}) = \{r\} \setminus \{r\}$. Then, $\mathsf{subj}(G_1 \restriction r) \setminus (R_2 \cap \{r\}) = \emptyset$.
  - ∗ Recall $r \notin \mathsf{subj}(G_2)$. Then, by induction, $\mathsf{subj}(G_2 \restriction r) = \emptyset$. Then, $\mathsf{subj}(G_2 \restriction r) \setminus (R_1 \cap \{r\}) = \emptyset \setminus (R_1 \cap \{r\})$. Then, $\mathsf{subj}(G_2 \restriction r) \setminus (R_1 \cap \{r\}) = \emptyset$.
  - ∗ Recall $\emptyset \cup \emptyset \cup \emptyset = \emptyset$. Then, $(\mathsf{subj}(\xi \restriction r) \setminus ((R_1 \cap \{r\}) \cup (R_2 \cap \{r\}))) \cup (\mathsf{subj}(G_1 \restriction r) \setminus (R_2 \cap \{r\})) \cup (\mathsf{subj}(G_2 \restriction r) \setminus (R_1 \cap \{r\})) = \emptyset$. Then, by Defn. 24, $\mathsf{subj}(\textbf{if } (\xi \restriction r)|_{|\mathsf{subj}(\xi)\setminus(R_1\cup R_2\cup\{r\})|}\ (G_1 \restriction r)|_{R_1\cap\{r\}}\ (G_2 \restriction r)|_{R_2\cap\{r\}}) = \emptyset$. Then, by Defn. 26, $\mathsf{subj}(\textbf{if } \xi|_0\ G_1|_{R_1}\ G_2|_{R_2} \restriction r) = \emptyset$ Then, $\mathsf{subj}(G \restriction r) = \emptyset$.
- **Case:** $r \in \mathsf{subj}(\xi)$ and $r \in R_1 \cup R_2$ and $r \in \mathsf{subj}(G_1)$ and $r \in R_2$ and $r \in \mathsf{subj}(G_2)$ and $r \in R_1$.
  - ∗ Recall $r \in R_1 \cup R_2$. Then, $\{r\} \cap (R_1 \cup R_2) = \{r\}$.
  - ∗ Recall $r \in \mathsf{subj}(\xi)$. Then, by induction, $\mathsf{subj}(\xi \restriction r) = \{r\}$. Then, $\mathsf{subj}(\xi \restriction r) \setminus \{r\} = \{r\} \setminus \{r\}$. Then, $\mathsf{subj}(\xi \restriction r) \setminus ((R_1 \cup R_2) \cap \{r\}) = \{r\} \setminus \{r\}$.

Then, $\mathsf{subj}(\xi \restriction r) \setminus ((R_1 \cap \{r\}) \cup (R_2 \cap \{r\})) = \{r\} \setminus \{r\}$. Then, $\mathsf{subj}(\xi \restriction r) \setminus ((R_1 \cap \{r\}) \cup (R_2 \cap \{r\})) = \emptyset$.

* Recall $r \in R_2$. Then, $\{r\} \cap R_2 = \{r\}$.
* Recall $r \in \mathsf{subj}(G_1)$. Then, by induction, $\mathsf{subj}(G_1 \restriction r) = \{r\}$. Then, $\mathsf{subj}(G_1 \restriction r) \setminus \{r\} = \{r\} \setminus \{r\}$. Then, $\mathsf{subj}(G_1 \restriction r) \setminus (R_2 \cap \{r\}) = \{r\} \setminus \{r\}$. Then, $\mathsf{subj}(G_1 \restriction r) \setminus (R_2 \cap \{r\}) = \emptyset$.
* Recall $r \in R_1$. Then, $\{r\} \cap R_1 = \{r\}$.
* Recall $r \in \mathsf{subj}(G_2)$. Then, by induction, $\mathsf{subj}(G_2 \restriction r) = \{r\}$. Then, $\mathsf{subj}(G_2 \restriction r) \setminus \{r\} = \{r\} \setminus \{r\}$. Then, $\mathsf{subj}(G_2 \restriction r) \setminus (R_1 \cap \{r\}) = \{r\} \setminus \{r\}$. Then, $\mathsf{subj}(G_2 \restriction r) \setminus (R_1 \cap \{r\}) = \emptyset$.
* Recall $\emptyset \cup \emptyset \cup \emptyset = \emptyset$. Then, $(\mathsf{subj}(\xi \restriction r) \setminus ((R_1 \cap \{r\}) \cup (R_2 \cap \{r\}))) \cup (\mathsf{subj}(G_1 \restriction r) \setminus (R_2 \cap \{r\})) \cup (\mathsf{subj}(G_2 \restriction r) \setminus (R_1 \cap \{r\})) = \emptyset$. Then, by Defn. 24, $\mathsf{subj}(\mathbf{if}\ (\xi \restriction r)|_{|\mathsf{subj}(\xi)\setminus(R_1 \cup R_2 \cup \{r\})|}\ (G_1 \restriction r)|_{R_1 \cap \{r\}}\ (G_2 \restriction r)|_{R_2 \cap \{r\}}) = \emptyset$. Then, by Defn. 26, $\mathsf{subj}(\mathbf{if}\ \xi|_0\ G_1|_{R_1}\ G_2|_{R_2} \restriction r) = \emptyset$ Then, $\mathsf{subj}(G \restriction r) = \emptyset$.

– **Step:** $G = \mathbf{while}\ \xi|_0\ \{\psi\}\ \hat{G}|_\emptyset$, for some $\hat{G}, \psi, \xi$. Similar to case "$G = G_1 \circ G_2$, for some G_1, G_2". □

Proof (of Lem. 43). Recall:

– **Case:** $r \in \mathsf{subj}(G)$.
  - Recall $r \in \mathsf{subj}(G)$. Then, by Lem. 42, $\mathsf{subj}(G \restriction r) = \{r\}$.
  - Recall $r \in \mathsf{subj}(G)$. Then, $\{r\} \subseteq \mathsf{subj}(G)$. Then, $\mathsf{subj}(G \restriction r) \subseteq \mathsf{subj}(G)$.

– **Case:** $r \notin \mathsf{subj}(G)$.
  - Recall $r \notin \mathsf{subj}(G)$. Then, by Lem. 42, $\mathsf{subj}(G \restriction r) = \emptyset$.
  - Recall $\emptyset \subseteq \mathsf{subj}(G)$. Then, $\mathsf{subj}(G \restriction r) \subseteq \mathsf{subj}(G)$. □

Proof (of Lem. 44).

– Recall $r \in \mathsf{subj}(G)$. Then, by Lem. 42, $\mathsf{subj}(G \restriction r) = \{r\}$.
– Recall $r \in \{r\}$. Then, $r \in \mathsf{subj}(G \restriction r)$. □

Proof (of Lem. 45). By Defn. 21:

– **Base:** $G = \xi$, for some ξ.
  Recall $\emptyset \subseteq \mathsf{chan}(G)$. Then, by Defn. 25, $\mathsf{chan}(\xi \restriction r) \subseteq \mathsf{chan}(G)$. Then, $\mathsf{chan}(G \restriction r) \subseteq \mathsf{chan}(G)$.
– **Base:** $G = \gamma$, for some γ.
  By Lem. 35, $\mathsf{chan}(\gamma \restriction r) \subseteq \mathsf{chan}(\gamma)$. Then, $\mathsf{chan}(G \restriction r) \subseteq \mathsf{chan}(G)$.
– **Base:** $G = \mathbf{skip}$.
  Recall $\mathsf{chan}(\mathbf{skip}) \subseteq \mathsf{chan}(\mathbf{skip})$. Then, by Defn. 26, $\mathsf{chan}(\mathbf{skip} \restriction r) \subseteq \mathsf{chan}(\mathbf{skip})$. Then, $\mathsf{chan}(G \restriction r) \subseteq \mathsf{chan}(G)$.
– **Step:** $G = G_1 \circ G_2$, for some G_1, G_2.
  By induction, $\mathsf{chan}(G_1 \restriction r) \subseteq \mathsf{chan}(G_1)$ and $\mathsf{chan}(G_2 \restriction r) \subseteq \mathsf{chan}(G_2)$. Then, $\mathsf{chan}(G_1 \restriction r) \cup \mathsf{chan}(G_2 \restriction r) \subseteq \mathsf{chan}(G_1) \cup \mathsf{chan}(G_2)$. Then, by Defn. 25, $\mathsf{chan}((G_1 \restriction r) \circ (G_2 \restriction r)) \subseteq \mathsf{chan}(G_1 \circ G_2)$. Then, by Defn. 26, $\mathsf{chan}(G_1 \circ G_2 \restriction r) \subseteq \mathsf{chan}(G_1 \circ G_2)$. Then, $\mathsf{chan}(G \restriction r) \subseteq \mathsf{chan}(G)$.
– **Step:** $G = R.\mathbf{if}\ \xi\ G_1\ G_2$, for some R, G_1, G_2, ξ.
  By induction, $\mathsf{chan}(G_1 \restriction r) \subseteq \mathsf{chan}(G_1)$ and $\mathsf{chan}(G_2 \restriction r) \subseteq \mathsf{chan}(G_2)$. Then, $\mathsf{chan}(G_1 \restriction r) \cup \mathsf{chan}(G_2 \restriction r) \subseteq \mathsf{chan}(G_1) \cup \mathsf{chan}(G_2)$. Then, $\{R\} \cup \mathsf{chan}(G_1 \restriction r) \cup \mathsf{chan}(G_2 \restriction r) \subseteq \{R\} \cup \mathsf{chan}(G_1) \cup \mathsf{chan}(G_2)$. Then, by Defn. 25, $\mathsf{chan}(R.\mathbf{if}\ (\xi \restriction r)\ (G_1 \restriction r)\ (G_2 \restriction r)) \subseteq \mathsf{chan}(R.\mathbf{if}\ \xi\ G_1\ G_2)$. Then, by Defn. 26, $\mathsf{chan}(R.\mathbf{if}\ \xi\ G_1\ G_2 \restriction r) \subseteq \mathsf{chan}(R.\mathbf{if}\ \xi\ G_1\ G_2)$. Then, $\mathsf{chan}(G \restriction r) \subseteq \mathsf{chan}(G)$.
– **Step:** $G = R.\mathbf{while}\ \xi\ \{\psi\}\ \hat{G}$, for some $R, \hat{G}, \psi, \xi$. Similar to case "$G = R.\mathbf{if}\ \xi\ G_1\ G_2$, for some R, G_1, G_2, ξ".
– **Step:** $G = \mathbf{if}\ \xi|_0\ G_1|_{R_1}\ G_2|_{R_2}$, for some R_1, R_2, G_1, G_2, ξ.

- By Lem. 10, $\mathsf{subj}(\xi \restriction r) \subseteq \mathsf{subj}(\xi)$. Then, $\{\{\hat{r}\} \mid \hat{r} \in \mathsf{subj}(\xi \restriction r)\} \subseteq \{\{\hat{r}\} \mid \hat{r} \in \mathsf{subj}(\xi)\}$.
- By induction, $\mathsf{chan}(G_1 \restriction r) \subseteq \mathsf{chan}(G_1)$ and $\mathsf{chan}(G_2 \restriction r) \subseteq \mathsf{chan}(G_2)$.
- Recall $\{\{\hat{r}\} \mid \hat{r} \in \mathsf{subj}(\xi \restriction r)\} \subseteq \{\{\hat{r}\} \mid \hat{r} \in \mathsf{subj}(\xi)\}$ and $\mathsf{chan}(G_1 \restriction r) \subseteq \mathsf{chan}(G_1)$ and $\mathsf{chan}(G_2 \restriction r) \subseteq \mathsf{chan}(G_2)$. Then, $\{\{\hat{r}\} \mid \hat{r} \in \mathsf{subj}(\xi{\restriction}r)\} \cup \mathsf{chan}(G_1{\restriction}r) \cup \mathsf{chan}(G_2{\restriction}r) \subseteq \{\{\hat{r}\} \mid \hat{r} \in \mathsf{subj}(\xi)\} \cup \mathsf{chan}(G_1) \cup \mathsf{chan}(G_2)$. Then, by Defn. 25, $\mathsf{chan}(\mathbf{if}\ (\xi{\restriction}r)|_{|\mathsf{subj}(\xi)\setminus(R_1 \cup R_2 \cup \{r\})|}\ (G_1{\restriction}r)|_{R_1 \cap \{r\}}\ (G_2{\restriction}r)|_{R_2 \cap \{r\}}) \subseteq \mathsf{chan}(\mathbf{if}\ \xi|_0\ G_1|_{R_1}\ G_2|_{R_2})$. Then, by Defn. 26, $\mathsf{chan}(\mathbf{if}\ \xi|_0\ G_1|_{R_1}\ G_2|_{R_2} \restriction r) \subseteq \mathsf{chan}(\mathbf{if}\ \xi|_0\ G_1|_{R_1}\ G_2|_{R_2})$. Then, $\mathsf{chan}(G \restriction r) \subseteq \mathsf{chan}(G)$.

– **Step:** $G = \mathbf{while}\ \xi|_0\ \{\psi\}\ \hat{G}|_\emptyset$, for some $\hat{G}, \psi, \xi$. Similar to case "$G = \mathbf{if}\ \xi|_0\ G_1|_{R_1}\ G_2|_{R_2}$, for some R_1, R_2, G_1, G_2, ξ". □

Proof (of Lem. 46).

– Recall $\checkmark_R(\mathbf{while}\ \xi|_0\ \{\psi\}\ G|_\emptyset)$. Then, by Defn. 27, $\checkmark_R(G)$ and $R = \mathsf{subj}(\xi)$.
– Recall $\checkmark_R(G)$ and $\checkmark_R(\mathbf{while}\ \xi|_0\ \{\psi\}\ G|_\emptyset)$. Then, by [✓1-Seq], $\checkmark_R(G\,;\mathbf{while}\ \xi|_0\ \{\psi\}\ G|_\emptyset)$.
– Recall $\checkmark_R(G)$. Then, by Lem. 47, $R \neq \emptyset$. Then, by [✓1-Skip], $\checkmark_R(\mathbf{skip})$.
– Recall $\checkmark_R(G\,;\mathbf{while}\ \xi|_0\ \{\psi\}\ G|_\emptyset)$ and $\checkmark_R(\mathbf{skip})$ and $R = \mathsf{subj}(\xi)$ and $\emptyset, \emptyset \subseteq \mathsf{subj}(\xi)$, and $\emptyset \neq \emptyset$ implies $\emptyset = \emptyset$, and $\emptyset \neq \emptyset$ implies $\emptyset = \emptyset$. Then, by [✓1-NIf], $\mathbf{if}\ \xi|_0\ (G\,;\mathbf{while}\ \xi|_0\ \{\psi\}\ G|_\emptyset)|_\emptyset\ \mathbf{skip}|_\emptyset$. □

Proof (of Lem. 47). Recall $\checkmark_R(G)$. Then, by Defn. 27:

– **Base:** [✓1-Act1], such that $G = q.y{:=}e$ and $q \in R$, for some q, y, e.
  - Recall $q \in R$. Then, $\{q\} \subseteq R$. Then, by Defn. 24, $\mathsf{subj}(q.y{:=}e) \subseteq R$. Then, $\mathsf{subj}(G) \subseteq R$.
  - Recall $q \in R$. Then, $R \neq \emptyset$.
– **Base:** [✓1-Act2]. Similar to case [✓1-Act1].
– **Base:** [✓1-Skip], such that $G = \mathbf{skip}$ and $R \neq \emptyset$. Recall $\emptyset \subseteq R$. Then, by Defn. 24, $\mathsf{subj}(\mathbf{skip}) \subseteq R$. Then, $\mathsf{subj}(G) \subseteq R$.
– **Step:** [✓1-Seq], such that $G = G_1\,;G_2$ and $\checkmark_R(G_1)$ and $\checkmark_R(G_2)$, for some G_1, G_2.
  - Recall $\checkmark_R(G_1)$ and $\checkmark_R(G_2)$. Then, by induction, $\mathsf{subj}(G_1) \subseteq R$ and $\mathsf{subj}(G_2) \subseteq R$ and $R \neq \emptyset$.
  - Recall $\mathsf{subj}(G_1) \subseteq R$ and $\mathsf{subj}(G_2) \subseteq R$. Then, $\mathsf{subj}(G_1) \cup \mathsf{subj}(G_2) \subseteq R$. Then, by Defn. 24, $\mathsf{subj}(G_1\,;G_2) \subseteq R$. Then, $\mathsf{subj}(G) \subseteq R$.
– **Step:** [✓1-Par]. Similar to case [✓1-Seq].
– **Step:** [✓1-If], such that $G = R.\mathbf{if}\ \xi\ G_1\ G_2$ and $\checkmark_R(G_1)$ and $\checkmark_R(G_2)$ and $R = \mathsf{subj}(\xi)$, for some G_1, G_2, ξ.
  - Recall $\checkmark_R(G_1)$ and $\checkmark_R(G_2)$. Then, by induction, $\mathsf{subj}(G_1) \subseteq R$ and $\mathsf{subj}(G_2) \subseteq R$ and $R \neq \emptyset$.
  - Recall $R \subseteq R$ and $\mathsf{subj}(G_1) \subseteq R$ and $\mathsf{subj}(G_2) \subseteq R$. Then, $R \cup \mathsf{subj}(G_1) \cup \mathsf{subj}(G_2) \subseteq R$. Then, $\mathsf{subj}(\xi) \cup \mathsf{subj}(G_1) \cup \mathsf{subj}(G_2) \subseteq R$. Then, by Defn. 24, $\mathsf{subj}(R.\mathbf{if}\ \xi\ G_1\ G_2) \subseteq R$. Then, $\mathsf{subj}(G) \subseteq R$.
– **Step:** [✓1-While]. Similar to case [✓1-If].
– **Step:** [✓1-NIf], such that $G = \mathbf{if}\ \xi|_0\ G_1|_{R_1}\ G_2|_{R_2}$ and $\checkmark_R(G_1)$ and $\checkmark_R(G_2)$ and $R = \mathsf{subj}(\xi)$, for some R_1, R_2, G_1, G_2, ξ.
  - Recall $R \subseteq R$. Then, $\mathsf{subj}(\xi) \subseteq R$. Then, $\mathsf{subj}(\xi) \setminus (R_1 \cup R_2) \subseteq R$.
  - Recall $\checkmark_R(G_1)$ and $\checkmark_R(G_2)$. Then, by induction, $\mathsf{subj}(G_1) \subseteq R$ and $\mathsf{subj}(G_2) \subseteq R$ and $R \neq \emptyset$.
  - Recall $\mathsf{subj}(G_1) \subseteq R$ and $\mathsf{subj}(G_2) \subseteq R$. Then, $\mathsf{subj}(G_1) \setminus R_2 \subseteq R$ and $\mathsf{subj}(G_2) \setminus R_1 \subseteq R$.
  - Recall $\mathsf{subj}(\xi) \setminus (R_1 \cup R_2) \subseteq R$ and $\mathsf{subj}(G_1) \setminus R_2 \subseteq R$ and $\mathsf{subj}(G_2) \setminus R_1 \subseteq R$. Then, $(\mathsf{subj}(\xi) \setminus (R_1 \cup R_2)) \cup (\mathsf{subj}(G_1) \setminus R_2) \cup (\mathsf{subj}(G_2) \setminus R_1) \subseteq R$. Then, by Defn. 24, $\mathsf{subj}(\mathbf{if}\ \xi|_0\ G_1|_{R_1}\ G_2|_{R_2}) \subseteq R$. Then, $\mathsf{subj}(G) \subseteq R$.
– **Step:** [✓1-NWhile], such that $G = \mathbf{while}\ \xi|_0\ \{\psi\}\ \hat{G}|_\emptyset$ and $\checkmark_R(\hat{G})$ and $R = \mathsf{subj}(\xi)$, for some $\hat{G}, \psi, \xi$.
  - Recall $\checkmark_R(\hat{G})$. Then, by induction, $\mathsf{subj}(\hat{G}) \subseteq R$ and $R \neq \emptyset$.
  - Recall $R \subseteq R$ and $\mathsf{subj}(\hat{G}) \subseteq R$. Then, $R \cup \mathsf{subj}(\hat{G}) \subseteq R$. Then, $\mathsf{subj}(\xi) \cup \mathsf{subj}(\hat{G}) \subseteq R$. Then, by Defn. 24, $\mathsf{subj}(\mathbf{while}\ \xi|_0\ \{\psi\}\ \hat{G}|_\emptyset) \subseteq R$. Then, $\mathsf{subj}(G) \subseteq R$. □

E Termination

Proof (of Lem. 48). Recall $G\downarrow$. Then, by Defn. 28:

- **Base:** [$\downarrow$1-Skip], such that $G = \textbf{skip}$.
  By Defn. 24, $\mathsf{subj}(\textbf{skip}) = \emptyset$. Then, $\mathsf{subj}(G) = \emptyset$.
- **Step:** [$\downarrow$1-Seq], such that $G = G_1 \,;\, G_2$ and $G_1\downarrow$ and $G_2\downarrow$, for some G_1, G_2.
  - Recall $\checkmark_R(G)$. Then, $\checkmark_R(G_1 \,;\, G_2)$. Then, by Defn. 27, $\checkmark_R(G_1)$ and $\checkmark_R(G_2)$.
  - Recall $\checkmark_R(G_1)$ and $G_1\downarrow$. Then, $\mathsf{subj}(G_1) = \emptyset$.
  - Recall $\checkmark_R(G_2)$ and $G_2\downarrow$. Then, $\mathsf{subj}(G_1) = \emptyset$.
  - Recall $\emptyset \cup \emptyset = \emptyset$. Then, $\mathsf{subj}(G_1) \cup \mathsf{subj}(G_2) = \emptyset$. Then, by Defn. 24, $\mathsf{subj}(G_1 \,;\, G_2) = \emptyset$. Then, $\mathsf{subj}(G) = \emptyset$.
- **Step:** [$\downarrow$1-Par]. Similar to case [$\downarrow$1-Seq].
- **Step:** [$\downarrow$1-NIf], such that $G = \textbf{if}\ \xi|_0\ G_1|_{R_1}\ G_2|_{R_2}$ and $\mathsf{subj}(\xi) = R_1 \cup R_2$, and $R_1 \neq \emptyset$ implies $G_1\downarrow$, and $R_2 \neq \emptyset$ implies $G_2\downarrow$, for some R_1, R_2, G_1, G_2, ξ.
  Recall:
  - **Case:** $R_1 = \emptyset = R_2$.
    * Recall $\checkmark_R(G)$. Then, $\checkmark_R(\textbf{if}\ \xi|_0\ G_1|_{R_1}\ G_2|_{R_2})$. Then, by Defn. 27, $R = \mathsf{subj}(\xi)$.
    * Recall $\checkmark_R(G)$. Then, by Lem. 47, $R \neq \emptyset$. Then, $\mathsf{subj}(\xi) \neq \emptyset$. Then, $R_1 \cup R_2 \neq \emptyset$. Then, $\emptyset \cup \emptyset \neq \emptyset$. Then, $\emptyset \neq \emptyset$. Then, **false**.
  - **Case:** $R_1 = \emptyset \neq R_2$.
    * Recall $\mathsf{subj}(\xi) \setminus \mathsf{subj}(\xi) = \emptyset$. Then, $\mathsf{subj}(\xi) \setminus (R_1 \cup R_2) = \emptyset$.
    * Recall $\checkmark_R(G)$. Then, $\checkmark_R(\textbf{if}\ \xi|_0\ G_1|_{R_1}\ G_2|_{R_2})$. Then, by Defn. 27, $\checkmark_R(G_1)$.
    * Recall $\checkmark_R(G)$. Then, $\checkmark_R(\textbf{if}\ \xi|_0\ G_1|_{R_1}\ G_2|_{R_2})$. Then, by Defn. 27, $R = \mathsf{subj}(\xi)$.
    * Recall $\mathsf{subj}(\xi) = R_1 \cup R_2$. Then, $\mathsf{subj}(\xi) = \emptyset \cup R_2$. Then, $\mathsf{subj}(\xi) = R_2$.
    * Recall $\checkmark_R(G_1)$. Then, by Lem. 47, $\mathsf{subj}(G_1) \subseteq R$. Then, $\mathsf{subj}(G_1) \setminus R \subseteq R \setminus R$. Then, $\mathsf{subj}(G_1) \setminus \mathsf{subj}(\xi) \subseteq R \setminus R$. Then, $\mathsf{subj}(G_1) \setminus R_2 \subseteq R \setminus R$. Then, $\mathsf{subj}(G_1) \setminus R_2 \subseteq \emptyset$. Then, $\mathsf{subj}(G_1) \setminus R_2 = \emptyset$.
    * Recall $R_2 \neq \emptyset$. Then, $G_2\downarrow$.
    * Recall $\checkmark_R(G)$. Then, $\checkmark_R(\textbf{if}\ \xi|_0\ G_1|_{R_1}\ G_2|_{R_2})$. Then, by Defn. 27, $\checkmark_R(G_2)$.
    * Recall $G_2\downarrow$ and $\checkmark_R(G_2)$. Then, by induction, $\mathsf{subj}(G_2) = \emptyset$. Then, $\mathsf{subj}(G_2) \setminus R_1 = \emptyset \setminus R_1$. Then, $\mathsf{subj}(G_2) \setminus R_1 = \emptyset$.
    * Recall $\emptyset \cup \emptyset \cup \emptyset = \emptyset$. Then, $(\mathsf{subj}(\xi) \setminus (R_1 \cup R_2)) \cup (\mathsf{subj}(G_1) \setminus R_2) \cup (\mathsf{subj}(G_2) \setminus R_1) = \emptyset$. Then, by Defn. 24, $\mathsf{subj}(\textbf{if}\ \xi|_0\ G_1|_{R_1}\ G_2|_{R_2}) = \emptyset$. Then, $\mathsf{subj}(G) = \emptyset$.
  - **Case:** $R_1 \neq \emptyset = R_2$. Similar to case $R_1 = \emptyset \neq R_2$.
  - **Case:** $R_1 \neq \emptyset \neq R_2$.
    * Recall $\mathsf{subj}(\xi) \setminus \mathsf{subj}(\xi) = \emptyset$. Then, $\mathsf{subj}(\xi) \setminus (R_1 \cup R_2) = \emptyset$.
    * Recall $R_1 \neq \emptyset$. Then, $G_1\downarrow$.
    * Recall $\checkmark_R(G)$. Then, $\checkmark_R(\textbf{if}\ \xi|_0\ G_1|_{R_1}\ G_2|_{R_2})$. Then, by Defn. 27, $\checkmark_R(G_1)$.
    * Recall $G_1\downarrow$ and $\checkmark_R(G_1)$. Then, by induction, $\mathsf{subj}(G_1) = \emptyset$. Then, $\mathsf{subj}(G_1) \setminus R_2 = \emptyset \setminus R_2$. Then, $\mathsf{subj}(G_1) \setminus R_2 = \emptyset$.
    * Recall $R_2 \neq \emptyset$. Then, $G_2\downarrow$.
    * Recall $\checkmark_R(G)$. Then, $\checkmark_R(\textbf{if}\ \xi|_0\ G_1|_{R_1}\ G_2|_{R_2})$. Then, by Defn. 27, $\checkmark_R(G_2)$.
    * Recall $G_2\downarrow$ and $\checkmark_R(G_2)$. Then, by induction, $\mathsf{subj}(G_2) = \emptyset$. Then, $\mathsf{subj}(G_2) \setminus R_1 = \emptyset \setminus R_1$. Then, $\mathsf{subj}(G_2) \setminus R_1 = \emptyset$.
    * Recall $\emptyset \cup \emptyset \cup \emptyset = \emptyset$. Then, $(\mathsf{subj}(\xi) \setminus (R_1 \cup R_2)) \cup (\mathsf{subj}(G_1) \setminus R_2) \cup (\mathsf{subj}(G_2) \setminus R_1) = \emptyset$. Then, by Defn. 24, $\mathsf{subj}(\textbf{if}\ \xi|_0\ G_1|_{R_1}\ G_2|_{R_2}) = \emptyset$. Then, $\mathsf{subj}(G) = \emptyset$. □

Proof (of Lem. 49). Recall $G\downarrow$. Then, by Defn. 28:

- **Base:** [$\downarrow$1-Skip], such that $G = \textbf{skip}$.
  By [$\downarrow$1-Skip], $\textbf{skip}\downarrow$. Then, by Defn. 26, $(\textbf{skip} \restriction r)\downarrow$. Then, $(G \restriction r)\downarrow$.
- **Step:** [$\downarrow$1-Seq], such that $G = G_1 \,;\, G_2$ and $G_1\downarrow$ and $G_2\downarrow$, for some G_1, G_2.
  - Recall $\checkmark_R(G)$. Then, $\checkmark_R(G_1 \,;\, G_2)$. Then, by Defn. 27, $\checkmark_R(G_1)$ and $\checkmark_R(G_2)$.

- Recall $\checkmark_R(G_1)$ and $G_1\downarrow$. Then, by induction, $(G_1 \upharpoonright r)\downarrow$.
- Recall $\checkmark_R(G_2)$ and $G_2\downarrow$. Then, by induction, $(G_2 \upharpoonright r)\downarrow$.
- Recall $(G_1 \upharpoonright r)\downarrow$ and $(G_2 \upharpoonright r)\downarrow$. Then, by [$\downarrow$1-Seq], $(G_1 \upharpoonright r) ; (G_2 \upharpoonright r)\downarrow$. Then, by Defn. 26, $(G_1 ; G_2 \upharpoonright r)\downarrow$. Then, $(G \upharpoonright r)\downarrow$.

– **Step:** [$\downarrow$1-Par]. Similar to case [$\downarrow$1-Seq].

– **Step:** [$\downarrow$1-NIf], such that $G = \textbf{if } \xi|_0\, G_1|_{R_1}\, G_2|_{R_2}$ and $\mathsf{subj}(\xi) = R_1 \cup R_2$, and $R_1 \neq \emptyset$ implies $G_1\downarrow$, and $R_2 \neq \emptyset$ implies $G_2\downarrow$, for some R_1, R_2, G_1, G_2, ξ.
Recall:

- **Case:** $R_1 = \emptyset = R_2$.
  - ∗ Recall $\checkmark_R(G)$. Then, $\checkmark_R(\textbf{if } \xi|_0\, G_1|_{R_1}\, G_2|_{R_2})$. Then, by Defn. 27, $R = \mathsf{subj}(\xi)$.
  - ∗ Recall $\checkmark_R(G)$. Then, by Lem. 47, $R \neq \emptyset$. Then, $\mathsf{subj}(\xi) \neq \emptyset$. Then, $R_1 \cup R_2 \neq \emptyset$. Then, $\emptyset \cup \emptyset \neq \emptyset$. Then, $\emptyset \neq \emptyset$. Then, **false**.
- **Case:** $R_1 = \emptyset \neq R_2$.
  - ∗ Recall $R_1 = \emptyset$. Then, $R_1 \cap \{r\} = \emptyset$. Then, $R_1 \cap \{r\} \neq \emptyset$ implies $(G_1 \upharpoonright r)\downarrow$.
  - ∗ Recall $R_2 \neq \emptyset$. Then, $G_2\downarrow$.
  - ∗ Recall $\checkmark_R(G)$. Then, $\checkmark_R(\textbf{if } \xi|_0\, G_1|_{R_1}\, G_2|_{R_2})$. Then, by Defn. 27, $\checkmark_R(G_2)$.
  - ∗ Recall $G_2\downarrow$ and $\checkmark_R(G_2)$. Then, by induction, $(G_2 \upharpoonright r)\downarrow$. Then, $R_2 \cap \{r\} \neq \emptyset$ implies $(G_2 \upharpoonright r)\downarrow$.
  - ∗ Recall $\mathsf{subj}(\xi) \subseteq \mathsf{subj}(\xi)$. Then, $\mathsf{subj}(\xi) \subseteq R_1 \cup R_2$. Then, $\mathsf{subj}(\xi) \subseteq R_1 \cup R_2 \cup \{r\}$. Then, $\mathsf{subj}(\xi) \setminus (R_1 \cup R_2 \cup \{r\}) = \emptyset$. Then, $|\mathsf{subj}(\xi) \setminus (R_1 \cup R_2 \cup \{r\})| = |\emptyset|$. Then, $|\mathsf{subj}(\xi) \setminus (R_1 \cup R_2 \cup \{r\})| = 0$.
  - ∗ Recall:
    - · **Case:** $r \in R_1 \cup R_2$.
      - · Recall $r \in R_1 \cup R_2$. Then, $r \in \mathsf{subj}(\xi)$. Then, by Lem. 9, $\mathsf{subj}(\xi \upharpoonright r) = \{r\}$.
      - · Recall $r \in R_1 \cup R_2$. Then, $\{r\} \subseteq R_1 \cup R_2$. Then, $\{r\} = (R_1 \cup R_2) \cap \{r\}$. Then, $\mathsf{subj}(\xi \upharpoonright r) = (R_1 \cup R_2) \cap \{r\}$. Then, $\mathsf{subj}(\xi \upharpoonright r) = (R_1 \cap \{r\}) \cup (R_2 \cap \{r\})$.
      - · Recall $\mathsf{subj}(\xi \upharpoonright r) = (R_1 \cap \{r\}) \cup (R_2 \cap \{r\})$, and $R_1 \cap \{r\} \neq \emptyset$ implies $(G_1 \upharpoonright r)\downarrow$, and $R_2 \cap \{r\} \neq \emptyset$ implies $(G_2 \upharpoonright r)\downarrow$. Then, by [$\downarrow$1-NIf], $\textbf{if } (\xi \upharpoonright r)|_0\, (G_1 \upharpoonright r)|_{R_1 \cap \{r\}}\, (G_2 \upharpoonright r)|_{R_2 \cap \{r\}}\downarrow$. Then, $\textbf{if } (\xi \upharpoonright r)|_{|\mathsf{subj}(\xi) \setminus (R_1 \cup R_2 \cup \{r\})|}\, (G_1 \upharpoonright r)|_{R_1 \cap \{r\}}\, (G_2 \upharpoonright r)|_{R_2 \cap \{r\}}\downarrow$. Then, by Defn. 26, $(\textbf{if } \xi|_0\, G_1|_{R_1}\, G_2|_{R_2} \upharpoonright r)\downarrow$. Then, $(G \upharpoonright r)\downarrow$.
    - · **Case:** $r \notin R_1 \cup R_2$.
      - · Recall $r \notin R_1 \cup R_2$. Then, $r \notin \mathsf{subj}(\xi)$. Then, by Lem. 9, $\mathsf{subj}(\xi \upharpoonright r) = \emptyset$.
      - · Recall $r \notin R_1 \cup R_2$. Then, $\emptyset = (R_1 \cup R_2) \cap \{r\}$. Then, $\mathsf{subj}(\xi \upharpoonright r) = (R_1 \cup R_2) \cap \{r\}$. Then, $\mathsf{subj}(\xi \upharpoonright r) = (R_1 \cap \{r\}) \cup (R_2 \cap \{r\})$.
      - · Recall $\mathsf{subj}(\xi \upharpoonright r) = (R_1 \cap \{r\}) \cup (R_2 \cap \{r\})$, and $R_1 \cap \{r\} \neq \emptyset$ implies $(G_1 \upharpoonright r)\downarrow$, and $R_2 \cap \{r\} \neq \emptyset$ implies $(G_2 \upharpoonright r)\downarrow$. Then, by [$\downarrow$1-NIf], $\textbf{if } (\xi \upharpoonright r)|_0\, (G_1 \upharpoonright r)|_{R_1 \cap \{r\}}\, (G_2 \upharpoonright r)|_{R_2 \cap \{r\}}\downarrow$. Then, $\textbf{if } (\xi \upharpoonright r)|_{|\mathsf{subj}(\xi) \setminus (R_1 \cup R_2 \cup \{r\})|}\, (G_1 \upharpoonright r)|_{R_1 \cap \{r\}}\, (G_2 \upharpoonright r)|_{R_2 \cap \{r\}}\downarrow$. Then, by Defn. 26, $(\textbf{if } \xi|_0\, G_1|_{R_1}\, G_2|_{R_2} \upharpoonright r)\downarrow$. Then, $(G \upharpoonright r)\downarrow$.
- **Case:** $R_1 \neq \emptyset = R_2$. Similar to case $R_1 = \emptyset \neq R_2$.
- **Case:** $R_1 \neq \emptyset \neq R_2$. Similar to case $R_1 = \emptyset \neq R_2$. □

F Reduction

Proof (of Lem. 50). Recall $A \xrightarrow{\xi,\alpha} A'$. Then, by Defn. 31:

- **Base:** [$\to$1-Act], such that $\xi = \texttt{true}$.
  Recall:
  - **Case:** $\alpha = \tau$.
    Recall $(\xi, \alpha) \neq (\texttt{true}, \tau)$. Then, $(\texttt{true}, \tau) \neq (\texttt{true}, \tau)$. Then, **false**.
  - **Case:** $\alpha \neq \tau$.
- **Base:** [$\to$1-If1], such that $\alpha \neq \tau$.
- **Base:** [$\to$1-If2]. Similar to case [$\to$1-If1].
- **Base:** [$\to$1-While1]. Similar to case [$\to$1-If1].
- **Base:** [$\to$1-While2]. Similar to case [$\to$1-If1].
- **Base:** [$\to$1-NIf1], such that $\xi = \texttt{true}$ and $\alpha = \tau$.
  Recall $(\xi, \alpha) \neq (\texttt{true}, \tau)$. Then, $(\texttt{true}, \tau) \neq (\texttt{true}, \tau)$. Then, **false**.
- **Base:** [$\to$1-NIf2]. Similar to case [$\to$1-If1].
- **Base:** [$\to$1-NIf3]. Similar to case [$\to$1-If1].
- **Step:** [$\to$1-Seq1], such that $A_1 \xrightarrow{\xi,\alpha} A'_1$, for some A_1, A'_1.
  Recall $A_1 \xrightarrow{\xi,\alpha} A'_1$ and $(\xi, \alpha) \neq (\texttt{true}, \tau)$. Then, by induction, $\alpha \neq \tau$.
- **Step:** [$\to$1-Seq2]. Similar to case [$\to$1-Seq1].
- **Step:** [$\to$1-Par1]. Similar to case [$\to$1-Seq1].
- **Step:** [$\to$1-Par2]. Similar to case [$\to$1-Seq1].
- **Step:** [$\to$1-NIf4]. Similar to case [$\to$1-Seq1].
- **Step:** [$\to$1-NIf5]. Similar to case [$\to$1-Seq1].
- **Step:** [$\to$1-NWhile]. Similar to case [$\to$1-Seq1]. $\square$

Proof (of Lem. 51).

1. Recall $A \xrightarrow{\xi,\alpha} A'$. Then, by Defn. 31:
   - **Base:** [$\to$1-Act], such that $A = \alpha$ and $A' = \textbf{skip}$ and $\xi = \texttt{true}$.
     Recall $\emptyset \cup \mathsf{read}(A) \cup \emptyset \subseteq \mathsf{read}(A)$. Then, by Lem. 3, $\mathsf{read}(\texttt{true}) \cup \mathsf{read}(A) \cup \emptyset \subseteq \mathsf{read}(A)$. Then, by Defn. 22, $\mathsf{read}(\texttt{true}) \cup \mathsf{read}(A) \cup \mathsf{read}(\textbf{skip}) \subseteq \mathsf{read}(A)$. Then,
     $\mathsf{read}(\xi) \cup \mathsf{read}(\alpha) \cup \mathsf{read}(A') \subseteq \mathsf{read}(A)$.
   - **Base:** [$\to$1-If1], such that $A = R.\textbf{if}\ \hat{\xi}\ A_1\ A_2$ and $A' = A_1$ and $\xi = \hat{\xi}^+$ and $\alpha = 1^R_{\mathsf{subj}(\hat{\xi})}$, for some $R, A_1, A_2, \hat{\xi}$.
     Recall $\mathsf{read}(\hat{\xi}) \cup \emptyset \cup \mathsf{read}(A_1) \subseteq \mathsf{read}(\hat{\xi}) \cup \mathsf{read}(A_1) \cup \mathsf{read}(A_2)$. Then, by Lem. 12, $\mathsf{read}(\hat{\xi}^+) \cup \emptyset \cup \mathsf{read}(A_1) \subseteq \mathsf{read}(\hat{\xi}) \cup \mathsf{read}(A_1) \cup \mathsf{read}(A_2)$. Then, by Defn. 15, $\mathsf{read}(\hat{\xi}^+) \cup \mathsf{read}(1^R_{\mathsf{subj}(\hat{\xi})}) \cup \mathsf{read}(A_1) \subseteq \mathsf{read}(\hat{\xi}) \cup \mathsf{read}(A_1) \cup \mathsf{read}(A_2)$. Then, by Defn. 22, $\mathsf{read}(\hat{\xi}^+) \cup \mathsf{read}(1^R_{\mathsf{subj}(\hat{\xi})}) \cup \mathsf{read}(A_1) \subseteq \mathsf{read}(R.\textbf{if}\ \hat{\xi}\ A_1\ A_2)$. Then,
     $\mathsf{read}(\xi) \cup \mathsf{read}(\alpha) \cup \mathsf{read}(A') \subseteq \mathsf{read}(A)$.
   - **Base:** [$\to$1-If2]. Similar to case [$\to$1-If1].
   - **Base:** [$\to$1-While1], such that $A = R.\textbf{while}\ \hat{\xi}\ \{\psi\}\ \hat{A}$ and $A' = \hat{A}\,;R.\textbf{while}\ \hat{\xi}\ \{\psi\}\ \hat{A}$ and $\xi = \hat{\xi}^+$ and $\alpha = 1^R_{\mathsf{subj}(\hat{\xi})}$, for some $R, \hat{A}, \psi, \hat{\xi}$.
     Recall $\mathsf{read}(\hat{\xi}) \cup \emptyset \cup \mathsf{read}(\hat{A}) \cup \mathsf{read}(\hat{\xi}) \cup \mathsf{read}(\hat{A}) \subseteq \mathsf{read}(\hat{\xi}) \cup \mathsf{read}(\hat{A})$. Then, by Lem. 12, $\mathsf{read}(\hat{\xi}^+) \cup \emptyset \cup \mathsf{read}(\hat{A}) \cup \mathsf{read}(\hat{\xi}) \cup \mathsf{read}(\hat{A}) \subseteq \mathsf{read}(\hat{\xi}) \cup \mathsf{read}(\hat{A})$. Then, by Defn. 15, $\mathsf{read}(\hat{\xi}^+) \cup \mathsf{read}(1^R_{\mathsf{subj}(\hat{\xi})}) \cup \mathsf{read}(\hat{A}) \cup \mathsf{read}(\hat{\xi}) \cup \mathsf{read}(\hat{A}) \subseteq \mathsf{read}(\hat{\xi}) \cup \mathsf{read}(\hat{A})$. Then, by Defn. 22,
     $\mathsf{read}(\hat{\xi}^+) \cup \mathsf{read}(1^R_{\mathsf{subj}(\hat{\xi})}) \cup \mathsf{read}(\hat{A}) \cup \mathsf{read}(R.\textbf{while}\ \hat{\xi}\ \{\psi\}\ \hat{A}) \subseteq \mathsf{read}(R.\textbf{while}\ \hat{\xi}\ \{\psi\}\ \hat{A})$.
     Then, by Defn. 22,
     $\mathsf{read}(\hat{\xi}^+) \cup \mathsf{read}(1^R_{\mathsf{subj}(\hat{\xi})}) \cup \mathsf{read}(\hat{A}\,;R.\textbf{while}\ \hat{\xi}\ \{\psi\}\ \hat{A}) \subseteq \mathsf{read}(R.\textbf{while}\ \hat{\xi}\ \{\psi\}\ \hat{A})$. Then,
     $\mathsf{read}(\xi) \cup \mathsf{read}(\alpha) \cup \mathsf{read}(A') \subseteq \mathsf{read}(A)$.
   - **Base:** [$\to$1-While2]. Similar to case [$\to$1-If1].

– **Base:** [$\rightarrow$1-NIF1], such that $A = \textbf{if } \hat{\xi}|_n \ A_1|_{R_1} \ A_2|_{R_2}$ and $A' = \textbf{if } \hat{\xi}|_{n-1} \ A_1|_{R_1} \ A_2|_{R_2}$ and $\xi = \mathtt{true}$ and $\alpha = \tau$, for some $R_1, R_2, A_1, A_2, \hat{\xi}, n$.
Recall $\emptyset \cup \emptyset \cup \mathsf{read}(\textbf{if } \hat{\xi}|_{n-1} \ A_1|_{R_1} \ A_2|_{R_2}) \subseteq \mathsf{read}(\textbf{if } \hat{\xi}|_{n-1} \ A_1|_{R_1} \ A_2|_{R_2})$. Then, by Lem. 3, $\mathsf{read}(\mathtt{true}) \cup \emptyset \cup \mathsf{read}(\textbf{if } \hat{\xi}|_{n-1} \ A_1|_{R_1} \ A_2|_{R_2}) \subseteq \mathsf{read}(\textbf{if } \hat{\xi}|_{n-1} \ A_1|_{R_1} \ A_2|_{R_2})$. Then, by Defn. 15, $\mathsf{read}(\mathtt{true}) \cup \mathsf{read}(\tau) \cup \mathsf{read}(\textbf{if } \hat{\xi}|_{n-1} \ A_1|_{R_1} \ A_2|_{R_2}) \subseteq \mathsf{read}(\textbf{if } \hat{\xi}|_{n-1} \ A_1|_{R_1} \ A_2|_{R_2})$. Then, by Defn. 22, $\mathsf{read}(\mathtt{true}) \cup \mathsf{read}(\tau) \cup \mathsf{read}(\textbf{if } \hat{\xi}|_{n-1} \ A_1|_{R_1} \ A_2|_{R_2}) \subseteq (\mathsf{read}(\hat{\xi}) \setminus \{\hat{r}.x \mid \hat{r} \in R_1 \cup R_2\}) \cup (\mathsf{read}(G_1) \setminus \{\hat{r}.x \mid \hat{r} \in R_2\}) \cup (\mathsf{read}(G_2) \setminus \{\hat{r}.x \mid \hat{r} \in R_1\})$. Then, by Defn. 22, $\mathsf{read}(\mathtt{true}) \cup \mathsf{read}(\tau) \cup \mathsf{read}(\textbf{if } \hat{\xi}|_{n-1} \ A_1|_{R_1} \ A_2|_{R_2}) \subseteq \mathsf{read}(\textbf{if } \hat{\xi}|_n \ A_1|_{R_1} \ A_2|_{R_2})$.
Then, $\mathsf{read}(\xi) \cup \mathsf{read}(\alpha) \cup \mathsf{read}(A') \subseteq \mathsf{read}(A)$.

– **Base:** [$\rightarrow$1-NIF2], such that $A = \textbf{if } \hat{\xi}|_n \ A_1|_{R_1} \ A_2|_{R_2}$ and $A' = \textbf{if } \hat{\xi}|_n \ A_1|_{R_1 \cup \{r\}} \ A_2|_{R_2}$ and $\xi = \hat{\xi}^+ \upharpoonright r$ and $\alpha = 1\{^r_r\}$ and $r \in \mathsf{subj}(\hat{\xi}) \setminus (R_1 \cup R_2)$, for some $R_1, R_2, A_1, A_2, r, \hat{\xi}, n$.
  - Recall $r \in \mathsf{subj}(\hat{\xi}) \setminus (R_1 \cup R_2)$. Then, $r \notin R_1 \cup R_2$. Then, $\{r\} \cap (R_1 \cup R_2) = \emptyset$.
  - By Lem. 5, $\mathsf{read}(\hat{\xi} \upharpoonright r) \subseteq \{\hat{r}.x \mid \hat{r} \in \mathsf{subj}(\hat{\xi} \upharpoonright r)\}$. Then, by Lem. 10, $\mathsf{read}(\hat{\xi} \upharpoonright r) \subseteq \{\hat{r}.x \mid \hat{r} \in \mathsf{subj}(\hat{\xi} \upharpoonright r) \subseteq \{r\}\}$. Then, $\mathsf{read}(\hat{\xi} \upharpoonright r) \subseteq \{\hat{r}.x \mid \hat{r} \in \{r\}\}$. Then, $\mathsf{read}(\hat{\xi} \upharpoonright r) \cap \{\hat{r}.x \mid \hat{r} \in R_1 \cup R_2\} \subseteq \{\hat{r}.x \mid \hat{r} \in \{r\}\} \cap \{\hat{r}.x \mid \hat{r} \in R_1 \cup R_2\}$. Then, $\mathsf{read}(\hat{\xi} \upharpoonright r) \cap \{\hat{r}.x \mid \hat{r} \in R_1 \cup R_2\} \subseteq \{\hat{r}.x \mid \hat{r} \in \{r\} \cap (R_1 \cup R_2)\}$. Then, $\mathsf{read}(\hat{\xi} \upharpoonright r) \cap \{\hat{r}.x \mid \hat{r} \in R_1 \cup R_2\} \subseteq \{\hat{r}.x \mid \hat{r} \in \emptyset\}$. Then, $\mathsf{read}(\hat{\xi} \upharpoonright r) \cap \{\hat{r}.x \mid \hat{r} \in R_1 \cup R_2\} \subseteq \emptyset$. Then, $\mathsf{read}(\hat{\xi} \upharpoonright r) \cap \{\hat{r}.x \mid \hat{r} \in R_1 \cup R_2\} = \emptyset$. Then, $\mathsf{read}(\hat{\xi} \upharpoonright r) \setminus \{\hat{r}.x \mid \hat{r} \in R_1 \cup R_2\} = \mathsf{read}(\hat{\xi} \upharpoonright r)$.
  - By Lem. 8, $\mathsf{read}(\hat{\xi} \upharpoonright r) \subseteq \mathsf{read}(\hat{\xi})$. Then, $\mathsf{read}(\hat{\xi} \upharpoonright r) \setminus \{\hat{r}.x \mid \hat{r} \in R_1 \cup R_2\} \subseteq \mathsf{read}(\hat{\xi}) \setminus \{\hat{r}.x \mid \hat{r} \in R_1 \cup R_2\}$. Then, $\mathsf{read}(\hat{\xi} \upharpoonright r) \subseteq \mathsf{read}(\hat{\xi}) \setminus \{\hat{r}.x \mid \hat{r} \in R_1 \cup R_2\}$. Then, by Lem. 12, $\mathsf{read}((\hat{\xi} \upharpoonright r)^+) \subseteq \mathsf{read}(\hat{\xi}) \setminus \{\hat{r}.x \mid \hat{r} \in R_1 \cup R_2\}$. Then, by Lem. 11, $\mathsf{read}(\hat{\xi}^+ \upharpoonright r) \subseteq \mathsf{read}(\hat{\xi}) \setminus \{\hat{r}.x \mid \hat{r} \in R_1 \cup R_2\}$.
  - Recall $\mathsf{read}(\hat{\xi}) \setminus \{\hat{r}.x \mid \hat{r} \in R_1 \cup R_2\} \subseteq \mathsf{read}(\hat{\xi}) \setminus \{\hat{r}.x \mid \hat{r} \in R_1 \cup R_2\}$. Then, $(\mathsf{read}(\hat{\xi}) \setminus \{\hat{r}.x \mid \hat{r} \in R_1 \cup R_2\}) \setminus \{\hat{r}.x \mid \hat{r} \in \{r\}\} \subseteq \mathsf{read}(\hat{\xi}) \setminus \{\hat{r}.x \mid \hat{r} \in R_1 \cup R_2\}$. Then, $\mathsf{read}(\hat{\xi}) \setminus (\{\hat{r}.x \mid \hat{r} \in R_1 \cup R_2\} \cup \{\hat{r}.x \mid \hat{r} \in \{r\}\}) \subseteq \mathsf{read}(\hat{\xi}) \setminus \{\hat{r}.x \mid \hat{r} \in R_1 \cup R_2\}$. Then, $\mathsf{read}(\hat{\xi}) \setminus \{\hat{r}.x \mid \hat{r} \in R_1 \cup R_2 \cup \{r\}\} \subseteq \mathsf{read}(\hat{\xi}) \setminus \{\hat{r}.x \mid \hat{r} \in R_1 \cup R_2\}$.
  - Recall $\mathsf{read}(G_2) \setminus \{\hat{r}.x \mid \hat{r} \in R_1\} \subseteq \mathsf{read}(G_2) \setminus \{\hat{r}.x \mid \hat{r} \in R_1\}$. Then, $(\mathsf{read}(G_2) \setminus \{\hat{r}.x \mid \hat{r} \in R_1\}) \setminus \{\hat{r}.x \mid \hat{r} \in \{r\}\} \subseteq \mathsf{read}(G_2) \setminus \{\hat{r}.x \mid \hat{r} \in R_1\}$. Then, $\mathsf{read}(G_2) \setminus (\{\hat{r}.x \mid \hat{r} \in R_1\} \cup \{\hat{r}.x \mid \hat{r} \in \{r\}\}) \subseteq \mathsf{read}(G_2) \setminus \{\hat{r}.x \mid \hat{r} \in R_1\}$. Then, $\mathsf{read}(G_2) \setminus \{\hat{r}.x \mid \hat{r} \in R_1 \cup \{r\}\} \subseteq \mathsf{read}(G_2) \setminus \{\hat{r}.x \mid \hat{r} \in R_1\}$.
  - Recall $\mathsf{read}(\hat{\xi}^+ \upharpoonright r) \subseteq \mathsf{read}(\hat{\xi}) \setminus \{\hat{r}.x \mid \hat{r} \in R_1 \cup R_2\}$ and $\mathsf{read}(\hat{\xi}) \setminus \{\hat{r}.x \mid \hat{r} \in R_1 \cup R_2 \cup \{r\}\} \subseteq \mathsf{read}(\hat{\xi}) \setminus \{\hat{r}.x \mid \hat{r} \in R_1 \cup R_2\}$ and $\mathsf{read}(G_2) \setminus \{\hat{r}.x \mid \hat{r} \in R_1 \cup \{r\}\} \subseteq \mathsf{read}(G_2) \setminus \{\hat{r}.x \mid \hat{r} \in R_1\}$. Then, $\mathsf{read}(\hat{\xi}^+ \upharpoonright r) \cup (\mathsf{read}(\hat{\xi}) \setminus \{\hat{r}.x \mid \hat{r} \in R_1 \cup R_2 \cup \{r\}\}) \cup (\mathsf{read}(G_2) \setminus \{\hat{r}.x \mid \hat{r} \in R_1 \cup \{r\}\}) \subseteq (\mathsf{read}(\hat{\xi}) \setminus \{\hat{r}.x \mid \hat{r} \in R_1 \cup R_2\}) \cup (\mathsf{read}(G_2) \setminus \{\hat{r}.x \mid \hat{r} \in R_1\})$. Then, $\mathsf{read}(\hat{\xi}^+ \upharpoonright r) \cup \emptyset \cup (\mathsf{read}(\hat{\xi}) \setminus \{\hat{r}.x \mid \hat{r} \in R_1 \cup R_2 \cup \{r\}\}) \cup (\mathsf{read}(G_1) \setminus \{\hat{r}.x \mid \hat{r} \in R_2\}) \cup (\mathsf{read}(G_2) \setminus \{\hat{r}.x \mid \hat{r} \in R_1 \cup \{r\}\}) \subseteq (\mathsf{read}(\hat{\xi}) \setminus \{\hat{r}.x \mid \hat{r} \in R_1 \cup R_2\}) \cup (\mathsf{read}(G_1) \setminus \{\hat{r}.x \mid \hat{r} \in R_2\}) \cup (\mathsf{read}(G_2) \setminus \{\hat{r}.x \mid \hat{r} \in R_1\})$. Then, by Defn. 15,
  $\mathsf{read}(\hat{\xi}^+ \upharpoonright r) \cup \mathsf{read}(1\{^r_r\}) \cup (\mathsf{read}(\hat{\xi}) \setminus \{\hat{r}.x \mid \hat{r} \in R_1 \cup R_2 \cup \{r\}\}) \cup (\mathsf{read}(G_1) \setminus \{\hat{r}.x \mid \hat{r} \in R_2\}) \cup (\mathsf{read}(G_2) \setminus \{\hat{r}.x \mid \hat{r} \in R_1 \cup \{r\}\}) \subseteq (\mathsf{read}(\hat{\xi}) \setminus \{\hat{r}.x \mid \hat{r} \in R_1 \cup R_2\}) \cup (\mathsf{read}(G_1) \setminus \{\hat{r}.x \mid \hat{r} \in R_2\}) \cup (\mathsf{read}(G_2) \setminus \{\hat{r}.x \mid \hat{r} \in R_1\})$. Then, by Defn. 22,
  $\mathsf{read}(\hat{\xi}^+ \upharpoonright r) \cup \mathsf{read}(1\{^r_r\}) \cup \mathsf{read}(\textbf{if } \hat{\xi}|_n \ A_1|_{R_1 \cup \{r\}} \ A_2|_{R_2}) \subseteq \mathsf{read}(\textbf{if } \hat{\xi}|_n \ A_1|_{R_1} \ A_2|_{R_2})$.
  Then, $\mathsf{read}(\xi) \cup \mathsf{read}(\alpha) \cup \mathsf{read}(A') \subseteq \mathsf{read}(A)$.

– **Base:** [$\rightarrow$1-NIF3]. Similar to case [$\rightarrow$1-NIF2].

– **Step:** [$\rightarrow$1-SEQ1], such that $A = A_1 \, ; A_2$ and $A' = A'_1 \, ; A_2$ and $A_1 \xrightarrow{\xi,\alpha} A'_1$, for some A_1, A'_1, A_2.
Recall $A_1 \xrightarrow{\xi,\alpha} A'_1$. Then, by induction, $\mathsf{read}(\xi) \cup \mathsf{read}(\alpha) \cup \mathsf{read}(A'_1) \subseteq \mathsf{read}(A_1)$. Then, $\mathsf{read}(\xi) \cup \mathsf{read}(\alpha) \cup \mathsf{read}(A'_1) \cup \mathsf{read}(A_2) \subseteq \mathsf{read}(A_1) \cup \mathsf{read}(A_2)$. Then, by Defn. 22, $\mathsf{read}(\xi) \cup \mathsf{read}(\alpha) \cup \mathsf{read}(A'_1 \, ; A_2) \subseteq \mathsf{read}(A_1 \, ; A_2)$. Then,
$\mathsf{read}(\xi) \cup \mathsf{read}(\alpha) \cup \mathsf{read}(A') \subseteq \mathsf{read}(A)$.

- **Step:** [$\rightarrow$1-Seq2]. Similar to case [$\rightarrow$1-Seq1].
- **Step:** [$\rightarrow$1-Par1]. Similar to case [$\rightarrow$1-Seq1].
- **Step:** [$\rightarrow$1-Par2]. Similar to case [$\rightarrow$1-Seq1].
- **Step:** [$\rightarrow$1-NIf4], such that $A = \textbf{if}\ \hat{\xi}|_n\ A_1|_{R_1}\ A_2|_{R_2}$ and $A' = \textbf{if}\ \hat{\xi}|_n\ A_1'|_{R_1}\ A_2|_{R_2}$ and $A_1 \xrightarrow{\xi,\alpha} A_1'$ and $\mathsf{subj}(\xi) \cup \mathsf{subj}(\alpha) \subseteq R_1 \setminus R_2$, for some $R_1, R_2, A_1, A_1', A_2, \hat{\xi}, n$.
  - Recall $\mathsf{subj}(\xi) \cup \mathsf{subj}(\alpha) \subseteq R_1 \setminus R_2$. Then, $\mathsf{subj}(\xi) \subseteq R_1 \setminus R_2$. Then, $\mathsf{subj}(\xi) \cap R_2 = \emptyset$.
  - By Lem. 5, $\mathsf{read}(\xi) \subseteq \{r.x \mid r \in \mathsf{subj}(\xi)\}$. Then,
    $\mathsf{read}(\xi) \cap \{r.x \mid r \in R_2\} \subseteq \{r.x \mid r \in \mathsf{subj}(\xi)\} \cap \{r.x \mid r \in R_2\}$. Then,
    $\mathsf{read}(\xi) \cap \{r.x \mid r \in R_2\} \subseteq \{r.x \mid r \in \mathsf{subj}(\xi) \cap R_2\}$. Then,
    $\mathsf{read}(\xi) \cap \{r.x \mid r \in R_2\} \subseteq \{r.x \mid r \in \emptyset\}$. Then, $\mathsf{read}(\xi) \cap \{r.x \mid r \in R_2\} \subseteq \emptyset$. Then,
    $\mathsf{read}(\xi) \cap \{r.x \mid r \in R_2\} = \emptyset$. Then, $\mathsf{read}(\xi) \setminus \{r.x \mid r \in R_2\} = \mathsf{read}(\xi)$.
  - Recall $\mathsf{subj}(\xi) \cup \mathsf{subj}(\alpha) \subseteq R_1 \setminus R_2$. Then, $\mathsf{subj}(\alpha) \subseteq R_1 \setminus R_2$. Then, $\mathsf{subj}(\alpha) \cap R_2 = \emptyset$.
  - By Lem. 31, $\mathsf{read}(\alpha) \subseteq \{r.x \mid r \in \mathsf{subj}(\alpha)\}$. Then,
    $\mathsf{read}(\alpha) \cap \{r.x \mid r \in R_2\} \subseteq \{r.x \mid r \in \mathsf{subj}(\alpha)\} \cap \{r.x \mid r \in R_2\}$. Then,
    $\mathsf{read}(\alpha) \cap \{r.x \mid r \in R_2\} \subseteq \{r.x \mid r \in \mathsf{subj}(\alpha) \cap R_2\}$. Then,
    $\mathsf{read}(\alpha) \cap \{r.x \mid r \in R_2\} \subseteq \{r.x \mid r \in \emptyset\}$. Then, $\mathsf{read}(\alpha) \cap \{r.x \mid r \in R_2\} \subseteq \emptyset$. Then,
    $\mathsf{read}(\alpha) \cap \{r.x \mid r \in R_2\} = \emptyset$. Then, $\mathsf{read}(\alpha) \setminus \{r.x \mid r \in R_2\} = \mathsf{read}(\alpha)$.
  - Recall $A_1 \xrightarrow{\xi,\alpha} A_1'$. Then, by induction, $\mathsf{read}(\xi) \cup \mathsf{read}(\alpha) \cup \mathsf{read}(A_1') \subseteq \mathsf{read}(A_1)$.
    Then, $(\mathsf{read}(\xi) \setminus \{r.x \mid r \in R_2\}) \cup (\mathsf{read}(\alpha) \setminus \{r.x \mid r \in R_2\}) \cup (\mathsf{read}(A_1') \setminus \{r.x \mid r \in R_2\}) \subseteq \mathsf{read}(A_1) \setminus \{r.x \mid r \in R_2\}$. Then,
    $\mathsf{read}(\xi) \cup \mathsf{read}(\alpha) \cup (\mathsf{read}(A_1') \setminus \{r.x \mid r \in R_2\}) \subseteq \mathsf{read}(A_1) \setminus \{r.x \mid r \in R_2\}$. Then,
    $\mathsf{read}(\xi) \cup \mathsf{read}(\alpha) \cup (\mathsf{read}(\hat{\xi}) \setminus \{r.x \mid r \in R_1 \cup R_2\}) \cup (\mathsf{read}(A_1') \setminus \{r.x \mid r \in R_2\}) \cup (\mathsf{read}(A_2) \setminus \{r.x \mid r \in R_1\}) \subseteq (\mathsf{read}(\hat{\xi}) \setminus \{r.x \mid r \in R_1 \cup R_2\}) \cup (\mathsf{read}(A_1) \setminus \{r.x \mid r \in R_2\}) \cup (\mathsf{read}(A_2) \setminus \{r.x \mid r \in R_1\})$. Then, by Defn. 22,
    $\mathsf{read}(\xi) \cup \mathsf{read}(\alpha) \cup \mathsf{read}(\textbf{if}\ \hat{\xi}|_n\ A_1'|_{R_1}\ A_2|_{R_2}) \subseteq \mathsf{read}(\textbf{if}\ \hat{\xi}|_n\ A_1|_{R_1}\ A_2|_{R_2})$. Then,
    $\mathsf{read}(\xi) \cup \mathsf{read}(\alpha) \cup \mathsf{read}(A') \subseteq \mathsf{read}(A)$.
- **Step:** [$\rightarrow$1-NIf4]. Similar to case [$\rightarrow$1-NIf5].
- **Step:** [$\rightarrow$1-NWhile], such that $A = \textbf{while}\ \hat{\xi}|_n\ \{\psi\}\ \hat{A}|_\emptyset$ and $\textbf{if}\ \hat{\xi}|_n\ (\hat{A}\ ;\textbf{while}\ \hat{\xi}|_n\ \{\psi\}\ \hat{A}|_\emptyset)|_\emptyset\ \textbf{skip}|_\emptyset \xrightarrow{\xi,\alpha} A'$, for some $\hat{A}, \psi, \hat{\xi}, n$.
  Recall $\textbf{if}\ \hat{\xi}|_n\ (\hat{A}\ ;\textbf{while}\ \hat{\xi}|_n\ \{\psi\}\ \hat{A}|_\emptyset)|_\emptyset\ \textbf{skip}|_\emptyset \xrightarrow{\xi,\alpha} A'$. Then, by induction, $\mathsf{read}(\xi) \cup \mathsf{read}(\alpha) \cup \mathsf{read}(A') \subseteq \mathsf{read}(\textbf{if}\ \hat{\xi}|_n\ (\hat{A}\ ;\textbf{while}\ \hat{\xi}|_n\ \{\psi\}\ \hat{A}|_\emptyset)|_\emptyset\ \textbf{skip}|_\emptyset)$. Then, by Lem. 36, $\mathsf{read}(\xi) \cup \mathsf{read}(\alpha) \cup \mathsf{read}(A') \subseteq \mathsf{read}(\hat{\xi}) \cup \mathsf{read}(\hat{A}\ ;\textbf{while}\ \hat{\xi}|_n\ \{\psi\}\ \hat{A}|_\emptyset) \cup \mathsf{read}(\textbf{skip})$. Then, by Defn. 22,
  $\mathsf{read}(\xi) \cup \mathsf{read}(\alpha) \cup \mathsf{read}(A') \subseteq \mathsf{read}(\hat{\xi}) \cup \mathsf{read}(\hat{A}) \cup \mathsf{read}(\textbf{while}\ \hat{\xi}|_n\ \{\psi\}\ \hat{A}|_\emptyset) \cup \emptyset$. Then, by Defn. 22,
  $\mathsf{read}(\xi) \cup \mathsf{read}(\alpha) \cup \mathsf{read}(A') \subseteq \mathsf{read}(\textbf{while}\ \hat{\xi}|_n\ \{\psi\}\ \hat{A}|_\emptyset) \cup \mathsf{read}(\textbf{while}\ \hat{\xi}|_n\ \{\psi\}\ \hat{A}|_\emptyset) \cup \emptyset$.
  Then, $\mathsf{read}(\xi) \cup \mathsf{read}(\alpha) \cup \mathsf{read}(A') \subseteq \mathsf{read}(\textbf{while}\ \hat{\xi}|_n\ \{\psi\}\ \hat{A}|_\emptyset)$. Then,
  $\mathsf{read}(\xi) \cup \mathsf{read}(\alpha) \cup \mathsf{read}(A') \subseteq \mathsf{read}(A)$. □

2. Similar to 1. □

Proof (of Lem. 52).

1.
   - Recall $A_1 \# A_2$. Then, by Defn. 23, $\mathsf{read}(A_1) \cap \mathsf{read}(A_2) = \emptyset$ and $\mathsf{read}(A_1) \cap \mathsf{write}(A_2) = \emptyset$ and $\mathsf{write}(A_1) \cap \mathsf{read}(A_2) = \emptyset$ and $\mathsf{write}(A_1) \cap \mathsf{write}(A_2) = \emptyset$.
   - Recall $A_2 \xrightarrow{\xi,\alpha} A_2'$. Then, by Lem. 51, $\mathsf{read}(\xi) \cup \mathsf{read}(\alpha) \cup \mathsf{read}(A_2') \subseteq \mathsf{read}(A_2)$ and $\mathsf{write}(\xi) \cup \mathsf{write}(\alpha) \cup \mathsf{write}(A_2') \subseteq \mathsf{write}(A_2)$. Then, $\mathsf{read}(\xi) \subseteq \mathsf{read}(A_2)$ and $\mathsf{write}(\xi) \subseteq \mathsf{write}(A_2)$.
   - Recall $\mathsf{read}(A_1) \cap \mathsf{read}(A_2) = \emptyset$ and $\mathsf{write}(A_1) \cap \mathsf{read}(A_2) = \emptyset$ and $\mathsf{read}(\xi) \subseteq \mathsf{read}(A_2)$. Then, $\mathsf{read}(A_1) \cap \mathsf{read}(\xi) = \emptyset$ and $\mathsf{write}(A_1) \cap \mathsf{read}(\xi) = \emptyset$.
   - Recall $\mathsf{read}(A_1) \cap \mathsf{write}(A_2) = \emptyset$ and $\mathsf{write}(A_1) \cap \mathsf{write}(A_2) = \emptyset$ and $\mathsf{write}(\xi) \subseteq \mathsf{write}(A_2)$. Then, $\mathsf{read}(A_1) \cap \mathsf{write}(\xi) = \emptyset$ and $\mathsf{write}(A_1) \cap \mathsf{write}(\xi) = \emptyset$.
   - Recall $\mathsf{read}(A_1) \cap \mathsf{read}(\xi) = \emptyset$ and $\mathsf{write}(A_1) \cap \mathsf{read}(\xi) = \emptyset$ and $\mathsf{read}(A_1) \cap \mathsf{write}(\xi) = \emptyset$ and $\mathsf{write}(A_1) \cap \mathsf{write}(\xi) = \emptyset$. Then, by [#], $A_1 \# \xi$. □
2. Similar to 1. □

3. Similar to 1. □
4. Similar to 1. □
5. Similar to 1. □
6. Similar to 1. □

Proof (of Lem. 53).

1. Recall $A \xrightarrow{\xi,\alpha} A'$. Then, by Defn. 31:
   - **Base:** [$\rightarrow$1-Act], such that $A = \alpha$ and $A' = \textbf{skip}$ and $\xi = \texttt{true}$.
     Recall $\emptyset \cup \mathsf{subj}(A) \cup \emptyset \subseteq \mathsf{subj}(A)$. Then, by Lem. 4, $\mathsf{subj}(\texttt{true}) \cup \mathsf{subj}(A) \cup \emptyset \subseteq \mathsf{subj}(A)$. Then, by Defn. 24, $\mathsf{subj}(\texttt{true}) \cup \mathsf{subj}(A) \cup \mathsf{subj}(\textbf{skip}) \subseteq \mathsf{subj}(A)$. Then, $\mathsf{subj}(\xi) \cup \mathsf{subj}(\alpha) \cup \mathsf{subj}(A') \subseteq \mathsf{subj}(A)$.
   - **Base:** [$\rightarrow$1-If1], such that $A = R.\textbf{if}\ \hat{\xi}\ A_1\ A_2$ and $A' = A_1$ and $\xi = \hat{\xi}^+$ and $\alpha = 1^R_{\mathsf{subj}(\hat{\xi})}$, for some $R, A_1, A_2, \hat{\xi}$.
     - Recall $\alpha = 1^R_{\mathsf{subj}(\hat{\xi})}$. Then, $\mathsf{subj}(\alpha) = \mathsf{subj}(1^R_{\mathsf{subj}(\hat{\xi})})$. Then, by Defn. 16, $\mathsf{subj}(\alpha) = \mathsf{subj}(\hat{\xi})$.
     - Recall $\mathsf{subj}(\hat{\xi}) \cup \mathsf{subj}(\hat{\xi}) \cup \mathsf{subj}(A_1) \subseteq \mathsf{subj}(\hat{\xi}) \cup \mathsf{subj}(A_1) \cup \mathsf{subj}(A_2)$. Then, by Lem. 13, $\mathsf{subj}(\hat{\xi}^+) \cup \mathsf{subj}(\hat{\xi}) \cup \mathsf{subj}(A_1) \subseteq \mathsf{subj}(\hat{\xi}) \cup \mathsf{subj}(A_1) \cup \mathsf{subj}(A_2)$. Then, $\mathsf{subj}(\hat{\xi}^+) \cup \mathsf{subj}(\alpha) \cup \mathsf{subj}(A_1) \subseteq \mathsf{subj}(\hat{\xi}) \cup \mathsf{subj}(A_1) \cup \mathsf{subj}(A_2)$. Then, by Defn. 24, $\mathsf{subj}(\hat{\xi}^+) \cup \mathsf{subj}(\alpha) \cup \mathsf{subj}(A_1) \subseteq \mathsf{subj}(R.\textbf{if}\ \hat{\xi}\ A_1\ A_2)$. Then, $\mathsf{subj}(\xi) \cup \mathsf{subj}(\alpha) \cup \mathsf{subj}(A') \subseteq \mathsf{subj}(A)$.
   - **Base:** [$\rightarrow$1-If2]. Similar to case [$\rightarrow$1-If1].
   - **Base:** [$\rightarrow$1-While1], such that $A = R.\textbf{while}\ \hat{\xi}\ \{\psi\}\ \hat{A}$ and $A' = \hat{A}\ ;\ R.\textbf{while}\ \hat{\xi}\ \{\psi\}\ \hat{A}$ and $\xi = \hat{\xi}^+$ and $\alpha = 1^R_{\mathsf{subj}(\hat{\xi})}$, for some $R, \hat{A}, \psi, \hat{\xi}$.
     - Recall $\alpha = 1^R_{\mathsf{subj}(\hat{\xi})}$. Then, $\mathsf{subj}(\alpha) = \mathsf{subj}(1^R_{\mathsf{subj}(\hat{\xi})})$. Then, by Defn. 16, $\mathsf{subj}(\alpha) = \mathsf{subj}(\hat{\xi})$.
     - Recall $\mathsf{subj}(\hat{\xi}) \cup \mathsf{subj}(\hat{\xi}) \cup \mathsf{subj}(\hat{A}) \cup \mathsf{subj}(\hat{\xi}) \cup \mathsf{subj}(\hat{A}) \subseteq \mathsf{subj}(\hat{\xi}) \cup \mathsf{subj}(\hat{A})$. Then, by Lem. 13, $\mathsf{subj}(\hat{\xi}^+) \cup \mathsf{subj}(\hat{\xi}) \cup \mathsf{subj}(\hat{A}) \cup \mathsf{subj}(\hat{\xi}) \cup \mathsf{subj}(\hat{A}) \subseteq \mathsf{subj}(\hat{\xi}) \cup \mathsf{subj}(\hat{A})$. Then, $\mathsf{subj}(\hat{\xi}^+) \cup \mathsf{subj}(\alpha) \cup \mathsf{subj}(\hat{A}) \cup \mathsf{subj}(\hat{\xi}) \cup \mathsf{subj}(\hat{A}) \subseteq \mathsf{subj}(\hat{\xi}) \cup \mathsf{subj}(\hat{A})$. Then, by Defn. 24, $\mathsf{subj}(\hat{\xi}^+) \cup \mathsf{subj}(\alpha) \cup \mathsf{subj}(\hat{A}) \cup \mathsf{subj}(R.\textbf{while}\ \hat{\xi}\ \{\psi\}\ \hat{A}) \subseteq \mathsf{subj}(R.\textbf{while}\ \hat{\xi}\ \{\psi\}\ \hat{A})$. Then, by Defn. 24, $\mathsf{subj}(\hat{\xi}^+) \cup \mathsf{subj}(\alpha) \cup \mathsf{subj}(\hat{A}\ ;\ R.\textbf{while}\ \hat{\xi}\ \{\psi\}\ \hat{A}) \subseteq \mathsf{subj}(R.\textbf{while}\ \hat{\xi}\ \{\psi\}\ \hat{A})$. Then, $\mathsf{subj}(\xi) \cup \mathsf{subj}(\alpha) \cup \mathsf{subj}(A') \subseteq \mathsf{subj}(A)$.
   - **Base:** [$\rightarrow$1-While2]. Similar to case [$\rightarrow$1-If1].
   - **Base:** [$\rightarrow$1-NIf1], such that $A = \textbf{if}\ \hat{\xi}|_n\ A_1|_{R_1}\ A_2|_{R_2}$ and $A' = \textbf{if}\ \hat{\xi}|_{n-1}\ A_1|_{R_1}\ A_2|_{R_2}$ and $\xi = \hat{\xi}^+ \upharpoonright r$ and $\xi = \texttt{true}$ and $\alpha = \tau$, for some $R_1, R_2, A_1, A_2, \hat{\xi}, n$.
     Recall $\emptyset \cup \emptyset \cup \mathsf{subj}(\textbf{if}\ \hat{\xi}|_{n-1}\ A_1|_{R_1}\ A_2|_{R_2}) \subseteq \mathsf{subj}(\textbf{if}\ \hat{\xi}|_{n-1}\ A_1|_{R_1}\ A_2|_{R_2})$. Then, by Lem. 4, $\mathsf{subj}(\texttt{true}) \cup \emptyset \cup \mathsf{subj}(\textbf{if}\ \hat{\xi}|_{n-1}\ A_1|_{R_1}\ A_2|_{R_2}) \subseteq \mathsf{subj}(\textbf{if}\ \hat{\xi}|_{n-1}\ A_1|_{R_1}\ A_2|_{R_2})$. Then, by Defn. 16, $\mathsf{subj}(\texttt{true}) \cup \mathsf{subj}(\tau) \cup \mathsf{subj}(\textbf{if}\ \hat{\xi}|_{n-1}\ A_1|_{R_1}\ A_2|_{R_2}) \subseteq \mathsf{subj}(\textbf{if}\ \hat{\xi}|_{n-1}\ A_1|_{R_1}\ A_2|_{R_2})$. Then, by Defn. 24, $\mathsf{subj}(\texttt{true}) \cup \mathsf{subj}(\tau) \cup \mathsf{subj}(\textbf{if}\ \hat{\xi}|_{n-1}\ A_1|_{R_1}\ A_2|_{R_2}) \subseteq (\mathsf{subj}(\xi) \setminus (R_1 \cup R_2)) \cup (\mathsf{subj}(A_1) \setminus R_2) \cup (\mathsf{subj}(A_2) \setminus R_1)$. Then, by Defn. 24, $\mathsf{subj}(\texttt{true}) \cup \mathsf{subj}(\tau) \cup \mathsf{subj}(\textbf{if}\ \hat{\xi}|_{n-1}\ A_1|_{R_1}\ A_2|_{R_2}) \subseteq \mathsf{subj}(\textbf{if}\ \hat{\xi}|_n\ A_1|_{R_1}\ A_2|_{R_2})$. Then, $\mathsf{subj}(\xi) \cup \mathsf{subj}(\alpha) \cup \mathsf{subj}(A') \subseteq \mathsf{subj}(A)$.
   - **Base:** [$\rightarrow$1-NIf2], such that $A = \textbf{if}\ \hat{\xi}|_n\ A_1|_{R_1}\ A_2|_{R_2}$ and $A' = \textbf{if}\ \hat{\xi}|_n\ A_1|_{R_1 \cup \{r\}}\ A_2|_{R_2}$ and $\xi = \hat{\xi}^+ \upharpoonright r$ and $\alpha = 1^{\{r\}}_{\{r\}}$ and $r \in \mathsf{subj}(\hat{\xi}) \setminus (R_1 \cup R_2)$, for some $R_1, R_2, A_1, A_2, r, \hat{\xi}, n$.
     - Recall $r \in \mathsf{subj}(\hat{\xi}) \setminus (R_1 \cup R_2)$. Then, $r \notin R_1 \cup R_2$. Then, $\{r\} \cap (R_1 \cup R_2) = \emptyset$.
     - By Lem. 10, $\mathsf{subj}(\hat{\xi} \upharpoonright r) \subseteq \{r\}$. Then, $\mathsf{subj}(\hat{\xi} \upharpoonright r) \cap (R_1 \cup R_2) \subseteq \{r\} \cap (R_1 \cup R_2)$. Then, $\mathsf{subj}(\hat{\xi} \upharpoonright r) \cap (R_1 \cup R_2) \subseteq \emptyset$. Then, $\mathsf{subj}(\hat{\xi} \upharpoonright r) \cap (R_1 \cup R_2) = \emptyset$. Then, $\mathsf{subj}(\hat{\xi} \upharpoonright r) \setminus (R_1 \cup R_2) = \mathsf{subj}(\hat{\xi} \upharpoonright r)$.
     - By Lem. 10, $\mathsf{subj}(\hat{\xi} \upharpoonright r) \subseteq \mathsf{subj}(\hat{\xi})$. Then, $\mathsf{subj}(\hat{\xi} \upharpoonright r) \setminus (R_1 \cup R_2) \subseteq \mathsf{subj}(\hat{\xi}) \setminus (R_1 \cup R_2)$. Then, $\mathsf{subj}(\hat{\xi} \upharpoonright r) \subseteq \mathsf{subj}(\hat{\xi}) \setminus (R_1 \cup R_2)$. Then, by Lem. 13, $\mathsf{subj}((\hat{\xi} \upharpoonright r)^+) \subseteq \mathsf{subj}(\hat{\xi}) \setminus (R_1 \cup R_2)$. Then, by Lem. 11, $\mathsf{subj}(\hat{\xi}^+ \upharpoonright r) \subseteq \mathsf{subj}(\hat{\xi}) \setminus (R_1 \cup R_2)$.
     - Recall $\mathsf{subj}(\hat{\xi}) \setminus (R_1 \cup R_2) \subseteq \mathsf{subj}(\hat{\xi}) \setminus (R_1 \cup R_2)$. Then, $(\mathsf{subj}(\hat{\xi}) \setminus (R_1 \cup R_2)) \setminus \{r\} \subseteq \mathsf{subj}(\hat{\xi}) \setminus (R_1 \cup R_2)$. Then, $\mathsf{subj}(\hat{\xi}) \setminus (R_1 \cup R_2 \cup \{r\}) \subseteq \mathsf{subj}(\hat{\xi}) \setminus (R_1 \cup R_2)$.

- Recall $\mathsf{subj}(G_2) \setminus R_1 \subseteq \mathsf{subj}(G_2) \setminus R_1$. Then, $(\mathsf{subj}(G_2) \setminus R_1) \setminus \{r\} \subseteq \mathsf{subj}(G_2) \setminus R_1$. Then, $\mathsf{subj}(G_2) \setminus (R_1 \cup \{r\}) \subseteq \mathsf{subj}(G_2) \setminus R_1$.
- Recall $\mathsf{subj}(\hat{\xi}^+ \restriction r) \subseteq \mathsf{subj}(\hat{\xi}) \setminus (R_1 \cup R_2)$ and $\mathsf{subj}(\hat{\xi}) \setminus (R_1 \cup R_2 \cup \{r\}) \subseteq \mathsf{subj}(\hat{\xi}) \setminus (R_1 \cup R_2)$ and $\mathsf{subj}(G_2) \setminus (R_1 \cup \{r\}) \subseteq \mathsf{subj}(G_2) \setminus R_1$. Then, $\mathsf{subj}(\hat{\xi}^+ \restriction r) \cup (\mathsf{subj}(\hat{\xi}) \setminus (R_1 \cup R_2 \cup \{r\})) \cup (\mathsf{subj}(G_2) \setminus (R_1 \cup \{r\})) \subseteq (\mathsf{subj}(\hat{\xi}) \setminus (R_1 \cup R_2)) \cup (\mathsf{subj}(G_2) \setminus R_1)$. Then, $\mathsf{subj}(\hat{\xi}^+ \restriction r) \cup \emptyset \cup (\mathsf{subj}(\hat{\xi}) \setminus (R_1 \cup R_2 \cup \{r\})) \cup (\mathsf{subj}(G_1) \setminus R_2) \cup (\mathsf{subj}(G_2) \setminus (R_1 \cup \{r\})) \subseteq (\mathsf{subj}(\hat{\xi}) \setminus (R_1 \cup R_2)) \cup (\mathsf{subj}(G_1) \setminus R_2) \cup (\mathsf{subj}(G_2) \setminus R_1)$. Then, by Defn. 16, $\mathsf{subj}(\hat{\xi}^+ \restriction r) \cup \mathsf{subj}(1\{^r_r\}) \cup (\mathsf{subj}(\hat{\xi}) \setminus (R_1 \cup R_2 \{r\})) \cup (\mathsf{subj}(G_1) \setminus R_2) \cup (\mathsf{subj}(G_2) \setminus (R_1 \cup \{r\})) \subseteq (\mathsf{subj}(\hat{\xi}) \setminus (R_1 \cup R_2)) \cup (\mathsf{subj}(G_1) \setminus R_2) \cup (\mathsf{subj}(G_2) \setminus R_1)$. Then, by Defn. 24, $\mathsf{subj}(\hat{\xi}^+ \restriction r) \cup \mathsf{subj}(1\{^r_r\}) \cup \mathsf{subj}(\textbf{if } \hat{\xi}|_n\ A_1|_{R_1 \cup \{r\}}\ A_2|_{R_2}) \subseteq \mathsf{subj}(\textbf{if } \hat{\xi}|_n\ A_1|_{R_1}\ A_2|_{R_2})$. Then, $\mathsf{subj}(\xi) \cup \mathsf{subj}(\alpha) \cup \mathsf{subj}(A') \subseteq \mathsf{subj}(A)$.

  - **Base:** [$\rightarrow$1-NIF3]. Similar to case [$\rightarrow$1-NIF2].
  - **Step:** [$\rightarrow$1-SEQ1], such that $A = A_1 \,;\, A_2$ and $A_1 \xrightarrow{\xi,\alpha} A_1'$, for some A_1, A_1', A_2. Recall $A_1 \xrightarrow{\xi,\alpha} A_1'$. Then, by induction, $\mathsf{subj}(\xi) \cup \mathsf{subj}(\alpha) \cup \mathsf{subj}(A_1') \subseteq \mathsf{subj}(A_1)$. Then, $\mathsf{subj}(\xi) \cup \mathsf{subj}(\alpha) \cup \mathsf{subj}(A_1') \cup \mathsf{subj}(A_2) \subseteq \mathsf{subj}(A_1) \cup \mathsf{subj}(A_2)$. Then, by Defn. 24, $\mathsf{subj}(\xi) \cup \mathsf{subj}(\alpha) \cup \mathsf{subj}(A_1' \,;\, A_2) \subseteq \mathsf{subj}(A_1 \,;\, A_2)$. Then, $\mathsf{subj}(\xi) \cup \mathsf{subj}(\alpha) \cup \mathsf{subj}(A') \subseteq \mathsf{subj}(A)$.
  - **Step:** [$\rightarrow$1-SEQ2]. Similar to case [$\rightarrow$1-SEQ1].
  - **Step:** [$\rightarrow$1-PAR1]. Similar to case [$\rightarrow$1-SEQ1].
  - **Step:** [$\rightarrow$1-PAR2]. Similar to case [$\rightarrow$1-SEQ1].
  - **Step:** [$\rightarrow$1-NIF4], such that $A = \textbf{if } \hat{\xi}|_n\ A_1|_{R_1}\ A_2|_{R_2}$ and $A' = \textbf{if } \hat{\xi}|_n\ A_1'|_{R_1}\ A_2|_{R_2}$ and $A_1 \xrightarrow{\xi,\alpha} A_1'$ and $\mathsf{subj}(\xi) \cup \mathsf{subj}(\alpha) \subseteq R_1 \setminus R_2$, for some $R_1, R_2, A_1, A_1', A_2, \hat{\xi}$.
    - Recall $\mathsf{subj}(\xi) \cup \mathsf{subj}(\alpha) \subseteq R_1 \setminus R_2$. Then, $\mathsf{subj}(\xi) \subseteq R_1 \setminus R_2$. Then, $\mathsf{subj}(\xi) \cap R_2 = \emptyset$. Then, $\mathsf{subj}(\xi) \setminus R_2 = \mathsf{subj}(\xi)$.
    - Recall $\mathsf{subj}(\xi) \cup \mathsf{subj}(\alpha) \subseteq R_1 \setminus R_2$. Then, $\mathsf{subj}(\alpha) \subseteq R_1 \setminus R_2$. Then, $\mathsf{subj}(\alpha) \cap R_2 = \emptyset$. Then, $\mathsf{subj}(\alpha) \setminus R_2 = \mathsf{subj}(\alpha)$.
    - Recall $A_1 \xrightarrow{\xi,\alpha} A_1'$. Then, by induction, $\mathsf{subj}(\xi) \cup \mathsf{subj}(\alpha) \cup \mathsf{subj}(A_1') \subseteq \mathsf{subj}(A_1)$. Then, $(\mathsf{subj}(\xi) \setminus R_2) \cup (\mathsf{subj}(\alpha) \setminus R_2) \cup (\mathsf{subj}(A_1') \setminus R_2) \subseteq \mathsf{subj}(A_1) \setminus R_2$. Then, $\mathsf{subj}(\xi) \cup \mathsf{subj}(\alpha) \cup (\mathsf{subj}(A_1') \setminus R_2) \subseteq \mathsf{subj}(A_1) \setminus R_2$. Then, $\mathsf{subj}(\xi) \cup \mathsf{subj}(\alpha) \cup (\mathsf{subj}(\hat{\xi}) \setminus (R_1 \cup R_2)) \cup (\mathsf{subj}(A_1') \setminus R_2) \cup (\mathsf{subj}(A_2) \setminus R_1) \subseteq (\mathsf{subj}(\hat{\xi}) \setminus (R_1 \cup R_2)) \cup (\mathsf{subj}(A_1) \setminus R_2) \cup (\mathsf{subj}(A_2) \setminus R_1)$. Then, by Defn. 24, $\mathsf{subj}(\xi) \cup \mathsf{subj}(\alpha) \cup \mathsf{subj}(\textbf{if } \hat{\xi}|_n\ A_1'|_{R_1}\ A_2|_{R_2}) \subseteq \mathsf{subj}(\textbf{if } \hat{\xi}|_n\ A_1|_{R_1}\ A_2|_{R_2})$. Then, $\mathsf{subj}(\xi) \cup \mathsf{subj}(\alpha) \cup \mathsf{subj}(A') \subseteq \mathsf{subj}(A)$.
  - **Step:** [$\rightarrow$1-NIF5]. Similar to case [$\rightarrow$1-NIF4].
  - **Step:** [$\rightarrow$1-NWHILE], such that $A = \textbf{while } \hat{\xi}|_n\ \{\psi\}\ \hat{A}|_\emptyset$ and $\textbf{if } \hat{\xi}|_n\ (\hat{A} \,;\, \textbf{while } \hat{\xi}|_n\ \{\psi\}\ \hat{A}|_\emptyset)|_\emptyset\ \textbf{skip}|_\emptyset \xrightarrow{\xi,\alpha} A'$, for some $\hat{A}, \psi, \hat{\xi}, n$. Recall $\textbf{if } \hat{\xi}|_n\ (\hat{A} \,;\, \textbf{while } \hat{\xi}|_n\ \{\psi\}\ \hat{A}|_\emptyset)|_\emptyset\ \textbf{skip}|_\emptyset \xrightarrow{\xi,\alpha} A'$. Then, by induction, $\mathsf{subj}(\xi) \cup \mathsf{subj}(\alpha) \cup \mathsf{subj}(A') \subseteq \mathsf{subj}(\textbf{if } \hat{\xi}|_n\ (\hat{A} \,;\, \textbf{while } \hat{\xi}|_n\ \{\psi\}\ \hat{A}|_\emptyset)|_\emptyset\ \textbf{skip}|_\emptyset)$. Then, by Defn. 24, $\mathsf{subj}(\xi) \cup \mathsf{subj}(\alpha) \cup \mathsf{subj}(A') \subseteq (\mathsf{subj}(\hat{\xi}) \setminus (\emptyset \cup \emptyset)) \cup (\mathsf{subj}(\hat{A} \,;\, \textbf{while } \hat{\xi}|_n\ \{\psi\}\ \hat{A}|_\emptyset) \setminus \emptyset) \cup (\mathsf{subj}(\textbf{skip}) \setminus \emptyset)$. Then, $\mathsf{subj}(\xi) \cup \mathsf{subj}(\alpha) \cup \mathsf{subj}(A') \subseteq \mathsf{subj}(\hat{\xi}) \cup \mathsf{subj}(\hat{A} \,;\, \textbf{while } \hat{\xi}|_n\ \{\psi\}\ \hat{A}|_\emptyset) \cup \mathsf{subj}(\textbf{skip})$. Then, by Defn. 24, $\mathsf{subj}(\xi) \cup \mathsf{subj}(\alpha) \cup \mathsf{subj}(A') \subseteq \mathsf{subj}(\hat{\xi}) \cup \mathsf{subj}(\hat{A}) \cup \mathsf{subj}(\textbf{while } \hat{\xi}|_n\ \{\psi\}\ \hat{A}|_\emptyset) \cup \emptyset$. Then, by Defn. 24, $\mathsf{subj}(\xi) \cup \mathsf{subj}(\alpha) \cup \mathsf{subj}(A') \subseteq \mathsf{subj}(\textbf{while } \hat{\xi}|_n\ \{\psi\}\ \hat{A}|_\emptyset) \cup \mathsf{subj}(\textbf{while } \hat{\xi}|_n\ \{\psi\}\ \hat{A}|_\emptyset) \cup \emptyset$. Then, $\mathsf{subj}(\xi) \cup \mathsf{subj}(\alpha) \cup \mathsf{subj}(A') \subseteq \mathsf{subj}(\textbf{while } \hat{\xi}|_n\ \{\psi\}\ \hat{A}|_\emptyset)$. Then, $\mathsf{subj}(\xi) \cup \mathsf{subj}(\alpha) \cup \mathsf{subj}(A') \subseteq \mathsf{subj}(A)$. □

2. Recall $A \xrightarrow{\xi,\alpha} A'$. Then, by Defn. 31:
  - **Base:** [$\rightarrow$1-ACT], such that $\xi = \mathtt{true}$. Recall $\emptyset \subseteq \mathsf{subj}(\alpha)$. Then, by Lem. 4, $\mathsf{subj}(\mathtt{true}) \subseteq \mathsf{subj}(\alpha)$. Then, $\mathsf{subj}(\xi) \subseteq \mathsf{subj}(\alpha)$.
  - **Base:** [$\rightarrow$1-IF1], such that $\xi = \hat{\xi}^+$ and $\alpha = 1^R_{\mathsf{subj}(\hat{\xi})}$, for some $R, \hat{\xi}$. Recall $\mathsf{subj}(\xi) \subseteq \mathsf{subj}(\xi)$. Then, $\mathsf{subj}(\xi) \subseteq \mathsf{subj}(\hat{\xi}^+)$. Then, by Lem. 13, $\mathsf{subj}(\xi) \subseteq \mathsf{subj}(\hat{\xi})$. Then, by Defn. 16, $\mathsf{subj}(\xi) \subseteq \mathsf{subj}(1^R_{\mathsf{subj}(\hat{\xi})})$. Then, $\mathsf{subj}(\xi) \subseteq \mathsf{subj}(\alpha)$.

- **Base:** [→1-If2]. Similar to case [→1-If1].
- **Base:** [→1-While1]. Similar to case [→1-If1].
- **Base:** [→1-While2]. Similar to case [→1-If1].
- **Base:** [→1-NIf1]. Similar to case [→1-Act].
- **Base:** [→1-NIf2], such that $\xi = \hat{\xi}^+ \restriction r$ and $\alpha = 1\{^r_r\}$ and $r \in \mathsf{subj}(\hat{\xi}) \setminus (R_1 \cup R_2)$, for some $R, R_1, R_2, r, \hat{\xi}$.
  - Recall $r \in \mathsf{subj}(\hat{\xi}) \setminus (R_1 \cup R_2)$. Then, $r \in \mathsf{subj}(\hat{\xi})$. Then, by Lem. 9, $\mathsf{subj}(\hat{\xi} \restriction r) = \{r\}$.
  - Recall $\mathsf{subj}(\xi) \subseteq \mathsf{subj}(\xi)$. Then, $\mathsf{subj}(\xi) \subseteq \mathsf{subj}(\hat{\xi}^+ \restriction r)$. Then, by Lem. 11, $\mathsf{subj}(\xi) \subseteq \mathsf{subj}((\hat{\xi} \restriction r)^+)$. Then, by Lem. 13, $\mathsf{subj}(\xi) \subseteq \mathsf{subj}(\hat{\xi} \restriction r)$. Then, $\mathsf{subj}(\xi) \subseteq \{r\}$. Then, by Defn. 16, $\mathsf{subj}(\xi) \subseteq \mathsf{subj}(1\{^r_r\})$. Then, $\mathsf{subj}(\xi) \subseteq \mathsf{subj}(\alpha)$.
- **Base:** [→1-NIf3]. Similar to case [→1-NIf2].
- **Step:** [→1-Seq1], such that $A_1 \xrightarrow{\xi,\alpha} A_1'$, for some A_1, A_1'.
  Recall $A_1 \xrightarrow{\xi,\alpha} A_1'$. Then, by induction, $\mathsf{subj}(\xi) \subseteq \mathsf{subj}(\alpha)$.
- **Step:** [→1-Seq2]. Similar to case [→1-Seq1].
- **Step:** [→1-Par1]. Similar to case [→1-Seq1].
- **Step:** [→1-Par2]. Similar to case [→1-Seq1].
- **Step:** [→1-NIf4]. Similar to case [→1-Seq1].
- **Step:** [→1-NIf5]. Similar to case [→1-Seq1].
- **Step:** [→1-NWhile]. Similar to case [→1-Seq1]. □

3. Recall $A \xrightarrow{\mathtt{true},\tau} A'$. Then, by Defn. 31:
   - **Base:** [→1-Act], such that $A = \tau$ and $A' = \mathbf{skip}$.
     Recall $\emptyset = \emptyset$. Then, by Defn. 24, $\mathsf{subj}(\tau) = \mathsf{subj}(\mathbf{skip})$. Then, $\mathsf{subj}(A) = \mathsf{subj}(A')$.
   - **Base:** [→1-If1], such that $\tau = i^R_{\mathsf{subj}(\hat{\xi})}$, for some $R, \hat{\xi}$. Then, **false**.
   - **Base:** [→1-If2]. Similar to case [→1-If1].
   - **Base:** [→1-While1]. Similar to case [→1-If1].
   - **Base:** [→1-While2]. Similar to case [→1-If1].
   - **Base:** [→1-NIf1], such that $A = \mathbf{if}\ \hat{\xi}|_n\ A_1|_{R_1}\ A_2|_{R_2}$ and $A' = \mathbf{if}\ \hat{\xi}|_{n-1}\ A_1|_{R_1}\ A_2|_{R_2}$, for some $R_1, R_2, A_1, A_2, \hat{\xi}, n$.
     By Defn. 24, $\mathsf{subj}(A = \mathbf{if}\ \hat{\xi}|_n\ A_1|_{R_1}\ A_2|_{R_2}) =$ $(\mathsf{subj}(\xi) \setminus (R_1 \cup R_2)) \cup (\mathsf{subj}(A_1) \setminus R_2) \cup (\mathsf{subj}(A_2) \setminus R_1) = \mathsf{subj}(\mathbf{if}\ \hat{\xi}|_{n-1}\ A_1|_{R_1}\ A_2|_{R_2})$.
     Then, $\mathsf{subj}(\mathbf{if}\ \hat{\xi}|_n\ A_1|_{R_1}\ A_2|_{R_2}) = \mathsf{subj}(\mathbf{if}\ \hat{\xi}|_{n-1}\ A_1|_{R_1}\ A_2|_{R_2})$. Then, $\mathsf{subj}(A) = \mathsf{subj}(A')$.
   - **Base:** [→1-NIf2]. Similar to case [→1-If1].
   - **Base:** [→1-NIf3]. Similar to case [→1-If1].
   - **Step:** [→1-Seq1], such that $A = A_1 \mathbin{;} A_2$ and $A' = A_1' \mathbin{;} A_2$ and $A_1 \xrightarrow{\mathtt{true},\tau} A_1'$, for some A_1, A_1', A_2.
     - Recall $A_1 \xrightarrow{\mathtt{true},\tau} A_1'$. Then, by induction, $\mathsf{subj}(A_1) = \mathsf{subj}(A_1')$.
     - By Defn. 24, $\mathsf{subj}(A_1 \mathbin{;} A_2) = \mathsf{subj}(A_1) \cup \mathsf{subj}(A_2)$. Then, $\mathsf{subj}(A_1 \mathbin{;} A_2) = \mathsf{subj}(A_1') \cup \mathsf{subj}(A_2)$. Then, by Defn. 24, $\mathsf{subj}(A_1 \mathbin{;} A_2) = \mathsf{subj}(A_1' \mathbin{;} A_2)$. Then, $\mathsf{subj}(A) = \mathsf{subj}(A')$.
   - **Step:** [→1-Seq2]. Similar to case [→1-Seq1].
   - **Step:** [→1-Par1]. Similar to case [→1-Seq1].
   - **Step:** [→1-Par2]. Similar to case [→1-Seq1].
   - **Step:** [→1-NIf4]. Similar to case [→1-Seq1].
   - **Step:** [→1-NIf5]. Similar to case [→1-Seq1].
   - **Step:** [→1-NWhile], such that $A = \mathbf{while}\ \hat{\xi}|_n\ \{\psi\}\ \hat{A}|_\emptyset$ and $\mathbf{if}\ \hat{\xi}|_n\ (\hat{A} \mathbin{;} \mathbf{while}\ \hat{\xi}|_n\ \{\psi\}\ \hat{A}|_\emptyset)|_\emptyset\ \mathbf{skip}|_\emptyset \xrightarrow{\xi,\alpha} A'$, for some $\hat{A}, \psi, \hat{\xi}, n$.
     Recall $\mathbf{if}\ \hat{\xi}|_n\ (\hat{A} \mathbin{;} \mathbf{while}\ \hat{\xi}|_n\ \{\psi\}\ \hat{A}|_\emptyset)|_\emptyset\ \mathbf{skip}|_\emptyset \xrightarrow{\xi,\alpha} A'$. Then, by induction, $\mathsf{subj}(\mathbf{if}\ \hat{\xi}|_n\ (\hat{A} \mathbin{;} \mathbf{while}\ \hat{\xi}|_n\ \{\psi\}\ \hat{A}|_\emptyset)|_\emptyset\ \mathbf{skip}|_\emptyset) = \mathsf{subj}(A')$. Then, by Defn. 24, $(\mathsf{subj}(\hat{\xi}) \setminus (\emptyset \cup \emptyset)) \cup (\mathsf{subj}(\hat{A} \mathbin{;} \mathbf{while}\ \hat{\xi}|_n\ \{\psi\}\ \hat{A}|_\emptyset) \setminus \emptyset) \cup (\mathsf{subj}(\mathbf{skip}) \setminus \emptyset) = \mathsf{subj}(A')$. Then, $\mathsf{subj}(\hat{\xi}) \cup \mathsf{subj}(\hat{A} \mathbin{;} \mathbf{while}\ \hat{\xi}|_n\ \{\psi\}\ \hat{A}|_\emptyset) \cup \mathsf{subj}(\mathbf{skip}) = \mathsf{subj}(A')$. Then, by Defn. 24, $\mathsf{subj}(\hat{\xi}) \cup \mathsf{subj}(\hat{A}) \cup \mathsf{subj}(\mathbf{while}\ \hat{\xi}|_n\ \{\psi\}\ \hat{A}|_\emptyset) \cup \emptyset = \mathsf{subj}(A')$. Then, by Defn. 24, $\mathsf{subj}(\mathbf{while}\ \hat{\xi}|_n\ \{\psi\}\ \hat{A}|_\emptyset) \cup \mathsf{subj}(\mathbf{while}\ \hat{\xi}|_n\ \{\psi\}\ \hat{A}|_\emptyset) \cup \emptyset = \mathsf{subj}(A')$. Then, $\mathsf{subj}(\mathbf{while}\ \hat{\xi}|_n\ \{\psi\}\ \hat{A}|_\emptyset) = \mathsf{subj}(A')$. Then, $\mathsf{subj}(A) = \mathsf{subj}(A')$. □

Proof (of Lem. 54). Recall $A \xrightarrow{\xi,\alpha} A'$. Then, by Defn. 31:

- **Base:** [$\to$1-Act], such that $A = \alpha$ and $A' = \textbf{skip}$.
  Recall $\mathsf{chan}(A) \cup \emptyset \subseteq \mathsf{chan}(A)$. Then, by Defn. 25, $\mathsf{chan}(A) \cup \mathsf{chan}(\textbf{skip}) \subseteq \mathsf{chan}(A)$. Then, $\mathsf{chan}(\alpha) \cup \mathsf{chan}(A') \subseteq \mathsf{chan}(A)$.
- **Base:** [$\to$1-If1], such that $A = R.\textbf{if}\ \hat{\xi}\ A_1\ A_2$ and $A' = A_1$ and $\alpha = 1^R_{\mathsf{subj}(\hat{\xi})}$, for some $R, A_1, A_2, \hat{\xi}$.
  Recall $\{R\} \cup \mathsf{chan}(A_1) \subseteq \{R\} \cup \mathsf{chan}(A_1) \cup \mathsf{chan}(A_2)$. Then, by Defn. 17, $\mathsf{chan}(1^R_{\mathsf{subj}(\hat{\xi})}) \cup \mathsf{chan}(A_1) \subseteq \{R\} \cup \mathsf{chan}(A_1) \cup \mathsf{chan}(A_2)$. Then, by Defn. 25, $\mathsf{chan}(1^R_{\mathsf{subj}(\hat{\xi})}) \cup \mathsf{chan}(A_1) \subseteq \mathsf{chan}(R.\textbf{if}\ \hat{\xi}\ A_1\ A_2)$. Then, $\mathsf{chan}(\alpha) \cup \mathsf{chan}(A') \subseteq \mathsf{chan}(A)$.
- **Base:** [$\to$1-If2]. Similar to case [$\to$1-If1].
- **Base:** [$\to$1-While1], such that $A = R.\textbf{while}\ \hat{\xi}\ \{\psi\}\ \hat{A}$ and $A' = \hat{A}\,; R.\textbf{while}\ \hat{\xi}\ \{\psi\}\ \hat{A}$ and $\alpha = 1^R_{\mathsf{subj}(\hat{\xi})}$, for some $R, \hat{A}, \psi, \hat{\xi}$.
  Recall $\{R\} \cup \mathsf{chan}(\hat{A}) \cup \{R\} \cup \mathsf{chan}(\hat{A}) \subseteq \{R\} \cup \mathsf{chan}(\hat{A})$. Then, by Defn. 17, $\mathsf{chan}(1^R_{\mathsf{subj}(\hat{\xi})}) \cup \mathsf{chan}(\hat{A}) \cup \{R\} \cup \mathsf{chan}(\hat{A}) \subseteq \{R\} \cup \mathsf{chan}(\hat{A})$. Then, by Defn. 25, $\mathsf{chan}(1^R_{\mathsf{subj}(\hat{\xi})}) \cup \mathsf{chan}(\hat{A}) \cup \mathsf{chan}(R.\textbf{while}\ \hat{\xi}\ \{\psi\}\ \hat{A}) \subseteq \mathsf{chan}(R.\textbf{while}\ \hat{\xi}\ \{\psi\}\ \hat{A})$. Then, by Defn. 25, $\mathsf{chan}(1^R_{\mathsf{subj}(\hat{\xi})}) \cup \mathsf{chan}(\hat{A}\,; R.\textbf{while}\ \hat{\xi}\ \{\psi\}\ \hat{A}) \subseteq \mathsf{chan}(R.\textbf{while}\ \hat{\xi}\ \{\psi\}\ \hat{A})$. Then, $\mathsf{chan}(\alpha) \cup \mathsf{chan}(A') \subseteq \mathsf{chan}(A)$.
- **Base:** [$\to$1-While2]. Similar to case [$\to$1-If1].
- **Base:** [$\to$1-NIf1], such that $A = \textbf{if}\ \hat{\xi}|_n\ A_1|_{R_1}\ A_2|_{R_2}$ and $A' = \textbf{if}\ \hat{\xi}|_{n-1}\ A_1|_{R_1}\ A_2|_{R_2}$, for some $R_1, R_2, A_1, A_2, \hat{\xi}, n$.
  Recall $\emptyset \cup \mathsf{chan}(A') \subseteq \mathsf{chan}(\textbf{if}\ \hat{\xi}|_{n-1}\ A_1|_{R_1}\ A_2|_{R_2})$. Then, by Defn. 17, $\mathsf{chan}(\tau) \cup \mathsf{chan}(\textbf{if}\ \hat{\xi}|_{n-1}\ A_1|_{R_1}\ A_2|_{R_2}) \subseteq \mathsf{chan}(\textbf{if}\ \hat{\xi}|_{n-1}\ A_1|_{R_1}\ A_2|_{R_2})$. Then, by Defn. 25, $\mathsf{chan}(\tau) \cup \mathsf{chan}(\textbf{if}\ \hat{\xi}|_{n-1}\ A_1|_{R_1}\ A_2|_{R_2}) \subseteq \{\{r\} \mid r \in \mathsf{subj}(\hat{\xi})\} \cup \mathsf{chan}(A)$. Then, by Defn. 25, $\mathsf{chan}(\tau) \cup \mathsf{chan}(\textbf{if}\ \hat{\xi}|_{n-1}\ A_1|_{R_1}\ A_2|_{R_2}) \subseteq \mathsf{chan}(\textbf{if}\ \hat{\xi}|_n\ A_1|_{R_1}\ A_2|_{R_2})$. Then, $\mathsf{chan}(\alpha) \cup \mathsf{chan}(A') \subseteq \mathsf{chan}(A)$.
- **Base:** [$\to$1-NIf2], such that $A = \textbf{if}\ \hat{\xi}|_n\ A_1|_{R_1}\ A_2|_{R_2}$ and $A' = \textbf{if}\ \hat{\xi}|_n\ A_1|_{R_1 \cup \{r\}}\ A_2|_{R_2}$ and $\alpha = 1^{\{r\}}_{\{r\}}$ and $r \in \mathsf{subj}(\hat{\xi}) \setminus (R_1 \cup R_2)$, for some $R_1, R_2, A_1, A_2, \hat{\xi}, r, n$,
  Recall $r \in \mathsf{subj}(\hat{\xi}) \setminus (R_1 \cup R_2)$. Then, $r \in \mathsf{subj}(\hat{\xi})$. Then, $\{r\} \in \{\{\hat{r}\} \mid \hat{r} \in \mathsf{subj}(\hat{\xi})\}$. Then, $\{\{r\}\} \subseteq \{\{\hat{r}\} \mid \hat{r} \in \mathsf{subj}(\hat{\xi})\}$. Then, by Defn. 17, $\mathsf{chan}(1^{\{r\}}_{\{r\}}) \subseteq \{\{\hat{r}\} \mid \hat{r} \in \mathsf{subj}(\hat{\xi})\}$. Then, $\mathsf{chan}(1^{\{r\}}_{\{r\}}) \cup \{\{\hat{r}\} \mid \hat{r} \in \mathsf{subj}(\hat{\xi})\} \cup \mathsf{chan}(A_1) \cup \mathsf{chan}(A_2) \subseteq \{\{\hat{r}\} \mid \hat{r} \in \mathsf{subj}(\hat{\xi})\} \cup \mathsf{chan}(A_1) \cup \mathsf{chan}(A_2)$. Then, by Defn. 25, $\mathsf{chan}(1^{\{r\}}_{\{r\}}) \cup \mathsf{chan}(\textbf{if}\ \hat{\xi}|_n\ A_1|_{R_1 \cup \{r\}}\ A_2|_{R_2}) \subseteq \mathsf{chan}(\textbf{if}\ \hat{\xi}|_n\ A_1|_{R_1}\ A_2|_{R_2})$. Then, $\mathsf{chan}(\alpha) \cup \mathsf{chan}(A') \subseteq \mathsf{chan}(A)$.
- **Base:** [$\to$1-NIf3]. Similar to case [$\to$1-NIf2].
- **Step:** [$\to$1-Seq1], such that $A = A_1\,; A_2$ and $A' = A_1'\,; A_2$ and $A_1 \xrightarrow{\xi,\alpha} A_1'$, for some A_1, A_1', A_2
  Recall $A_1 \xrightarrow{\xi,\alpha} A_1'$. Then, by induction, $\mathsf{chan}(\alpha) \cup \mathsf{chan}(A_1') \subseteq \mathsf{chan}(A_1)$. Then, $\mathsf{chan}(\alpha) \cup \mathsf{chan}(A_1') \cup \mathsf{chan}(A_2) \subseteq \mathsf{chan}(A_1) \cup \mathsf{chan}(A_2)$. Then, by Defn. 25, $\mathsf{chan}(\alpha) \cup \mathsf{chan}(A_1'\,; A_2) \subseteq \mathsf{chan}(A_1\,; A_2)$. Then, $\mathsf{chan}(\alpha) \cup \mathsf{chan}(A') \subseteq \mathsf{chan}(A)$.
- **Step:** [$\to$1-Seq2]. Similar to case [$\to$1-Seq1].
- **Step:** [$\to$1-Par1]. Similar to case [$\to$1-Seq1].
- **Step:** [$\to$1-Par2]. Similar to case [$\to$1-Seq1].
- **Step:** [$\to$1-NIf4]. Similar to case [$\to$1-Seq1].
- **Step:** [$\to$1-NIf5]. Similar to case [$\to$1-Seq1].
- **Step:** [$\to$1-NWhile], such that $A = \textbf{while}\ \hat{\xi}|_n\ \{\psi\}\ \hat{A}|_\emptyset$ and $\textbf{if}\ \hat{\xi}|_n\ (\hat{A}\,; \textbf{while}\ \hat{\xi}|_n\ \{\psi\}\ \hat{A}|_\emptyset)|_\emptyset\ \textbf{skip}|_\emptyset \xrightarrow{\xi,\alpha} A'$, for some $\hat{A}, \psi, \hat{\xi}, n$.
  Recall $\textbf{if}\ \hat{\xi}|_n\ (\hat{A}\,; \textbf{while}\ \hat{\xi}|_n\ \{\psi\}\ \hat{A}|_\emptyset)|_\emptyset\ \textbf{skip}|_\emptyset \xrightarrow{\xi,\alpha} A'$. Then, by induction, $\mathsf{chan}(\alpha) \cup \mathsf{chan}(A') \subseteq \mathsf{chan}(\textbf{if}\ \hat{\xi}|_n\ (\hat{A}\,; \textbf{while}\ \hat{\xi}|_n\ \{\psi\}\ \hat{A}|_\emptyset)|_\emptyset\ \textbf{skip}|_\emptyset)$. Then, by Defn. 25, $\mathsf{chan}(\alpha) \cup \mathsf{chan}(A') \subseteq \{\{r\} \mid r \in \mathsf{subj}(\hat{\xi})\} \cup \mathsf{chan}(\hat{A}\,; \textbf{while}\ \hat{\xi}|_n\ \{\psi\}\ \hat{A}|_\emptyset) \cup \mathsf{chan}(\textbf{skip})$. Then, by Defn. 25, $\mathsf{chan}(\alpha) \cup \mathsf{chan}(A') \subseteq \{\{r\} \mid r \in \mathsf{subj}(\hat{\xi})\} \cup \mathsf{chan}(\hat{A}) \cup \mathsf{chan}(\textbf{while}\ \hat{\xi}|_n\ \{\psi\}\ \hat{A}|_\emptyset) \cup \emptyset$. Then, by Defn. 25, $\mathsf{chan}(\alpha) \cup \mathsf{chan}(A') \subseteq \mathsf{chan}(\textbf{while}\ \hat{\xi}|_n\ \{\psi\}\ \hat{A}|_\emptyset) \cup \mathsf{chan}(\textbf{while}\ \hat{\xi}|_n\ \{\psi\}\ \hat{A}|_\emptyset) \cup \emptyset$. Then, $\mathsf{chan}(\alpha) \cup \mathsf{chan}(A') \subseteq \mathsf{chan}(\textbf{while}\ \hat{\xi}|_n\ \{\psi\}\ \hat{A}|_\emptyset)$. Then, $\mathsf{chan}(\alpha) \cup \mathsf{chan}(A') \subseteq \mathsf{chan}(A)$. $\square$

Proof (of Lem. 55). Recall $A\downarrow$. Then, by Defn. 28:

- **Base:** [$\downarrow$1-Skip], such that $A = \textbf{skip}$.
  By Defn. 31, not $\textbf{skip} \xrightarrow{\xi,\alpha} A'$. Then, not $A \xrightarrow{\xi,\alpha} A'$. Then, **false**.
- **Step:** [$\downarrow$1-Seq], such that $A = A_1 ; A_2$ and $A_1\downarrow$ and $A_2\downarrow$, for some A_1, A_2.
  Recall $A \xrightarrow{\xi,\alpha} A'$. Then, $A_1 ; A_2 \xrightarrow{\xi,\alpha} A'$. Then, by Defn. 31:
  - **Case:** [$\rightarrow$1-Seq1], such that $A' = A_1 ; A_2$ and $A_1 \xrightarrow{\xi,\alpha} A_1'$, for some A_1'.
    - Recall $A_1\downarrow$ and $A_1 \xrightarrow{\xi,\alpha} A_1'$. Then, by induction, $A_1'\downarrow$ and $\xi = \texttt{true}$ and $\alpha = \tau$.
    - Recall $A_1'\downarrow$ and $A_2\downarrow$. Then, by [$\downarrow$1-Seq], $A_1' ; A_2\downarrow$. Then, $A'\downarrow$.
  - **Case:** [$\rightarrow$1-Seq2]. Similar to case [$\rightarrow$1-Seq1].
- **Step:** [$\downarrow$1-Par]. Similar to case [$\downarrow$1-Seq].
- **Step:** [$\downarrow$1-NIf], such that $A = \textbf{if}\ \hat{\xi}|_0\ A_1|_{R_1}\ A_2|_{R_2}$ and $\mathsf{subj}(\hat{\xi}) = R_1 \cup R_2$, and $R_1 \neq \emptyset$ implies $A_1\downarrow$, and $R_2 \neq \emptyset$ implies $A_2\downarrow$, for some $R_1, R_2, A_1, A_2, \hat{\xi}$.
  Recall $A \xrightarrow{\xi,\alpha} A'$. Then, $\textbf{if}\ \hat{\xi}|_0\ A_1|_{R_1}\ A_2|_{R_2} \xrightarrow{\xi,\alpha} A'$. Then, by Defn. 31:
  - **Case:** [$\rightarrow$1-NIf1], such that $0 > 0$. Then, **false**.
  - **Case:** [$\rightarrow$1-NIf2], such that $r \in \mathsf{subj}(\xi) \setminus (R_1 \cup R_2)$, for some r. Then, $r \in \mathsf{subj}(\xi) \setminus \mathsf{subj}(\xi)$. Then, $r \in \emptyset$. Then, **false**.
  - **Case:** [$\rightarrow$1-NIf3]. Similar to case [$\rightarrow$1-NIf2].
  - **Case:** [$\rightarrow$1-NIf4], such that $A' = \textbf{if}\ \hat{\xi}|_0\ A_1'|_{R_1}\ A_2|_{R_2}$ and $A_1 \xrightarrow{\xi,\alpha} A_1'$ and $\mathsf{subj}(\xi) \cup \mathsf{subj}(\alpha) \subseteq R_1 \setminus R_2$, for some A_1'.
    Recall:
    - **Case:** $R_1 = \emptyset$.
      - Recall $R_1 = \emptyset$. Then, $R_1 \neq \emptyset$ implies $A_1'\downarrow$.
      - Recall $\mathsf{subj}(\hat{\xi}) = R_1 \cup R_2$, and $R_1' \neq \emptyset$ implies $A_1\downarrow$, and $R_2 \neq \emptyset$ implies $A_2\downarrow$. Then, by [$\downarrow$1-NIf], $\textbf{if}\ \hat{\xi}|_0\ A_1'|_{R_1}\ A_2|_{R_2}\downarrow$. Then, $A'\downarrow$.
      - Recall $\mathsf{subj}(\xi) \cup \mathsf{subj}(\alpha) \subseteq R_1 \setminus R_2$. Then, $\mathsf{subj}(\alpha) \subseteq R_1 \setminus R_2$. Then, $\mathsf{subj}(\alpha) \subseteq \emptyset \setminus R_2$. Then, $\mathsf{subj}(\alpha) \subseteq \emptyset$. Then, $\mathsf{subj}(\alpha) = \emptyset$. Then, by Lem. 30, $\alpha = \tau$.
      - Recall $A_1 \xrightarrow{\xi,\alpha} A_1'$ and $\alpha = \tau$. Then, by Lem. 50, $(\xi, \alpha) = (\texttt{true}, \tau)$. Then, $\xi = \texttt{true}$.
    - **Case:** $R_1 \neq \emptyset$.
      - Recall $R_1 \neq \emptyset$. Then, $A_1\downarrow$.
      - Recall $A_1\downarrow$ and $A_1 \xrightarrow{\xi,\alpha} A_1'$. Then, by induction, $A_1'\downarrow$ and $\xi = \texttt{true}$ and $\alpha = \tau$.
      - Recall $A_1'\downarrow$. Then, $R_1 \neq \emptyset$ implies $A_1'\downarrow$.
      - Recall $\mathsf{subj}(\hat{\xi}) = R_1 \cup R_2$, and $R_1' \neq \emptyset$ implies $A_1\downarrow$, and $R_2 \neq \emptyset$ implies $A_2\downarrow$. Then, by [$\downarrow$1-NIf], $\textbf{if}\ \hat{\xi}|_0\ A_1'|_{R_1}\ A_2|_{R_2}\downarrow$. Then, $A'\downarrow$.
  - **Case:** [$\rightarrow$1-NIf5]. Similar to case [$\rightarrow$1-NIf4]. □

Proof (of Lem. 56).

1. Recall $A \xrightarrow{\texttt{true},\tau} A_1'$. Then, by Defn. 31:
   - **Base:** [$\rightarrow$1-Act], such that $A = \tau$ and $A_1' = \textbf{skip}$.
     - Recall $A \xrightarrow{\xi,\alpha} A_2'$. Then, $\tau \xrightarrow{\xi,\alpha} A_2'$. Then, by Defn. 31, $A_2' = \textbf{skip}$ and $\xi = \texttt{true}$ and $\alpha = \tau$.
     - Recall $\textbf{skip} = \textbf{skip}$. Then, $A_1' = A_2'$.
   - **Base:** [$\rightarrow$1-If1], such that $\tau = i^R_{\mathsf{subj}(\hat{\xi})}$, for some $R, \hat{\xi}$. Then, **false**.
   - **Base:** [$\rightarrow$1-If2]. Similar to case [$\rightarrow$1-If1].
   - **Base:** [$\rightarrow$1-While1]. Similar to case [$\rightarrow$1-If1].
   - **Base:** [$\rightarrow$1-While2]. Similar to case [$\rightarrow$1-If1].
   - **Base:** [$\rightarrow$1-NIf1], such that $A = \textbf{if}\ \hat{\xi}|_n\ A_{11}|_{R_1}\ A_{12}|_{R_2}$ and $A_1' = \textbf{if}\ \hat{\xi}|_{n-1}\ A_{11}|_{R_1}\ A_{12}|_{R_2}$ and $n > 0$, for some $R_1, R_2, A_{11}, A_{12}, \hat{\xi}, n$.
     Recall $A \xrightarrow{\xi,\alpha} A_2'$. Then, $\textbf{if}\ \hat{\xi}|_n\ A_{11}|_{R_1}\ A_{12}|_{R_2} \xrightarrow{\xi,\alpha} A_2'$. Then, by Defn. 31:
     - **Case:** [$\rightarrow$1-NIf1], such that $A_2' = \textbf{if}\ \hat{\xi}|_{n-1}\ A_{11}|_{R_1}\ A_{12}|_{R_2}$ and $\xi = \texttt{true}$ and $\alpha = \tau$.
       Recall $A_1' = \textbf{if}\ \hat{\xi}|_{n-1}\ A_{11}|_{R_1}\ A_{12}|_{R_2} = A_2'$. Then, $A_1' = A_2'$.

- **Case:** [$\to$1-NIF2], such that $A_2' = \textbf{if } \hat{\xi}|_n\ A_{11}|_{R_1\cup\{r\}}\ A_{12}|_{R_2}$ and $\xi = \hat{\xi}^+ \restriction r$ and $\alpha = 1_{\{r\}}^{\{r\}}$ and $r \in \mathsf{subj}(\hat{\xi}) \setminus (R_1 \cup R_2)$, for some r.
  - Recall $r \in \mathsf{subj}(\hat{\xi}) \setminus (R_1 \cup R_2)$. Then, by [$\to$1-NIF2], $\textbf{if } \hat{\xi}|_{n-1}\ A_{11}|_{R_1}\ A_{12}|_{R_2} \xrightarrow{\hat{\xi}^+\restriction r, 1_{\{r\}}^{\{r\}}} \textbf{if } \hat{\xi}|_{n-1}\ A_{11}|_{R_1\cup\{r\}}\ A_{12}|_{R_2}$. Then, $A_1' \xrightarrow{\xi,\alpha} \textbf{if } \hat{\xi}|_{n-1}\ A_{11}|_{R_1\cup\{r\}}\ A_{12}|_{R_2}$. Then, $A'' = \textbf{if } \hat{\xi}|_{n-1}\ A_{11}|_{R_1\cup\{r\}}\ A_{12}|_{R_2}$ and $A_1' \xrightarrow{\xi,\alpha} A''$, for some A''.
  - Recall $n > 0$. Then, by [$\to$1-NIF1], $\textbf{if } \hat{\xi}|_n\ A_{11}|_{R_1\cup\{r\}}\ A_{12}|_{R_2} \xrightarrow{\mathtt{true},\tau} \textbf{if } \hat{\xi}|_{n-1}\ A_{11}|_{R_1\cup\{r\}}\ A_{12}|_{R_2}$. Then, $A_2' \xrightarrow{\mathtt{true},\tau} A''$.
- **Case:** [$\to$1-NIF3]. Similar to case [$\to$1-NIF2].
- **Case:** [$\to$1-NIF4]. Similar to case [$\to$1-NIF2].
- **Case:** [$\to$1-NIF5]. Similar to case [$\to$1-NIF2].

– **Base:** [$\to$1-NIF2]. Similar to case [$\to$1-IF1].

– **Base:** [$\to$1-NIF3]. Similar to case [$\to$1-IF1].

– **Step:** [$\to$1-SEQ1], such that $A = A_{11} ; A_{12}$ and $A_1' = A_{11}' ; A_{12}$ and $A_{11} \xrightarrow{\mathtt{true},\tau} A_{11}'$, for some A_{11}, A_{11}', A_{12}.
Recall $A \xrightarrow{\xi,\alpha} A_2'$. Then, $A_{11} ; A_{12} \xrightarrow{\xi,\alpha} A_2'$. Then, by Defn. 31:

- **Case:** [$\to$1-SEQ1], such that $A_2' = A_{21}' ; A_{12}$ and $A_{11} \xrightarrow{\xi,\alpha} A_{21}'$, for some A_{21}'.
Recall $A_{11} \xrightarrow{\mathtt{true},\tau} A_{11}'$ and $A_{11} \xrightarrow{\xi,\alpha} A_{21}'$. Then, by induction:
  - **Case:** $A_{11}' \xrightarrow{\xi,\alpha} A_{11}''$ and $A_{21}' \xrightarrow{\mathtt{true},\tau} A_{11}''$, for some A_{11}''.
    - Recall $A_{11}' \xrightarrow{\xi,\alpha} A_{11}''$. Then, by [$\to$1-SEQ1], $A_{11}' ; A_{12} \xrightarrow{\xi,\alpha} A_{11}'' ; A_{12}$. Then, $A_1' \xrightarrow{\xi,\alpha} A_{11}'' ; A_{12}$. Then, $A'' = A_{11}'' ; A_{12}$ and $A_1' \xrightarrow{\xi,\alpha} A''$, for some A''.
    - Recall $A_{21}' \xrightarrow{\mathtt{true},\tau} A_{11}''$. Then, by [$\to$1-SEQ1], $A_{21}' ; A_{12} \xrightarrow{\mathtt{true},\tau} A_{11}'' ; A_{12}$. Then, $A_2' \xrightarrow{\mathtt{true},\tau} A''$.
  - **Case:** $\xi = \mathtt{true}$ and $\alpha = \tau$ and $A_{11}' = A_{21}'$.
Recall $A_{11}' = A_{21}'$. Then, $A_{11}' ; A_{12} = A_{21}' ; A_{12}$. Then, $A_1' = A_2'$.
- **Case:** [$\to$1-SEQ2], such that $A_2' = A_{11} ; A_{22}'$ and $\mathsf{subj}(A_{11}) \cap (\mathsf{subj}(\xi) \cup \mathsf{subj}(\alpha)) = \emptyset$ and $A_{12} \xrightarrow{\xi,\alpha} A_{22}'$, for some A_{22}'.
  - Recall $A_{11} \xrightarrow{\mathtt{true},\tau} A_{11}'$. Then, by Lem. 53, $\mathsf{subj}(A_{11}) = \mathsf{subj}(A_{11}')$.
  - Recall $\mathsf{subj}(A_{11}) \cap (\mathsf{subj}(\xi) \cup \mathsf{subj}(\alpha)) = \emptyset$. Then, $\mathsf{subj}(A_{11}') \cap (\mathsf{subj}(\xi) \cup \mathsf{subj}(\alpha)) = \emptyset$.
  - Recall $\mathsf{subj}(A_{11}') \cap (\mathsf{subj}(\xi) \cup \mathsf{subj}(\alpha)) = \emptyset$ and $A_{12} \xrightarrow{\xi,\alpha} A_{22}'$. Then, by [$\to$1-SEQ2], $A_{11}' ; A_{12} \xrightarrow{\xi,\alpha} A_{11}' ; A_{22}'$. Then, $A_1' \xrightarrow{\xi,\alpha} A_{11}' ; A_{22}'$. Then, $A'' = A_{11}' ; A_{22}'$ and $A_1' \xrightarrow{\xi,\alpha} A''$, for some A''.
  - Recall $A_{11} \xrightarrow{\mathtt{true},\tau} A_{11}'$. Then, by [$\to$1-SEQ1], $A_{11} ; A_{22}' \xrightarrow{\mathtt{true},\tau} A_{11}' ; A_{22}'$. Then, $A_2' \xrightarrow{\mathtt{true},\tau} A''$.

– **Step:** [$\to$1-SEQ2], such that $A = A_{11} ; A_{12}$ and $A_1' = A_{11} ; A_{12}'$ and $\mathsf{subj}(A_{11}) \cap (\mathsf{subj}(\mathtt{true}) \cup \mathsf{subj}(\tau)) = \emptyset$ and $A_{12} \xrightarrow{\mathtt{true},\tau} A_{12}'$, for some A_{11}, A_{12}, A_{12}'.
Recall $A \xrightarrow{\xi,\alpha} A_2'$. Then, $A_{11} ; A_{12} \xrightarrow{\xi,\alpha} A_2'$. Then, by Defn. 31:

- **Case:** [$\to$1-SEQ1], such that $A_2' = A_{21}' ; A_{12}$ and $A_{11} \xrightarrow{\xi,\alpha} A_{21}'$, for some A_{21}'.
  - Recall $A_{11} \xrightarrow{\xi,\alpha} A_{21}'$. Then, by [$\to$1-SEQ1], $A_{11} ; A_{12}' \xrightarrow{\xi,\alpha} A_{21}' ; A_{12}'$. Then, $A_1' \xrightarrow{\xi,\alpha} A_{21}' ; A_{12}'$. Then, $A'' = A_{21}' ; A_{12}'$ and $A_1' \xrightarrow{\xi,\alpha} A''$, for some A''.
  - Recall $\mathsf{subj}(A_{21}') \cap (\emptyset \cup \emptyset) = \emptyset$. Then, by Lem. 4, $\mathsf{subj}(A_{21}') \cap (\mathsf{subj}(\mathtt{true}) \cup \emptyset) = \emptyset$. Then, by Defn. 16, $\mathsf{subj}(A_{21}') \cap (\mathsf{subj}(\mathtt{true}) \cup \mathsf{subj}(\tau)) = \emptyset$.
  - Recall $\mathsf{subj}(A_{21}') \cap (\mathsf{subj}(\mathtt{true}) \cup \mathsf{subj}(\tau)) = \emptyset$ and $A_{12} \xrightarrow{\mathtt{true},\tau} A_{12}'$. Then, $A_{21}' ; A_{12} \xrightarrow{\mathtt{true},\tau} A_{21}' ; A_{12}'$. Then, $A_2' \xrightarrow{\mathtt{true},\tau} A''$.
- **Case:** [$\to$1-SEQ2], such that $A_2' = A_{11} ; A_{22}'$ and $\mathsf{subj}(A_{11}) \cap (\mathsf{subj}(\xi) \cup \mathsf{subj}(\alpha)) = \emptyset$ and $A_{12} \xrightarrow{\xi,\alpha} A_{22}'$, for some A_{22}'.
Recall $A_{12} \xrightarrow{\mathtt{true},\tau} A_{12}'$ and $A_{12} \xrightarrow{\xi,\alpha} A_{22}'$. Then, by induction:
  - **Case:** $A_{12}' \xrightarrow{\xi,\alpha} A_{12}''$ and $A_{22}' \xrightarrow{\mathtt{true},\tau} A_{12}''$, for some A_{12}''.
    - Recall $\mathsf{subj}(A_{11}) \cap (\mathsf{subj}(\xi) \cup \mathsf{subj}(\alpha)) = \emptyset$ and $A_{12}' \xrightarrow{\xi,\alpha} A_{12}''$. Then, by [$\to$1-SEQ2], $A_{11} ; A_{12}' \xrightarrow{\xi,\alpha} A_{11} ; A_{12}''$. Then, $A_1' \xrightarrow{\xi,\alpha} A_{11} ; A_{12}''$. Then, $A'' = A_{11} ; A_{12}''$ and $A_1' \xrightarrow{\xi,\alpha} A''$, for some A''.
    - Recall $\mathsf{subj}(A_{11}) \cap (\mathsf{subj}(\mathtt{true}) \cup \mathsf{subj}(\tau)) = \emptyset$ and $A_{22}' \xrightarrow{\mathtt{true},\tau} A_{12}''$. Then, by [$\to$1-SEQ2], $A_{11} ; A_{22}' \xrightarrow{\mathtt{true},\tau} A_{11} ; A_{12}''$. Then, $A_2' \xrightarrow{\mathtt{true},\tau} A''$.

    * **Case:** $\xi = \mathtt{true}$ and $\alpha = \tau$ and $A'_{12} = A'_{22}$.
      Recall $A'_{12} = A'_{22}$. Then, $A_{11} \,;\, A'_{12} = A_{11} \,;\, A'_{22}$. Then, $A'_1 = A'_2$.

- **Step:** [$\rightarrow$1-Par1], such that $A = A_{11} \parallel A_{12}$ and $A'_1 = A'_{11} \parallel A_{12}$ and $A_{11} \xrightarrow{\mathtt{true},\tau} A'_{11}$, for some A_{11}, A'_{11}, A_{12}.
  Recall $A \xrightarrow{\xi,\alpha} A'_2$. Then, $A_{11} \parallel A_{12} \xrightarrow{\xi,\alpha} A'_2$. Then, by Defn. 31:
  - **Case:** [$\rightarrow$1-Par1], such that $A'_2 = A'_{21} \parallel A_{12}$ and $A_{11} \xrightarrow{\xi,\alpha} A'_{21}$, for some A'_{21}.
    Recall $A_{11} \xrightarrow{\mathtt{true},\tau} A'_{11}$ and $A_{11} \xrightarrow{\xi,\alpha} A'_{21}$. Then, by induction:
    * **Case:** $A'_{11} \xrightarrow{\xi,\alpha} A''_{11}$ and $A'_{21} \xrightarrow{\mathtt{true},\tau} A''_{11}$, for some A''_{11}.
      · Recall $A'_{11} \xrightarrow{\xi,\alpha} A''_{11}$. Then, by [$\rightarrow$1-Par1], $A'_{11} \parallel A_{12} \xrightarrow{\xi,\alpha} A''_{11} \parallel A_{12}$. Then, $A'_1 \xrightarrow{\xi,\alpha} A''_{11} \parallel A_{12}$. Then, $A'' = A''_{11} \parallel A_{12}$ and $A'_1 \xrightarrow{\xi,\alpha} A''$, for some A''.
      · Recall $A'_{21} \xrightarrow{\mathtt{true},\tau} A''_{11}$. Then, by [$\rightarrow$1-Par1], $A'_{21} \parallel A_{12} \xrightarrow{\mathtt{true},\tau} A''_{11} \parallel A_{12}$. Then, $A'_2 \xrightarrow{\mathtt{true},\tau} A''$.
    * **Case:** $\xi = \mathtt{true}$ and $\alpha = \tau$ and $A'_{11} = A'_{21}$.
      Recall $A'_{11} = A'_{21}$. Then, $A'_{11} \parallel A_{12} = A'_{21} \parallel A_{12}$. Then, $A'_1 = A'_2$.
  - **Case:** [$\rightarrow$1-Par2], such that $A'_2 = A_{11} \parallel A'_{22}$ and $A_{12} \xrightarrow{\xi,\alpha} A'_{22}$, for some A'_{22}.
    * Recall $A_{12} \xrightarrow{\xi,\alpha} A'_{22}$. Then, by [$\rightarrow$1-Par2], $A'_{11} \parallel A_{12} \xrightarrow{\xi,\alpha} A'_{11} \parallel A'_{22}$. Then, $A'_1 \xrightarrow{\xi,\alpha} A'_{11} \parallel A'_{22}$. Then, $A'' = A'_{11} \parallel A'_{22}$ and $A'_1 \xrightarrow{\xi,\alpha} A''$, for some A''.
    * Recall $A_{11} \xrightarrow{\mathtt{true},\tau} A'_{11}$. Then, by [$\rightarrow$1-Par1], $A_{11} \parallel A'_{22} \xrightarrow{\mathtt{true},\tau} A'_{11} \parallel A'_{22}$. Then, $A'_2 \xrightarrow{\mathtt{true},\tau} A''$.

- **Step:** [$\rightarrow$1-Par2]. Similar to case [$\rightarrow$1-Par1].
- **Step:** [$\rightarrow$1-NIf4], such that $A = \mathbf{if}\ \hat{\xi}|_n\ A_{11}|_{R_1}\ A_{12}|_{R_2}$ and $A'_1 = \mathbf{if}\ \hat{\xi}|_n\ A'_{11}|_{R_1}\ A_{12}|_{R_2}$ and $A_{11} \xrightarrow{\mathtt{true},\tau} A'_{11}$ and $\mathsf{subj}(\mathtt{true}) \cup \mathsf{subj}(\tau) \subseteq R_1 \setminus R_2$, for some $R_1, R_2, A_{11}, A'_{11}, A_{12}, \hat{\xi}, n$.
  Recall $A \xrightarrow{\xi,\alpha} A'_2$. Then, $\mathbf{if}\ \hat{\xi}|_n\ A_{11}|_{R_1}\ A_{12}|_{R_2} \xrightarrow{\xi,\alpha} A'_2$. Then, by Defn. 31:
  - **Case:** [$\rightarrow$1-NIf1], such that $A'_2 = \mathbf{if}\ \hat{\xi}|_{n-1}\ A_{11}|_{R_1}\ A_{12}|_{R_2}$ and $\xi = \mathtt{true}$ and $\alpha = \tau$ and $n > 0$.
    * Recall $n > 0$. Then, by [$\rightarrow$1-NIf1], $\mathbf{if}\ \hat{\xi}|_n\ A'_{11}|_{R_1}\ A_{12}|_{R_2} \xrightarrow{\mathtt{true},\tau} \mathbf{if}\ \hat{\xi}|_{n-1}\ A'_{11}|_{R_1}\ A_{12}|_{R_2}$. Then, $A'_1 \xrightarrow{\xi,\alpha} \mathbf{if}\ \hat{\xi}|_{n-1}\ A'_{11}|_{R_1}\ A_{12}|_{R_2}$. Then, $A'' = \mathbf{if}\ \hat{\xi}|_{n-1}\ A'_{11}|_{R_1}\ A_{12}|_{R_2}$ and $A'_1 \xrightarrow{\xi,\alpha} A''$, for some A''.
    * Recall $A_{11} \xrightarrow{\mathtt{true},\tau} A'_{11}$ and $\mathsf{subj}(\mathtt{true}) \cup \mathsf{subj}(\tau) \subseteq R_1 \setminus R_2$. Then, by [$\rightarrow$1-NIf4], $\mathbf{if}\ \hat{\xi}|_{n-1}\ A_{11}|_{R_1}\ A_{12}|_{R_2} \xrightarrow{\mathtt{true},\tau} \mathbf{if}\ \hat{\xi}|_{n-1}\ A'_{11}|_{R_1}\ A_{12}|_{R_2}$. Then, $A'_2 \xrightarrow{\mathtt{true},\tau} A''$.
  - **Case:** [$\rightarrow$1-NIf2], such that $A'_2 = \mathbf{if}\ \hat{\xi}|_n\ A_{11}|_{R_1 \cup \{r\}}\ A_{12}|_{R_2}$ and $\xi = \hat{\xi}^+ \upharpoonright r$ and $\alpha = 1\{^r_r\}$ and $r \in \mathsf{subj}(\hat{\xi}) \setminus (R_1 \cup R_2)$, for some r.
    * Recall $r \in \mathsf{subj}(\hat{\xi}) \setminus (R_1 \cup R_2)$. Then, by [$\rightarrow$1-NIf2], $\mathbf{if}\ \hat{\xi}|_n\ A'_{11}|_{R_1}\ A_{12}|_{R_2} \xrightarrow{\hat{\xi}^+ \upharpoonright r, 1\{^r_r\}} \mathbf{if}\ \hat{\xi}|_n\ A'_{11}|_{R_1 \cup \{r\}}\ A_{12}|_{R_2}$. Then, $A'_1 \xrightarrow{\xi,\alpha} \mathbf{if}\ \hat{\xi}|_n\ A'_{11}|_{R_1 \cup \{r\}}\ A_{12}|_{R_2}$. Then, $A'' = \mathbf{if}\ \hat{\xi}|_n\ A'_{11}|_{R_1 \cup \{r\}}\ A_{12}|_{R_2}$ and $A'_1 \xrightarrow{\xi,\alpha} A''$, for some A''.
    * Recall $\emptyset \cup \emptyset \subseteq (R_1 \cup \{r\}) \setminus R_2$. Then, by Lem. 4, $\mathsf{subj}(\mathtt{true}) \cup \emptyset \subseteq (R_1 \cup \{r\}) \setminus R_2$. Then, by Defn. 16, $\mathsf{subj}(\mathtt{true}) \cup \mathsf{subj}(\tau) \subseteq (R_1 \cup \{r\}) \setminus R_2$.
    * Recall $A_{11} \xrightarrow{\mathtt{true},\tau} A'_{11}$ and $\mathsf{subj}(\mathtt{true}) \cup \mathsf{subj}(\tau) \subseteq (R_1 \cup \{r\}) \setminus R_2$. Then, by [$\rightarrow$1-NIf4], $\mathbf{if}\ \hat{\xi}|_{A_{11}}\ n|_{R_1 \cup \{r\}}\ A_{12}|_{R_2} \xrightarrow{\mathtt{true},\tau} \mathbf{if}\ \hat{\xi}|_{A'_{11}}\ n|_{R_1 \cup \{r\}}\ A_{12}|_{R_2}$. Then, $A'_2 \xrightarrow{\mathtt{true},\tau} A''$.
  - **Case:** [$\rightarrow$1-NIf3]. Similar to case [$\rightarrow$1-NIf2].
  - **Case:** [$\rightarrow$1-NIf4], such that $A'_2 = \mathbf{if}\ \hat{\xi}|_n\ A'_{21}|_{R_1}\ A_{12}|_{R_2}$ and $A_{11} \xrightarrow{\xi,\alpha} A'_{21}$ and $\mathsf{subj}(\xi) \cup \mathsf{subj}(\alpha) \subseteq R_1 \setminus R_2$, for some A'_{21}.
    Recall $A_{11} \xrightarrow{\mathtt{true},\tau} A'_{11}$ and $A_{11} \xrightarrow{\xi,\alpha} A'_{21}$. Then, by induction:
    * **Case:** $A'_{11} \xrightarrow{\xi,\alpha} A''_{11}$ and $A'_{21} \xrightarrow{\mathtt{true},\tau} A''_{11}$, for some A''_{11}.
      · Recall $A'_{11} \xrightarrow{\xi,\alpha} A''_{11}$ and $\mathsf{subj}(\xi) \cup \mathsf{subj}(\alpha) \subseteq R_1 \setminus R_2$. Then, by [$\rightarrow$1-NIf4], $\mathbf{if}\ \hat{\xi}|_n\ A'_{11}|_{R_1}\ A_{12}|_{R_2} \xrightarrow{\xi,\alpha} \mathbf{if}\ \hat{\xi}|_n\ A''_{11}|_{R_1}\ A_{12}|_{R_2}$. Then, $A'_1 \xrightarrow{\xi,\alpha} \mathbf{if}\ \hat{\xi}|_n\ A''_{11}|_{R_1}\ A_{12}|_{R_2}$. Then, $A'' = \mathbf{if}\ \hat{\xi}|_n\ A''_{11}|_{R_1}\ A_{12}|_{R_2}$ and $A'_1 \xrightarrow{\xi,\alpha} A''$, for some A''.
      · Recall $A'_{21} \xrightarrow{\mathtt{true},\tau} A''_{11}$ and $\mathsf{subj}(\mathtt{true}) \cup \mathsf{subj}(\tau) \subseteq R_1 \setminus R_2$. Then, by [$\rightarrow$1-NIf4], $\mathbf{if}\ \hat{\xi}|_n\ A'_{21}|_{R_1}\ A_{12}|_{R_2} \xrightarrow{\mathtt{true},\tau} \mathbf{if}\ \hat{\xi}|_n\ A''_{11}|_{R_1}\ A_{12}|_{R_2}$. Then, $A'_2 \xrightarrow{\mathtt{true},\tau} A''$.

  - ∗ **Case:** $\xi = \texttt{true}$ and $\alpha = \tau$ and $A'_{11} = A'_{21}$.
    Recall $A'_{11} = A'_{21}$. Then, $\textbf{if}\ \hat{\xi}|_n\ A'_{11}|_{R_1}\ A_{12}|_{R_2} = \textbf{if}\ \hat{\xi}|_n\ A'_{21}|_{R_1}\ A_{12}|_{R_2}$. Then, $A'_1 = A'_2$.
  - • **Case:** [→1-NIf5], such that $A'_2 = \textbf{if}\ \hat{\xi}|_n\ A_{11}|_{R_1}\ A'_{22}|_{R_2}$ and $A_{12} \xrightarrow{\xi,\alpha} A'_{22}$ and $\mathsf{subj}(\xi) \cup \mathsf{subj}(\alpha) \subseteq R_2 \setminus R_1$, for some A'_{22}.
    - ∗ Recall $A_{12} \xrightarrow{\xi,\alpha} A'_{22}$ and $\mathsf{subj}(\xi) \cup \mathsf{subj}(\alpha) \subseteq R_2 \setminus R_1$. Then, by [→1-NIf5], $\textbf{if}\ \hat{\xi}|_n\ A'_{11}|_{R_1}\ A_{12}|_{R_2} \xrightarrow{\xi,\alpha} \textbf{if}\ \hat{\xi}|_n\ A'_{11}|_{R_1}\ A'_{22}|_{R_2}$. Then, $A'_1 \xrightarrow{\xi,\alpha} \textbf{if}\ \hat{\xi}|_n\ A'_{11}|_{R_1}\ A'_{22}|_{R_2}$. Then, $A'' = \textbf{if}\ \hat{\xi}|_n\ A'_{11}|_{R_1}\ A'_{22}|_{R_2}$ and $A'_1 \xrightarrow{\xi,\alpha} A''$, for some A''.
    - ∗ Recall $A_{11} \xrightarrow{\texttt{true},\tau} A'_{11}$ and $\mathsf{subj}(\texttt{true}) \cup \mathsf{subj}(\tau) \subseteq R_1 \setminus R_2$. Then, by [→1-NIf4], $\textbf{if}\ \hat{\xi}|_n\ A_{11}|_{R_1}\ A'_{22}|_{R_2} \xrightarrow{\texttt{true},\tau} \textbf{if}\ \hat{\xi}|_n\ A'_{11}|_{R_1}\ A'_{22}|_{R_2}$. Then, $A'_2 \xrightarrow{\texttt{true},\tau} A''$.
- – **Step:** [→1-NIf5]. Similar to case [→1-NIf4].
- – **Step:** [→1-NWhile], such that $A = \textbf{while}\ \hat{\xi}|_n\ \{\psi\}\ \hat{A}|_\emptyset$ and $\textbf{if}\ \hat{\xi}|_n\ (\hat{A}\,;\textbf{while}\ \hat{\xi}|_n\ \{\psi\}\ \hat{A}|_\emptyset)|_\emptyset\ \textbf{skip}|_\emptyset \xrightarrow{\texttt{true},\tau} A'_1$, for some $\hat{A}, \psi, \hat{\xi}, n$.
  - • Recall $A \xrightarrow{\xi,\alpha} A'_2$. Then, $\textbf{while}\ \hat{\xi}|_n\ \{\psi\}\ \hat{A}|_\emptyset \xrightarrow{\xi,\alpha} A'_2$. Then, by Defn. 31, $\textbf{if}\ \hat{\xi}|_n\ (\hat{A}\,;\textbf{while}\ \hat{\xi}|_n\ \{\psi\}\ \hat{A}|_\emptyset)|_\emptyset\ \textbf{skip}|_\emptyset \xrightarrow{\xi,\alpha} A'_2$.
  - • Recall $\textbf{if}\ \hat{\xi}|_n\ (\hat{A}\,;\textbf{while}\ \hat{\xi}|_n\ \{\psi\}\ \hat{A}|_\emptyset)|_\emptyset\ \textbf{skip}|_\emptyset \xrightarrow{\texttt{true},\tau} A'_1$ and $\textbf{if}\ \hat{\xi}|_n\ (\hat{A}\,;\textbf{while}\ \hat{\xi}|_n\ \{\psi\}\ \hat{A}|_\emptyset)|_\emptyset\ \textbf{skip}|_\emptyset \xrightarrow{\xi,\alpha} A'_2$. Then, by induction:
    - ∗ **Case:** $\xi = \texttt{true}$ and $\alpha = \tau$ and $A'_1 = A'_2$.
    - ∗ **Case:** $A'_1 \xrightarrow{\xi,\alpha} A''$ and $A'_2 \xrightarrow{\xi,\alpha} A''$, for some A''. □

2. Recall $A \xrightarrow{\texttt{true},\tau} A'_1$. Then, by Defn. 31:
- – **Base:** [→1-Act], such that $A'_1 = \textbf{skip}$.
  By Defn. 31, not $\textbf{skip} \xrightarrow{\xi,\alpha} A'_1$. Then, not $A'_1 \xrightarrow{\xi,\alpha} A'_1$. Then, **false**.
- – **Base:** [→1-If1], such that $\tau = i^R_{\mathsf{subj}(\hat{\xi})}$, for some $R, \hat{\xi}$. Then, **false**.
- – **Base:** [→1-If2]. Similar to case [→1-If1].
- – **Base:** [→1-While1]. Similar to case [→1-If1].
- – **Base:** [→1-While2]. Similar to case [→1-If1].
- – **Base:** [→1-NIf1], such that $A = \textbf{if}\ \hat{\xi}|_n\ A_{11}|_{R_1}\ A_{12}|_{R_2}$ and $A'_1 = \textbf{if}\ \hat{\xi}|_{n-1}\ A_{11}|_{R_1}\ A_{12}|_{R_2}$ and $n > 0$, for some $R_1, R_2, A_{11}, A_{12}, \hat{\xi}, n$.
  Recall $A'_1 \xrightarrow{\xi,\alpha} A''$. Then, $\textbf{if}\ \hat{\xi}|_{n-1}\ A_{11}|_{R_1}\ A_{12}|_{R_2} \xrightarrow{\xi,\alpha} A''$. Then, by Defn. 31:
  - • **Case:** [→1-NIf1], such that $A'' = \textbf{if}\ \hat{\xi}|_{n-1-1}\ A_{11}|_{R_1}\ A_{12}|_{R_2}$ and $\xi = \texttt{true}$ and $\alpha = \tau$.
    - ∗ Recall $A \xrightarrow{\texttt{true},\tau} A'_1$. Then, $A \xrightarrow{\xi,\alpha} A'_1$. Then, $A'_2 = A'_1$ and $A \xrightarrow{\xi,\alpha} A'_2$, for some A'_2.
    - ∗ Recall $A'_1 \xrightarrow{\xi,\alpha} A''$. Then, $A'_2 \xrightarrow{\texttt{true},\tau} A''$.
  - • **Case:** [→1-NIf2], $A'' = \textbf{if}\ \hat{\xi}|_{n-1}\ A_{11}|_{R_1\cup\{r\}}\ A_{12}|_{R_2}$ and $\xi = \hat{\xi}^+ \restriction r$ and $\alpha = 1^{\{r\}}_{\{r\}}$ and $r \in \mathsf{subj}(\hat{\xi}) \setminus (R_1 \cup R_2)$, for some r.
    - ∗ Recall $r \in \mathsf{subj}(\hat{\xi}) \setminus (R_1 \cup R_2)$. Then, by [→1-NIf2], $\textbf{if}\ \hat{\xi}|_n\ A_{11}|_{R_1}\ A_{12}|_{R_2} \xrightarrow{\hat{\xi}^+\restriction r,1^{\{r\}}_{\{r\}}} \textbf{if}\ \hat{\xi}|_n\ A_{11}|_{R_1\cup\{r\}}\ A_{12}|_{R_2}$. Then, $A \xrightarrow{\xi,\alpha} \textbf{if}\ \hat{\xi}|_n\ A_{11}|_{R_1\cup\{r\}}\ A_{12}|_{R_2}$. Then $A'_2 = \textbf{if}\ \hat{\xi}|_n\ A_{11}|_{R_1\cup\{r\}}\ A_{12}|_{R_2}$ and $A \xrightarrow{\xi,\alpha} A'_2$, for some A'_2.
    - ∗ Recall $n > 0$. Then, by [→1-NIf1], $\textbf{if}\ \hat{\xi}|_n\ A_{11}|_{R_1\cup\{r\}}\ A_{12}|_{R_2} \xrightarrow{\texttt{true},\tau} \textbf{if}\ \hat{\xi}|_{n-1}\ A_{11}|_{R_1\cup\{r\}}\ A_{12}|_{R_2}$. Then, $A'_2 \xrightarrow{\texttt{true},\tau} A''$.
  - • **Case:** [→1-NIf3]. Similar to case [→1-NIf2].
  - • **Case:** [→1-NIf4]. Similar to case [→1-NIf2].
  - • **Case:** [→1-NIf5]. Similar to case [→1-NIf2].
- – **Base:** [→1-NIf2]. Similar to case [→1-If1].
- – **Base:** [→1-NIf3]. Similar to case [→1-If1].
- – **Step:** [→1-Seq1], such that $A = A_{11}\,;A_{12}$ and $A'_1 = A'_{11}\,;A_{12}$ and $A_{11} \xrightarrow{\texttt{true},\tau} A'_{11}$, for some A_{11}, A'_{11}, A_{12}.
  Recall $A'_1 \xrightarrow{\xi,\alpha} A''$. Then, $A'_{11}\,;A_{12} \xrightarrow{\xi,\alpha} A''$. Then, by Defn. 31:
  - • **Case:** [→1-Seq1], such that $A'' = A''_{11}\,;A_{12}$ and $A'_{11} \xrightarrow{\xi,\alpha} A''_{11}$, for some A''_{11}.
    - ∗ Recall $A_{11} \xrightarrow{\texttt{true},\tau} A'_{11} \xrightarrow{\xi,\alpha} A''_{11}$. Then, by induction, $A_{11} \xrightarrow{\xi,\alpha} A'_{21} \xrightarrow{\texttt{true},\tau} A''_{11}$, for some A'_{21}.
    - ∗ Recall $A_{11} \xrightarrow{\xi,\alpha} A'_{21}$. Then, by [→1-Seq1], $A_{11}\,;A_{12} \xrightarrow{\xi,\alpha} A'_{21}\,;A_{12}$. Then, $A \xrightarrow{\xi,\alpha} A'_{21}\,;A_{12}$. Then, $A'_2 = A'_{21}\,;A_{12}$ and $A \xrightarrow{\xi,\alpha} A'_2$, for some A'_2.

  * Recall $A'_{21} \xrightarrow{\texttt{true},\tau} A''_{11}$. Then, by [→1-Seq1], $A'_{21} ; A_{12} \xrightarrow{\texttt{true},\tau} A''_{11} ; A_{12}$. Then, $A'_2 \xrightarrow{\texttt{true},\tau} A''$.
- **Case:** [→1-Seq2], such that $A'' = A'_{11} ; A'_{12}$ and $\mathsf{subj}(A'_{11}) \cap (\mathsf{subj}(\xi) \cup \mathsf{subj}(\alpha)) = \emptyset$ and $A_{12} \xrightarrow{\xi,\alpha} A'_{12}$, for some A'_{12}.
  * Recall $A_{11} \xrightarrow{\texttt{true},\tau} A'_{11}$. Then, by Lem. 53, $\mathsf{subj}(A_{11}) = \mathsf{subj}(A'_{11})$.
  * Recall $\mathsf{subj}(A'_{11}) \cap (\mathsf{subj}(\xi) \cup \mathsf{subj}(\alpha)) = \emptyset$. Then, $\mathsf{subj}(A_{11}) \cap (\mathsf{subj}(\xi) \cup \mathsf{subj}(\alpha)) = \emptyset$.
  * Recall $\mathsf{subj}(A_{11}) \cap (\mathsf{subj}(\xi) \cup \mathsf{subj}(\alpha)) = \emptyset$ and $A_{12} \xrightarrow{\xi,\alpha} A'_{12}$. Then, by [→1-Seq2], $A_{11} ; A_{12} \xrightarrow{\xi,\alpha} A_{11} ; A'_{12}$. Then, $A \xrightarrow{\xi,\alpha} A_{11} ; A'_{12}$. Then, $A'_2 = A_{11} ; A'_{12}$ and $A \xrightarrow{\xi,\alpha} A'_2$, for some A'_2.
  * Recall $A_{11} \xrightarrow{\texttt{true},\tau} A'_{11}$. Then, by [→1-Seq1], $A_{11} ; A'_{12} \xrightarrow{\texttt{true},\tau} A'_{11} ; A'_{12}$. Then, $A'_2 \xrightarrow{\texttt{true},\tau} A''$.

– **Step:** [→1-Seq2], such that $A = A_{11} ; A_{12}$ and $A'_1 = A_{11} ; A'_{12}$ and $\mathsf{subj}(A_{11}) \cap (\mathsf{subj}(\texttt{true}) \cup \mathsf{subj}(\tau)) = \emptyset$ and $A_{12} \xrightarrow{\texttt{true},\tau} A'_{12}$, for some A_{11}, A_{12}, A'_{12}.
Recall $A'_1 \xrightarrow{\xi,\alpha} A''$. Then, $A_{11} ; A'_{12} \xrightarrow{\xi,\alpha} A''$. Then, by Defn. 31:
- **Case:** [→1-Seq1], such that $A'' = A'_{11} ; A'_{12}$ and $A_{11} \xrightarrow{\xi,\alpha} A'_{11}$, for some A'_{11}.
  * Recall $A_{11} \xrightarrow{\xi,\alpha} A'_{11}$. Then, by [→1-Seq1], $A_{11} ; A_{12} \xrightarrow{\xi,\alpha} A'_{11} ; A_{12}$. Then, $A \xrightarrow{\xi,\alpha} A'_{11} ; A_{12}$. Then, $A'_2 = A'_{11} ; A_{12}$ and $A \xrightarrow{\xi,\alpha} A'_2$, for some A'_2.
  * Recall $\mathsf{subj}(A'_{11}) \cap (\emptyset \cup \emptyset) = \emptyset$. Then, by Lem. 4, $\mathsf{subj}(A'_{11}) \cap (\mathsf{subj}(\texttt{true}) \cup \emptyset) = \emptyset$. Then, by Defn. 16, $\mathsf{subj}(A'_{11}) \cap (\mathsf{subj}(\texttt{true}) \cup \mathsf{subj}(\tau)) = \emptyset$.
  * Recall $\mathsf{subj}(A'_{11}) \cap (\mathsf{subj}(\texttt{true}) \cup \mathsf{subj}(\tau)) = \emptyset$ and $A_{12} \xrightarrow{\texttt{true},\tau} A'_{12}$. Then, by [→1-Seq2], $A'_{11} ; A_{12} \xrightarrow{\texttt{true},\tau} A'_{11} ; A'_{12}$. Then, $A'_2 \xrightarrow{\texttt{true},\tau} A''$.
- **Case:** [→1-Seq2], such that $A'' = A_{11} ; A''_{12}$ and $\mathsf{subj}(A_{11}) \cap (\mathsf{subj}(\xi) \cup \mathsf{subj}(\alpha)) = \emptyset$ and $A'_{12} \xrightarrow{\xi,\alpha} A''_{12}$, for some A''_{12}.
  * Recall $A_{12} \xrightarrow{\texttt{true},\tau} A'_{12} \xrightarrow{\xi,\alpha} A''_{12}$. Then, by induction, $A_{12} \xrightarrow{\xi,\alpha} A'_{22} \xrightarrow{\texttt{true},\tau} A''_{12}$, for some A'_{22}.
  * Recall $\mathsf{subj}(A_{11}) \cap (\mathsf{subj}(\xi) \cup \mathsf{subj}(\alpha)) = \emptyset$ and $A_{12} \xrightarrow{\xi,\alpha} A'_{22}$. Then, by [→1-Seq2], $A_{11} ; A_{12} \xrightarrow{\xi,\alpha} A_{11} ; A'_{22}$. Then, $A \xrightarrow{\xi,\alpha} A_{11} ; A'_{22}$. Then, $A'_2 = A_{11} ; A'_{22}$ and $A \xrightarrow{\xi,\alpha} A'_2$, for some A'_2.
  * Recall $\mathsf{subj}(A_{11}) \cap (\mathsf{subj}(\texttt{true}) \cup \mathsf{subj}(\tau)) = \emptyset$ and $A'_{22} \xrightarrow{\texttt{true},\tau} A''_{12}$. Then, by [→1-Seq2], $A_{11} ; A'_{22} \xrightarrow{\texttt{true},\tau} A_{11} ; A''_{12}$. Then, $A'_2 \xrightarrow{\texttt{true},\tau} A''$.

– **Step:** [→1-Par1], such that $A = A_{11} \parallel A_{12}$ and $A'_1 = A'_{11} \parallel A_{12}$ and $A_{11} \xrightarrow{\texttt{true},\tau} A'_{11}$, for some A_{11}, A'_{11}, A_{12}.
Recall $A'_1 \xrightarrow{\xi,\alpha} A''$. Then, $A'_{11} \parallel A_{12} \xrightarrow{\xi,\alpha} A''$. Then, by Defn. 31:
- **Case:** [→1-Par1], such that $A'' = A''_{11} \parallel A_{12}$ and $A'_{11} \xrightarrow{\xi,\alpha} A''_{11}$, for some A''_{11}.
  * Recall $A_{11} \xrightarrow{\texttt{true},\tau} A'_{11} \xrightarrow{\xi,\alpha} A''_{11}$. Then, by induction, $A_{11} \xrightarrow{\xi,\alpha} A'_{21} \xrightarrow{\texttt{true},\tau} A''_{11}$, for some A'_{21}.
  * Recall $A_{11} \xrightarrow{\xi,\alpha} A'_{21}$. Then, by [→1-Par1], $A_{11} \parallel A_{12} \xrightarrow{\xi,\alpha} A'_{21} \parallel A_{12}$. Then, $A \xrightarrow{\xi,\alpha} A'_{21} \parallel A_{12}$. Then, $A'_2 = A'_{21} \parallel A_{12}$ and $A \xrightarrow{\xi,\alpha} A'_2$, for some A'_2.
  * Recall $A'_{21} \xrightarrow{\texttt{true},\tau} A''_{11}$. Then, by [→1-Par1], $A'_{21} \parallel A_{12} \xrightarrow{\texttt{true},\tau} A''_{11} \parallel A_{12}$. Then, $A'_2 \xrightarrow{\texttt{true},\tau} A''$.
- **Case:** [→1-Par2], such that $A'' = A'_{11} \parallel A'_{12}$ and $A_{12} \xrightarrow{\xi,\alpha} A'_{12}$, for some A'_{12}.
  * Recall $A_{12} \xrightarrow{\xi,\alpha} A'_{12}$. Then, by [→1-Par2], $A_{11} \parallel A_{12} \xrightarrow{\xi,\alpha} A_{11} \parallel A'_{12}$. Then, $A \xrightarrow{\xi,\alpha} A_{11} \parallel A'_{12}$. Then, $A'_2 = A_{11} \parallel A'_{12}$ and $A \xrightarrow{\xi,\alpha} A'_2$, for some A'_2.
  * Recall $A_{11} \xrightarrow{\texttt{true},\tau} A'_{11}$. Then, by [→1-Par1], $A_{11} \parallel A'_{12} \xrightarrow{\texttt{true},\tau} A'_{11} \parallel A'_{12}$. Then, $A'_2 \xrightarrow{\texttt{true},\tau} A''$.

– **Step:** [→1-Par2]. Similar to case [→1-Par1].

– **Step:** [→1-NIf4], such that $A = \textbf{if } \hat{\xi}|_n \, A_{11}|_{R_1} \, A_{12}|_{R_2}$ and $A'_1 = \textbf{if } \hat{\xi}|_n \, A'_{11}|_{R_1} \, A_{12}|_{R_2}$ and $A_{11} \xrightarrow{\texttt{true},\tau} A'_{11}$ and $\mathsf{subj}(\texttt{true}) \cup \mathsf{subj}(\tau) \subseteq R_1 \setminus R_2$, for some $R_1, R_2, A_{11}, A'_{11}, A_{12}, \hat{\xi}, n$.
Recall $A'_1 \xrightarrow{\xi,\alpha} A''$. Then, $\textbf{if } \hat{\xi}|_n \, A'_{11}|_{R_1} \, A_{12}|_{R_2} \xrightarrow{\xi,\alpha} A''$. Then, by Defn. 31:
- **Case:** [→1-NIf1], such that $A'' = \textbf{if } \hat{\xi}|_{n-1} \, A'_{11}|_{R_1} \, A_{12}|_{R_2}$ and $\xi = \texttt{true}$ and $\alpha = \tau$ and $n > 0$.

  * Recall $n > 0$. Then, by [$\rightarrow$1-NIf1], $\mathbf{if}\ \hat{\xi}|_n\ A_{11}|_{R_1}\ A_{12}|_{R_2} \xrightarrow{\mathtt{true},\tau} \mathbf{if}\ \hat{\xi}|_{n-1}\ A_{11}|_{R_1}\ A_{12}|_{R_2}$. Then, $A \xrightarrow{\xi,\alpha} \mathbf{if}\ \hat{\xi}|_{n-1}\ A_{11}|_{R_1}\ A_{12}|_{R_2}$. Then, $A'_2 = \mathbf{if}\ \hat{\xi}|_{n-1}\ A_{11}|_{R_1}\ A_{12}|_{R_2}$ and $A \xrightarrow{\xi,\alpha} A'_2$, for some A'_2.
  * Recall $A_{11} \xrightarrow{\mathtt{true},\tau} A'_{11}$ and $\mathsf{subj}(\mathtt{true}) \cup \mathsf{subj}(\tau) \subseteq R_1 \setminus R_2$. Then, by [$\rightarrow$1-NIf4], $\mathbf{if}\ \hat{\xi}|_{n-1}\ A_{11}|_{R_1}\ A_{12}|_{R_2} \xrightarrow{\mathtt{true},\tau} \mathbf{if}\ \hat{\xi}|_{n-1}\ A'_{11}|_{R_1}\ A_{12}|_{R_2}$. Then, $A'_2 \xrightarrow{\mathtt{true},\tau} A''$.
- **Case:** [$\rightarrow$1-NIf2], $A'' = \mathbf{if}\ \hat{\xi}|_n\ A'_{11}|_{R_1\cup\{r\}}\ A_{12}|_{R_2}$ and $\xi = \hat{\xi}^+ \restriction r$ and $\alpha = 1\{^r_r\}$ and $r \in \mathsf{subj}(\hat{\xi}) \setminus (R_1 \cup R_2)$, for some r.
  * Recall $r \in \mathsf{subj}(\hat{\xi}) \setminus (R_1 \cup R_2)$. Then, by [$\rightarrow$1-NIf2], $\mathbf{if}\ \hat{\xi}|_n\ A_{11}|_{R_1}\ A_{12}|_{R_2} \xrightarrow{\hat{\xi}^+\restriction r, 1\{^r_r\}} \mathbf{if}\ \hat{\xi}|_n\ A_{11}|_{R_1\cup\{r\}}\ A_{12}|_{R_2}$. Then, $A \xrightarrow{\xi,\alpha} \mathbf{if}\ \hat{\xi}|_n\ A_{11}|_{R_1\cup\{r\}}\ A_{12}|_{R_2}$. Then $A'_2 = \mathbf{if}\ \hat{\xi}|_n\ A_{11}|_{R_1\cup\{r\}}\ A_{12}|_{R_2}$ and $A \xrightarrow{\xi,\alpha} A'_2$, for some A'_2.
  * Recall $\emptyset \cup \emptyset \subseteq (R_1 \cup \{r\}) \setminus R_2$. Then, by Lem. 4, $\mathsf{subj}(\mathtt{true}) \cup \emptyset \subseteq (R_1 \cup \{r\}) \setminus R_2$. Then, by Defn. 16, $\mathsf{subj}(\mathtt{true}) \cup \mathsf{subj}(\tau) \subseteq (R_1 \cup \{r\}) \setminus R_2$.
  * Recall $A_{11} \xrightarrow{\mathtt{true},\tau} A'_{11}$ and $\mathsf{subj}(\mathtt{true}) \cup \mathsf{subj}(\tau) \subseteq (R_1 \cup \{r\}) \setminus R_2$. Then, by [$\rightarrow$1-NIf4], $\mathbf{if}\ \hat{\xi}|_n\ A_{11}|_{R_1\cup\{r\}}\ A_{12}|_{R_2} \xrightarrow{\mathtt{true},\tau} \mathbf{if}\ \hat{\xi}|_n\ A'_{11}|_{R_1\cup\{r\}}\ A_{12}|_{R_2}$. Then, $A'_2 \xrightarrow{\mathtt{true},\tau} A''$.
- **Case:** [$\rightarrow$1-NIf3]. Similar to case [$\rightarrow$1-NIf2].
- **Case:** [$\rightarrow$1-NIf4], such that $A'' = \mathbf{if}\ \hat{\xi}|_n\ A''_{11}|_{R_1}\ A_{12}|_{R_2}$ and $A'_{11} \xrightarrow{\xi,\alpha} A''_{11}$ and $\mathsf{subj}(\xi) \cup \mathsf{subj}(\alpha) \subseteq R_1 \setminus R_2$, for some A''_{11}.
  * Recall $A_{11} \xrightarrow{\mathtt{true},\tau} A'_{11} \xrightarrow{\xi,\alpha} A''_{11}$. Then, by induction, $A_{11} \xrightarrow{\xi,\alpha} A'_{21} \xrightarrow{\mathtt{true},\tau} A''_{11}$, for some A'_{21}.
  * Recall $A_{11} \xrightarrow{\xi,\alpha} A'_{21}$ and $\mathsf{subj}(\xi) \cup \mathsf{subj}(\alpha) \subseteq R_1 \setminus R_2$. Then, by [$\rightarrow$1-NIf4], $\mathbf{if}\ \hat{\xi}|_n\ A_{11}|_{R_1}\ A_{12}|_{R_2} \xrightarrow{\xi,\alpha} \mathbf{if}\ \hat{\xi}|_n\ A'_{21}|_{R_1}\ A_{12}|_{R_2}$. Then, $A \xrightarrow{\xi,\alpha} \mathbf{if}\ \hat{\xi}|_n\ A'_{21}|_{R_1}\ A_{12}|_{R_2}$. Then, $A'_2 = \mathbf{if}\ \hat{\xi}|_n\ A'_{21}|_{R_1}\ A_{12}|_{R_2}$ and $A \xrightarrow{\xi,\alpha} A'_2$, for some A'_2.
  * Recall $A'_{21} \xrightarrow{\mathtt{true},\tau} A''_{11}$ and $\mathsf{subj}(\mathtt{true}) \cup \mathsf{subj}(\tau) \subseteq R_1 \setminus R_2$. Then, by [$\rightarrow$1-NIf4], $\mathbf{if}\ \hat{\xi}|_n\ A'_{21}|_{R_1}\ A_{12}|_{R_2} \xrightarrow{\mathtt{true},\tau} \mathbf{if}\ \hat{\xi}|_n\ A''_{11}|_{R_1}\ A_{12}|_{R_2}$. Then, $A'_2 \xrightarrow{\mathtt{true},\tau} A''$.
- **Case:** [$\rightarrow$1-NIf5], such that $A'' = \mathbf{if}\ \hat{\xi}|_n\ A'_{11}|_{R_1}\ A'_{12}|_{R_2}$ and $A_{12} \xrightarrow{\xi,\alpha} A'_{12}$ and $\mathsf{subj}(\xi) \cup \mathsf{subj}(\alpha) \subseteq R_2 \setminus R_1$, for some A'_{12}.
  * Recall $A_{12} \xrightarrow{\xi,\alpha} A'_{12}$ and $\mathsf{subj}(\xi) \cup \mathsf{subj}(\alpha) \subseteq R_2 \setminus R_1$. Then, by [$\rightarrow$1-NIf5], $\mathbf{if}\ \hat{\xi}|_n\ A_{11}|_{R_1}\ A_{12}|_{R_2} \xrightarrow{\xi,\alpha} \mathbf{if}\ \hat{\xi}|_n\ A_{11}|_{R_1}\ A'_{12}|_{R_2}$. Then, $A \xrightarrow{\xi,\alpha} \mathbf{if}\ \hat{\xi}|_n\ A_{11}|_{R_1}\ A'_{12}|_{R_2}$. Then, $A'_2 = \mathbf{if}\ \hat{\xi}|_n\ A_{11}|_{R_1}\ A'_{12}|_{R_2}$ and $A \xrightarrow{\xi,\alpha} A'_2$, for some A'_2.
  * Recall $A_{11} \xrightarrow{\mathtt{true},\tau} A'_{11}$ and $\mathsf{subj}(\mathtt{true}) \cup \mathsf{subj}(\tau) \subseteq R_1 \setminus R_2$. Then, by [$\rightarrow$1-NIf4], $\mathbf{if}\ \hat{\xi}|_n\ A_{11}|_{R_1}\ A'_{12}|_{R_2} \xrightarrow{\mathtt{true},\tau} \mathbf{if}\ \hat{\xi}|_n\ A'_{11}|_{R_1}\ A'_{12}|_{R_2}$. Then, $A'_2 \xrightarrow{\mathtt{true},\tau} A''$.

– **Step:** [$\rightarrow$1-NIf5]. Similar to case [$\rightarrow$1-NIf4].

– **Step:** [$\rightarrow$1-NWhile], such that $A = \mathbf{while}\ \hat{\xi}|_n\ \{\psi\}\ \hat{A}|_\emptyset$ and $\mathbf{if}\ \hat{\xi}|_n\ (\hat{A}\,;\mathbf{while}\ \hat{\xi}|_n\ \{\psi\}\ \hat{A}|_\emptyset)|_\emptyset\ \mathbf{skip}|_\emptyset \xrightarrow{\mathtt{true},\tau} A'_1$, for some $\hat{A}, \psi, \hat{\xi}, n$.
- Recall $\mathbf{if}\ \hat{\xi}|_n\ (\hat{A}\,;\mathbf{while}\ \hat{\xi}|_n\ \{\psi\}\ \hat{A}|_\emptyset)|_\emptyset\ \mathbf{skip}|_\emptyset \xrightarrow{\mathtt{true},\tau} A'_1 \xrightarrow{\xi,\alpha} A''$. Then, by induction, $\mathbf{if}\ \hat{\xi}|_n\ (\hat{A}\,;\mathbf{while}\ \hat{\xi}|_n\ \{\psi\}\ \hat{A}|_\emptyset)|_\emptyset\ \mathbf{skip}|_\emptyset \xrightarrow{\xi,\alpha} A'_2$ and $A'_2 \xrightarrow{\mathtt{true},\tau} A''$, for some A'_2.
- Recall $\mathbf{if}\ \hat{\xi}|_n\ (\hat{A}\,;\mathbf{while}\ \hat{\xi}|_n\ \{\psi\}\ \hat{A}|_\emptyset)|_\emptyset\ \mathbf{skip}|_\emptyset \xrightarrow{\xi,\alpha} A'_2$. Then, by [$\rightarrow$1-NWhile], $\mathbf{while}\ \hat{\xi}|_n\ \{\psi\}\ \hat{A}|_\emptyset \xrightarrow{\xi,\alpha} A'_2$. Then, $A \xrightarrow{\xi,\alpha} A'_2$. □

Proof (of Lem. 57). Recall $\checkmark_R(G)$. Then, by Defn. 27:

– **Base:** [✓1-Act1], such that $G = q.y{:=}e$, for some q, y, e.
By [$\rightarrow$1-Act], $q.y{:=}e \xrightarrow{\mathtt{true}, q.y:=e} \mathbf{skip}$. Then, $G' = \mathbf{skip}$ and $\xi = \mathtt{true}$ and $\gamma = q.y{:=}e$ and $G \xrightarrow{\xi,\gamma} G'$, for some G', ξ, γ.

– **Base:** [✓1-Act2]. Similar to case [✓1-Act1].

– **Base:** [✓1-Skip], such that $G = \mathbf{skip}$.
By [↓1-Skip], $\mathbf{skip}\downarrow$. Then, $G\downarrow$.

– **Step:** [✓1-Seq], such that $G = G_1\,; G_2$ and $\checkmark_R(G_1)$ and $\checkmark_R(G_2)$, for some G_1, G_2.
Recall $\checkmark_R(G_1)$ and $\checkmark_R(G_2)$. Then, by induction:
- **Case:** $G_1\downarrow$ and $G_2\downarrow$. Then, by [↓1-Seq], $G_1\,; G_2\downarrow$. Then, $G\downarrow$.

- **Case:** $G_1\downarrow$ and $G_2 \xrightarrow{\xi,\gamma} G_2'$, for some ξ,γ, for some G_2'.
  - ∗ Recall $G_1\downarrow$ and $\checkmark_R(G_1)$. Then, by Lem. 48, $\mathsf{subj}(G_1)=\emptyset$.
  - ∗ Recall $\emptyset\cap(\mathsf{subj}(\xi)\cup\mathsf{subj}(\gamma))=\emptyset$. Then, $\mathsf{subj}(G_1)\cap(\mathsf{subj}(\xi)\cup\mathsf{subj}(\gamma))=\emptyset$.
  - ∗ Recall $\mathsf{subj}(G_1)\cap(\mathsf{subj}(\xi)\cup\mathsf{subj}(\gamma))=\emptyset$ and $G_2 \xrightarrow{\xi,\gamma} G_2'$. Then, by [$\rightarrow$1-Seq2], $G_1\,;G_2 \xrightarrow{\xi,\gamma} G_1\,;G_2'$. Then, $G \xrightarrow{\xi,\gamma} G_1\,;G_2'$. Then, $G'=G_1\,;G_2'$ and $G \xrightarrow{\xi,\gamma} G'$, for some G'.
- **Case:** $G_1 \xrightarrow{\xi,\gamma} G_1'$.
  Recall $G_1 \xrightarrow{\xi,\gamma} G_1'$. Then, by [$\rightarrow$1-Seq1], $G_1\,;G_2 \xrightarrow{\xi,\gamma} G_1'\,;G_2$. Then, $G \xrightarrow{\xi,\gamma} G_1'\,;G_2$. Then, $G'=G_1'\,;G_2$ and $G \xrightarrow{\xi,\gamma} G'$, for some G'.

– **Step:** [$\checkmark$1-Par], such that $G=G_1\parallel G_2$ and $\checkmark_R(G_1)$ and $\checkmark_R(G_2)$, for some G_1,G_2.
Recall $\checkmark_R(G_1)$ and $\checkmark_R(G_2)$. Then, by induction:

- **Case:** $G_1\downarrow$ and $G_2\downarrow$. Then, by [$\downarrow$1-Par], $G_1\parallel G_2\downarrow$. Then, $G\downarrow$.
- **Case:** $G_1\downarrow$ and $G_2 \xrightarrow{\xi,\gamma} G_2'$, for some ξ,γ, for some G_2'.
  Recall $G_2 \xrightarrow{\xi,\gamma} G_2'$. Then, by [$\rightarrow$1-Par2], $G_1\parallel G_2 \xrightarrow{\xi,\gamma} G_1\parallel G_2'$. Then, $G \xrightarrow{\xi,\gamma} G_1\parallel G_2'$. Then, $G'=G_1\parallel G_2'$ and $G \xrightarrow{\xi,\gamma} G'$, for some G'.
- **Case:** $G_1 \xrightarrow{\xi,\gamma} G_1'$.
  Recall $G_1 \xrightarrow{\xi,\gamma} G_1'$. Then, by [$\rightarrow$1-Par1], $G_1\parallel G_2 \xrightarrow{\xi,\gamma} G_1'\parallel G_2$. Then, $G \xrightarrow{\xi,\gamma} G_1'\parallel G_2$. Then, $G'=G_1'\parallel G_2$ and $G \xrightarrow{\xi,\gamma} G'$, for some G'.

– **Step:** [$\checkmark$1-If], such that $G=R.\textbf{if}\ \hat\xi\ G_1\ G_2$, for some $G_1,G_2,\hat\xi$.
By [$\rightarrow$1-If1], $R.\textbf{if}\ \hat\xi\ G_1\ G_2 \xrightarrow{\hat\xi^+,1^R_{\mathsf{subj}(\hat\xi)}} G_1$. Then, $G \xrightarrow{\hat\xi^+,1^R_{\mathsf{subj}(\hat\xi)}} G_1$. Then, $G'=G_1$ and $\xi=\hat\xi^+$ and $\gamma=1^R_{\mathsf{subj}(\hat\xi)}$ and $G \xrightarrow{\xi,\gamma} G'$, for some G',ξ,γ.

– **Step:** [$\checkmark$1-While], such that $G=R.\textbf{while}\ \hat\xi\ \{\psi\}\ \hat G$, for some $\hat G,\psi,\hat\xi$.
By [$\rightarrow$1-While1], $R.\textbf{while}\ \hat\xi\ \{\psi\}\ \hat G \xrightarrow{\hat\xi^+,1^R_{\mathsf{subj}(\hat\xi)}} \hat G\,;R.\textbf{while}\ \hat\xi\ \{\psi\}\ \hat G$. Then, $G \xrightarrow{\hat\xi^+,1^R_{\mathsf{subj}(\hat\xi)}} \hat G\,;R.\textbf{while}\ \hat\xi\ \{\psi\}\ \hat G$. Then, $G'=\hat G\,;R.\textbf{while}\ \hat\xi\ \{\psi\}\ \hat G$ and $\xi=\hat\xi^+$ and $\gamma=1^R_{\mathsf{subj}(\hat\xi)}$ and $G \xrightarrow{\xi,\gamma} G'$, for some G',ξ,γ.

– **Step:** [$\checkmark$1-NIf], such that $G=\textbf{if}\ \hat\xi|_0\ G_1|_{R_1}\ G_2|_{R_2}$ and $\checkmark_R(G_1)$ and $\checkmark_R(G_2)$ and $R=\mathsf{subj}(\hat\xi)$ and $R_1,R_2\subseteq\mathsf{subj}(\xi)$, and $R_1\neq\emptyset$ implies $R_2=\emptyset$, and $R_2\neq\emptyset$ implies $R_1=\emptyset$, for some $R_1,R_2,G_1,G_2,\hat\xi$.
Recall:

- **Case:** $R_1\cup R_2=\mathsf{subj}(\hat\xi)$.
  Recall $\checkmark_R(G_1)$ and $\checkmark_R(G_2)$. Then, by induction:
  - ∗ **Case:** $G_1\downarrow$ and $G_2\downarrow$.
    - · Recall $G_1\downarrow$. Then, $R_1\neq\emptyset$ implies $G_1\downarrow$.
    - · Recall $G_2\downarrow$. Then, $R_2\neq\emptyset$ implies $G_2\downarrow$.
    - · Recall $R_1\cup R_2=\mathsf{subj}(\hat\xi)$, and $R_1\neq\emptyset$ implies $G_1\downarrow$, and $R_2\neq\emptyset$ implies $G_2\downarrow$. Then, by [$\downarrow$1-NIf], $\textbf{if}\ \hat\xi|_0\ G_1|_{R_1}\ G_2|_{R_2}\downarrow$. Then, $G\downarrow$.
  - ∗ **Case:** $G_1 \xrightarrow{\xi,\gamma} G_1'$, for some G_1', for some ξ,γ.
    - · Recall $G_1 \xrightarrow{\xi,\gamma} G_1'$. Then, by Lem. 53, $\mathsf{subj}(\xi)\cup\mathsf{subj}(\gamma)\cup\mathsf{subj}(G_1')\subseteq\mathsf{subj}(G_1)$. Then, $\mathsf{subj}(\xi)\cup\mathsf{subj}(\gamma)\subseteq\mathsf{subj}(G_1)$.
    - · Recall $\checkmark_R(G_1)$. Then, by Lem. 47, $\mathsf{subj}(G_1)\subseteq R$.
    - · Recall $R_1\neq\emptyset$ implies $R_2=\emptyset$. Then, $R_1\setminus R_2=R_1$.
    - · Recall $\mathsf{subj}(\xi)\cup\mathsf{subj}(\gamma)\subseteq\mathsf{subj}(G_1)\subseteq R$. Then, $\mathsf{subj}(\xi)\cup\mathsf{subj}(\gamma)\subseteq R$. Then, $\mathsf{subj}(\xi)\cup\mathsf{subj}(\gamma)\subseteq\mathsf{subj}(\hat\xi)$. Then, $\mathsf{subj}(\xi)\cup\mathsf{subj}(\gamma)\subseteq R_1$. Then, $\mathsf{subj}(\xi)\cup\mathsf{subj}(\gamma)\subseteq R_1\setminus R_2$.
    - · Recall $G_1 \xrightarrow{\xi,\gamma} G_1'$ and $\mathsf{subj}(\xi)\cup\mathsf{subj}(\gamma)\subseteq R_1\setminus R_2$. Then, by [$\rightarrow$1-NIf4], $\textbf{if}\ \hat\xi|_0\ G_1|_{R_1}\ G_2|_{R_2} \xrightarrow{\xi,\gamma} \textbf{if}\ \hat\xi|_0\ G_1'|_{R_1}\ G_2|_{R_2}$. Then, $G \xrightarrow{\xi,\gamma} \textbf{if}\ \hat\xi|_0\ G_1'|_{R_1}\ G_2|_{R_2}$. Then, $G'=\textbf{if}\ \hat\xi|_0\ G_1'|_{R_1}\ G_2|_{R_2}$ and $G \xrightarrow{\xi,\gamma} G'$, for some G'.
  - ∗ **Case:** $G_2 \xrightarrow{\xi,\gamma} G_2'$, for some G_2',ξ,γ. Similar to case $G_1 \xrightarrow{\xi,\gamma} G_1'$, for some G_1',ξ,γ.
- **Case:** $R_1\cup R_2\neq\mathsf{subj}(\hat\xi)$.
  - ∗ Recall $R_1,R_2\subseteq\mathsf{subj}(\hat\xi)$. Then, $R_1\cup R_2\subseteq\mathsf{subj}(\hat\xi)$.
  - ∗ Recall $R_1\cup R_2\subseteq\mathsf{subj}(\hat\xi)$ and $R_1\cup R_2\neq\mathsf{subj}(\hat\xi)$. Then, $\mathsf{subj}(\hat\xi)\setminus(R_1\cup R_2)\neq\emptyset$. Then, $r\in\mathsf{subj}(\hat\xi)\setminus(R_1\cup R_2)$, for some r. Then, by [$\rightarrow$1-NIf3], $\textbf{if}\ \hat\xi|_0\ G_1|_{R_1}\ G_2|_{R_2} \xrightarrow{\hat\xi^-\restriction r,2^{\{r\}}_{\{r\}}} \textbf{if}\ \hat\xi|_0\ G_1|_{R_1}\ G_2|_{R_2\cup\{r\}}$. Then, $G \xrightarrow{\hat\xi^-\restriction r,2^{\{r\}}_{\{r\}}} \textbf{if}\ \hat\xi|_0\ G_1|_{R_1}\ G_2|_{R_2\cup\{r\}}$. Then, $G'=\textbf{if}\ \hat\xi|_0\ G_1|_{R_1}\ G_2|_{R_2\cup\{r\}}$ and $\xi=\hat\xi^-\restriction r$ and $\gamma=2^{\{r\}}_{\{r\}}$ and $G \xrightarrow{\xi,\gamma} G'$, for some G',ξ,γ.

– **Step:** [✓1-NWhile], such that $G = \textbf{while}\ \hat{\xi}|_0\ \{\psi\}\ \hat{G}|_\emptyset$ and $R = \mathsf{subj}(\hat{\xi})$, for some $\hat{G}, \psi, \hat{\xi}$. Recall $\checkmark_R(G)$. Then, by Lem. 47, $R \neq \emptyset$. Then, $\mathsf{subj}(\hat{\xi}) \neq \emptyset$. Then, $r \in \mathsf{subj}(\hat{\xi})$, for some r. Then, $r \in \mathsf{subj}(\hat{\xi}) \setminus (\emptyset \cup \emptyset)$. Then, by [→1-NIf2], $\textbf{if}\ \hat{\xi}|_0\ (\hat{G}\,;\textbf{while}\ \hat{\xi}|_0\ \{\psi\}\ \hat{G}|_\emptyset)|_\emptyset\ \textbf{skip}|_\emptyset \xrightarrow{\hat{\xi}^+\restriction r, 1\{^r_r\}} \textbf{if}\ \hat{\xi}|_0\ (\hat{G}\,;\textbf{while}\ \hat{\xi}|_0\ \{\psi\}\ \hat{G}|_\emptyset)|_{\emptyset\cup\{r\}}\ \textbf{skip}|_\emptyset$. Then, by [→1-NWhile], $\textbf{while}\ \hat{\xi}|_0\ \{\psi\}\ \hat{G}|_\emptyset \xrightarrow{\hat{\xi}^+\restriction r, 1\{^r_r\}} \textbf{if}\ \hat{\xi}|_0\ (\hat{G}\,;\textbf{while}\ \hat{\xi}|_0\ \{\psi\}\ \hat{G}|_\emptyset)|_{\emptyset\cup\{r\}}\ \textbf{skip}|_\emptyset$. Then, $G \xrightarrow{\hat{\xi}^+\restriction r, 1\{^r_r\}} \textbf{if}\ \hat{\xi}|_0\ (\hat{G}\,;\textbf{while}\ \hat{\xi}|_0\ \{\psi\}\ \hat{G}|_\emptyset)|_{\emptyset\cup\{r\}}\ \textbf{skip}|_\emptyset$. Then, $G' = \textbf{if}\ \hat{\xi}|_0\ (\hat{G}\,;\textbf{while}\ \hat{\xi}|_0\ \{\psi\}\ \hat{G}|_\emptyset)|_{\emptyset\cup\{r\}}\ \textbf{skip}|_\emptyset$ and $\xi = \hat{\xi}^+ \restriction r$ and $\gamma = 1\{^r_r\}$ and $G \xrightarrow{\xi,\gamma} G'$, for some G', ξ, γ. □

Proof (of Lem. 58).

1. Recall $G \xrightarrow{\xi,\gamma} G'$. Then, by Defn. 31:
   – **Base:** [→1-Act], such that $G = \gamma$.
   Recall $\checkmark_R(G)$. Then, $\checkmark_R(\gamma)$. Then, by Defn. 27:
     • **Case:** $\gamma = q.y{:=}e$, for some q, y, e.
     Recall $\{q\} \neq \emptyset$. Then, by Defn. 16, $\mathsf{subj}(q.y{:=}e) \neq \emptyset$. Then, $\mathsf{subj}(\gamma) \neq \emptyset$.
     • **Case:** $\gamma = p.e \twoheadrightarrow q.y$, for some p, q, y, e. Similar to case "$\gamma = q.y{:=}e$, for some q, y, e".
   – **Base:** [→1-If1], such that $G = \hat{R}.\textbf{if}\ \hat{\xi}\ G_1\ G_2$ and $\gamma = 1^{\hat{R}}_{\mathsf{subj}(\hat{\xi})}$, for some $\hat{R}, G_1, G_2, \hat{\xi}$.
     • Recall $\checkmark_R(G)$. Then, $\checkmark_R(\hat{R}.\textbf{if}\ \hat{\xi}\ G_1\ G_2)$. Then, by Defn. 27, $R = \mathsf{subj}(\hat{\xi})$.
     • Recall $\checkmark_R(G)$. Then, by Lem. 47, $R \neq \emptyset$. Then, $\mathsf{subj}(\hat{\xi}) \neq \emptyset$. Then, by Defn. 16, $\mathsf{subj}(1^{\hat{R}}_{\mathsf{subj}(\hat{\xi})}) \neq \emptyset$. Then, $\mathsf{subj}(\gamma) \neq \emptyset$.
   – **Base:** [→1-If2]. Similar to case [→1-If1].
   – **Base:** [→1-While1]. Similar to case [→1-If1].
   – **Base:** [→1-While2]. Similar to case [→1-If1].
   – **Base:** [→1-NIf1], such that $0 > 0$. Then, **false**.
   – **Base:** [→1-NIf2], such that $\gamma = 1\{^r_r\}$, for some r.
   Recall $\{r\} \neq \emptyset$. Then, by Defn. 16, $\mathsf{subj}(1\{^r_r\}) \neq \emptyset$. Then, $\mathsf{subj}(\gamma) \neq \emptyset$.
   – **Base:** [→1-NIf3]. Similar to case [→1-NIf2].
   – **Step:** [→1-Seq1], such that $G = G_1\,; G_2$ and $G_1 \xrightarrow{\xi,\gamma} G_1'$, for some G_1, G_1', G_2.
     • Recall $\checkmark_R(G)$. Then, $\checkmark_R(G_1\,; G_2)$. Then, by Defn. 27, $\checkmark_R(G_1)$.
     • Recall $G_1 \xrightarrow{\xi,\gamma} G_1'$ and $\checkmark_R(G_1)$. Then, by induction, $\mathsf{subj}(\gamma) \neq \emptyset$.
   – **Step:** [→1-Seq2]. Similar to case [→1-Seq1].
   – **Step:** [→1-Par1]. Similar to case [→1-Seq1].
   – **Step:** [→1-Par2]. Similar to case [→1-Seq1].
   – **Step:** [→1-NIf4]. Similar to case [→1-Seq1].
   – **Step:** [→1-NIf5]. Similar to case [→1-Seq1].
   – **Step:** [→1-NWhile], such that $G = \textbf{while}\ \hat{\xi}|_0\ \{\psi\}\ \hat{G}|_\emptyset$ and $\textbf{if}\ \hat{\xi}|_0\ (\hat{G}\,;\textbf{while}\ \hat{\xi}|_0\ \{\psi\}\ \hat{G}|_\emptyset)|_\emptyset\ \textbf{skip}|_\emptyset \xrightarrow{\xi,\gamma} G'$, for some $\hat{G}, \psi, \hat{\xi}$.
     • Recall $\checkmark_R(G)$. Then, $\checkmark_R(\textbf{while}\ \hat{\xi}|_0\ \{\psi\}\ \hat{G}|_\emptyset)$. Then, by Lem. 46, $\checkmark_R(\textbf{if}\ \hat{\xi}|_0\ (\hat{G}\,;\textbf{while}\ \hat{\xi}|_0\ \{\psi\}\ \hat{G}|_\emptyset)|_\emptyset\ \textbf{skip}|_\emptyset)$.
     • Recall $\textbf{if}\ \hat{\xi}|_0\ (\hat{G}\,;\textbf{while}\ \hat{\xi}|_0\ \{\psi\}\ \hat{G}|_\emptyset)|_\emptyset\ \textbf{skip}|_\emptyset \xrightarrow{\xi,\gamma} G'$ and $\checkmark_R(\textbf{if}\ \hat{\xi}|_0\ (\hat{G}\,;\textbf{while}\ \hat{\xi}|_0\ \{\psi\}\ \hat{G}|_\emptyset)|_\emptyset\ \textbf{skip}|_\emptyset)$. Then, by induction, $\mathsf{subj}(\gamma) \neq \emptyset$. □
2. Recall $G \xrightarrow{\xi,\gamma} G'$ and $\checkmark_R(G)$. Then, by 1, $\mathsf{subj}(\gamma) \neq \emptyset$. Then, by Defn. 16, $\gamma \neq \tau$.

Proof (of Lem. 59).

1. Recall $G \xrightarrow{\xi,\gamma} G'$. Then, by Defn. 31:
   – **Base:** [→1-Act], such that $G = \gamma$ and $G' = \textbf{skip}$ and $\xi = \mathtt{true}$.
   By [→1-Act], $(\gamma \restriction r) \xrightarrow{\mathtt{true}, \gamma\restriction r} \textbf{skip}$. Then, by Defn. 11, $(\gamma \restriction r) \xrightarrow{\mathtt{true}\restriction r, \gamma\restriction r} \textbf{skip}$. Then by Defn. 26, $(\gamma \restriction r) \xrightarrow{\mathtt{true}\restriction r, \gamma\restriction r} (\textbf{skip} \restriction r)$. Then, $(G \restriction r) \xrightarrow{\xi\restriction r, \gamma\restriction r} (G' \restriction r)$.
   – **Base:** [→1-If1], such that $G = \hat{R}.\textbf{if}\ \hat{\xi}\ G_1\ G_2$ and $G' = G_1$ and $\xi = \hat{\xi}^+$ and $\gamma = 1^{\hat{R}}_{\mathsf{subj}(\hat{\xi})}$, for some $\hat{R}, G_1, G_2, \hat{\xi}$.
     • Recall $\checkmark_R(G)$. Then, $\checkmark_R(\hat{R}.\textbf{if}\ \hat{\xi}\ G_1\ G_2)$. Then, by Defn. 27, $R = \mathsf{subj}(\hat{\xi})$.

- Recall $r \in R \cap \mathsf{subj}(\gamma)$. Then, $r \in R$. Then, $r \in \mathsf{subj}(\hat{\xi})$.
- Recall $r \in \mathsf{subj}(\hat{\xi})$. Then, by Lem. 9, $\mathsf{subj}(\hat{\xi} \upharpoonright r) = \{r\}$.
- Recall $r \in \mathsf{subj}(\hat{\xi})$. Then, by Defn. 18, $1^{\hat{R}}_{\mathsf{subj}(\hat{\xi})} \upharpoonright r = 1^{\hat{R}}_{\{r\}}$.
- By [$\to$1-If1], $\hat{R}.\textbf{if}\ (\hat{\xi} \upharpoonright r)\ (G_1 \upharpoonright r)\ (G_2 \upharpoonright r) \xrightarrow{(\hat{\xi}\upharpoonright r)^+, 1^{\hat{R}}_{\mathsf{subj}(\hat{\xi}\upharpoonright r)}} (G_1 \upharpoonright r)$. Then, by Defn. 26, $(\hat{R}.\textbf{if}\ \hat{\xi}\ G_1\ G_2 \upharpoonright r) \xrightarrow{(\hat{\xi}\upharpoonright r)^+, 1^{\hat{R}}_{\mathsf{subj}(\hat{\xi}\upharpoonright r)}} (G_1 \upharpoonright r)$. Then, by Lem. 11, $(\hat{R}.\textbf{if}\ \hat{\xi}\ G_1\ G_2 \upharpoonright r) \xrightarrow{\hat{\xi}^+\upharpoonright r, 1^{\hat{R}}_{\mathsf{subj}(\hat{\xi}\upharpoonright r)}} (G_1 \upharpoonright r)$. Then, $(\hat{R}.\textbf{if}\ \hat{\xi}\ G_1\ G_2 \upharpoonright r) \xrightarrow{\hat{\xi}^+\upharpoonright r, 1^{\hat{R}}_{\{r\}}} (G_1 \upharpoonright r)$. Then, $(\hat{R}.\textbf{if}\ \hat{\xi}\ G_1\ G_2 \upharpoonright r) \xrightarrow{\hat{\xi}^+\upharpoonright r, 1^{\hat{R}}_{\mathsf{subj}(\hat{\xi})}\upharpoonright r} (G_1 \upharpoonright r)$. Then, $(G \upharpoonright r) \xrightarrow{\xi\upharpoonright r, \gamma\upharpoonright r} (G' \upharpoonright r)$.

– **Base:** [$\to$1-If2]. Similar to case [$\to$1-If1].

– **Base:** [$\to$1-While1], such that $G = \hat{R}.\textbf{while}\ \hat{\xi}\ \{\psi\}\ \hat{G}$ and $G' = \hat{G}\ ;\ \hat{R}.\textbf{while}\ \hat{\xi}\ \{\psi\}\ \hat{G}$ and $\xi = \hat{\xi}^+$ and $\gamma = 1^{\hat{R}}_{\mathsf{subj}(\hat{\xi})}$, for some $\hat{R}, \hat{G}, \psi, \hat{\xi}$.
- Recall $\checkmark_R(G)$. Then, $\checkmark_R(\hat{R}.\textbf{while}\ \hat{\xi}\ \{\psi\}\ \hat{G})$. Then, by Defn. 27, $R = \mathsf{subj}(\hat{\xi})$.
- Recall $r \in R \cap \mathsf{subj}(\gamma)$. Then, $r \in R$. Then, $r \in \mathsf{subj}(\hat{\xi})$.
- Recall $r \in \mathsf{subj}(\hat{\xi})$. Then, by Lem. 9, $\mathsf{subj}(\hat{\xi} \upharpoonright r) = \{r\}$.
- Recall $r \in \mathsf{subj}(\hat{\xi})$. Then, by Defn. 18, $1^{\hat{R}}_{\mathsf{subj}(\hat{\xi})} \upharpoonright r = 1^{\hat{R}}_{\{r\}}$.
- By [$\to$1-While1], $\hat{R}.\textbf{while}\ (\hat{\xi} \upharpoonright r)\ \{\mathtt{true}\}\ (\hat{G} \upharpoonright r) \xrightarrow{(\hat{\xi}\upharpoonright r)^+, 1^{\hat{R}}_{\mathsf{subj}(\hat{\xi}\upharpoonright r)}} (\hat{G} \upharpoonright r)\ ;\ \hat{R}.\textbf{while}\ (\hat{\xi} \upharpoonright r)\ \{\mathtt{true}\}\ (\hat{G} \upharpoonright r)$. Then, by Defn. 26, $(\hat{R}.\textbf{while}\ \hat{\xi}\ \{\psi\}\ \hat{G} \upharpoonright r) \xrightarrow{(\hat{\xi}\upharpoonright r)^+, 1^{\hat{R}}_{\mathsf{subj}(\hat{\xi}\upharpoonright r)}} (\hat{G} \upharpoonright r)\ ;\ (\hat{R}.\textbf{while}\ \hat{\xi}\ \{\psi\}\ \hat{G} \upharpoonright r)$. Then, by Lem. 11, $(\hat{R}.\textbf{while}\ \hat{\xi}\ \{\psi\}\ \hat{G} \upharpoonright r) \xrightarrow{\hat{\xi}^+\upharpoonright r, 1^{\hat{R}}_{\mathsf{subj}(\hat{\xi}\upharpoonright r)}} (\hat{G} \upharpoonright r)\ ;\ (\hat{R}.\textbf{while}\ \hat{\xi}\ \{\psi\}\ \hat{G} \upharpoonright r)$. Then, $(\hat{R}.\textbf{while}\ \hat{\xi}\ \{\psi\}\ \hat{G} \upharpoonright r) \xrightarrow{\hat{\xi}^+\upharpoonright r, 1^{\hat{R}}_{\{r\}}} (\hat{G} \upharpoonright r)\ ;\ (\hat{R}.\textbf{while}\ \hat{\xi}\ \{\psi\}\ \hat{G} \upharpoonright r)$. Then, $(\hat{R}.\textbf{while}\ \hat{\xi}\ \{\psi\}\ \hat{G} \upharpoonright r) \xrightarrow{\hat{\xi}^+\upharpoonright r, 1^{\hat{R}}_{\mathsf{subj}(\hat{\xi})}\upharpoonright r} (\hat{G} \upharpoonright r)\ ;\ (\hat{R}.\textbf{while}\ \hat{\xi}\ \{\psi\}\ \hat{G} \upharpoonright r)$. Then, by Defn. 26, $(\hat{R}.\textbf{while}\ \hat{\xi}\ \{\psi\}\ \hat{G} \upharpoonright r) \xrightarrow{\hat{\xi}^+\upharpoonright r, 1^{\hat{R}}_{\mathsf{subj}(\hat{\xi})}\upharpoonright r} (\hat{G}\ ;\ \hat{R}.\textbf{while}\ \hat{\xi}\ \{\psi\}\ \hat{G} \upharpoonright r)$. Then, $(G \upharpoonright r) \xrightarrow{\xi\upharpoonright r, \gamma\upharpoonright r} (G' \upharpoonright r)$.

– **Base:** [$\to$1-While2]. Similar to case [$\to$1-While1].

– **Base:** [$\to$1-NIf1], such that $0 > 0$. Then, **false**.

– **Base:** [$\to$1-NIf2], such that $G = \textbf{if}\ \hat{\xi}|_0\ G_1|_{R_1}\ G_2|_{R_2}$ and $G' = \textbf{if}\ \hat{\xi}|_0\ G_1|_{R_1 \cup \{\hat{r}\}}\ G_2|_{R_2}$ and $\xi = \hat{\xi}^+ \upharpoonright \hat{r}$ and $\gamma = 1^{\{\hat{r}\}}_{\{\hat{r}\}}$ and $\hat{r} \in \mathsf{subj}(\hat{\xi}) \setminus (R_1 \cup R_2)$, for some $R_1, R_2, G_1, G_2, \hat{r}, \hat{\xi}$.
- Recall $\hat{r} \in \mathsf{subj}(\hat{\xi}) \setminus (R_1 \cup R_2)$. Then, $\hat{r} \in \mathsf{subj}(\hat{\xi})$ and $\hat{r} \notin R_1 \cup R_2$.
- Recall $\hat{r} \in \mathsf{subj}(\hat{\xi})$. Then, by Lem. 9, $\mathsf{subj}(\hat{\xi} \upharpoonright \hat{r}) = \{\hat{r}\}$.
- Recall $\hat{r} \in \{\hat{r}\}$. Then, $\hat{r} \in \mathsf{subj}(\hat{\xi} \upharpoonright \hat{r})$.
- Recall $\hat{r} \notin R_1 \cup R_2$. Then, $\hat{r} \notin (R_1 \cup R_2) \cap \{\hat{r}\}$. Then, $\hat{r} \notin ((R_1 \cap \{\hat{r}\}) \cup (R_2 \cap \{\hat{r}\}))$.
- Recall $\hat{r} \in \{\hat{r}\}$. Then, by Defn. 18, $1^{\{\hat{r}\}}_{\{\hat{r}\}} \upharpoonright \hat{r} = 1^{\{\hat{r}\}}_{\{\hat{r}\}}$.
- Recall $r \in R \cap \mathsf{subj}(\gamma)$. Then, $r \in R \cap \mathsf{subj}(1^{\{\hat{r}\}}_{\{\hat{r}\}})$. Then, $r \in \mathsf{subj}(1^{\{\hat{r}\}}_{\{\hat{r}\}})$. Then, by Defn. 16, $r \in \{\hat{r}\}$. Then, $r = \hat{r}$.
- Recall $\hat{r} \in \mathsf{subj}(\hat{\xi} \upharpoonright \hat{r})$ and $\hat{r} \notin ((R_1 \cap \{\hat{r}\}) \cup (R_2 \cap \{\hat{r}\}))$. Then, $\hat{r} \in \mathsf{subj}(\hat{\xi} \upharpoonright \hat{r}) \setminus ((R_1 \cap \{\hat{r}\}) \cup (R_2 \cap \{\hat{r}\}))$. Then, by [$\to$1-NIf2], $\textbf{if}\ (\hat{\xi} \upharpoonright \hat{r})|_{|\mathsf{subj}(\hat{\xi})\setminus(R_1\cup R_2\cup\{\hat{r}\})|}\ (G_1 \upharpoonright \hat{r})|_{R_1\cap\{\hat{r}\}}\ (G_2 \upharpoonright \hat{r})|_{R_2\cap\{\hat{r}\}} \xrightarrow{(\hat{\xi}\upharpoonright\hat{r})^+\upharpoonright\hat{r}, 1^{\{\hat{r}\}}_{\{\hat{r}\}}}$ $\textbf{if}\ (\hat{\xi} \upharpoonright \hat{r})|_{|\mathsf{subj}(\hat{\xi})\setminus(R_1\cup R_2\cup\{\hat{r}\})|}\ (G_1 \upharpoonright \hat{r})|_{(R_1\cap\{\hat{r}\})\cup\{\hat{r}\}}\ (G_2 \upharpoonright \hat{r})|_{R_2\cap\{\hat{r}\}}$. Then, $\textbf{if}\ (\hat{\xi} \upharpoonright \hat{r})|_{|\mathsf{subj}(\hat{\xi})\setminus(R_1\cup R_2\cup\{\hat{r}\})|}\ (G_1 \upharpoonright \hat{r})|_{R_1\cap\{\hat{r}\}}\ (G_2 \upharpoonright \hat{r})|_{R_2\cap\{\hat{r}\}} \xrightarrow{(\hat{\xi}\upharpoonright\hat{r})^+\upharpoonright\hat{r}, 1^{\{\hat{r}\}}_{\{\hat{r}\}}}$ $\textbf{if}\ (\hat{\xi} \upharpoonright \hat{r})|_{|\mathsf{subj}(\hat{\xi})\setminus(R_1\cup\{\hat{r}\}\cup R_2\cup\{\hat{r}\})|}\ (G_1 \upharpoonright \hat{r})|_{(R_1\cap\{\hat{r}\})\cup\{\hat{r}\}}\ (G_2 \upharpoonright \hat{r})|_{R_2\cap\{\hat{r}\}}$. Then, $\textbf{if}\ (\hat{\xi} \upharpoonright \hat{r})|_{|\mathsf{subj}(\hat{\xi})\setminus(R_1\cup R_2\cup\{\hat{r}\})|}\ (G_1 \upharpoonright \hat{r})|_{R_1\cap\{\hat{r}\}}\ (G_2 \upharpoonright \hat{r})|_{R_2\cap\{\hat{r}\}} \xrightarrow{(\hat{\xi}\upharpoonright\hat{r})^+\upharpoonright\hat{r}, 1^{\{\hat{r}\}}_{\{\hat{r}\}}}$ $\textbf{if}\ (\hat{\xi} \upharpoonright \hat{r})|_{|\mathsf{subj}(\hat{\xi})\setminus(R_1\cup\{\hat{r}\}\cup R_2\cup\{\hat{r}\})|}\ (G_1 \upharpoonright \hat{r})|_{(R_1\cup\{\hat{r}\})\cap(\{\hat{r}\}\cup\{\hat{r}\})}\ (G_2 \upharpoonright \hat{r})|_{R_2\cap\{\hat{r}\}}$. Then, $\textbf{if}\ (\hat{\xi} \upharpoonright \hat{r})|_{|\mathsf{subj}(\hat{\xi})\setminus(R_1\cup R_2\cup\{\hat{r}\})|}\ (G_1 \upharpoonright \hat{r})|_{R_1\cap\{\hat{r}\}}\ (G_2 \upharpoonright \hat{r})|_{R_2\cap\{\hat{r}\}} \xrightarrow{(\hat{\xi}\upharpoonright\hat{r})^+\upharpoonright\hat{r}, 1^{\{\hat{r}\}}_{\{\hat{r}\}}}$ $\textbf{if}\ (\hat{\xi} \upharpoonright \hat{r})|_{|\mathsf{subj}(\hat{\xi})\setminus(R_1\cup\{\hat{r}\}\cup R_2\cup\{\hat{r}\})|}\ (G_1 \upharpoonright \hat{r})|_{(R_1\cup\{\hat{r}\})\cap\{\hat{r}\}}\ (G_2 \upharpoonright \hat{r})|_{R_2\cap\{\hat{r}\}}$. Then, by Defn. 26, $(\textbf{if}\ \hat{\xi}|_0\ G_1|_{R_1}\ G_2|_{R_2} \upharpoonright \hat{r}) \xrightarrow{(\hat{\xi}\upharpoonright\hat{r})^+\upharpoonright\hat{r}, 1^{\{\hat{r}\}}_{\{\hat{r}\}}} (\textbf{if}\ \hat{\xi}|_0\ G_1|_{R_1\cup\{\hat{r}\}}\ G_2|_{R_2} \upharpoonright \hat{r})$. Then, by Lem. 11, $(\textbf{if}\ \hat{\xi}|_0\ G_1|_{R_1}\ G_2|_{R_2} \upharpoonright \hat{r}) \xrightarrow{(\hat{\xi}^+\upharpoonright\hat{r})\upharpoonright\hat{r}, 1^{\{\hat{r}\}}_{\{\hat{r}\}}} (\textbf{if}\ \hat{\xi}|_0\ G_1|_{R_1\cup\{\hat{r}\}}\ G_2|_{R_2} \upharpoonright \hat{r})$. Then, $(\textbf{if}\ \hat{\xi}|_0\ G_1|_{R_1}\ G_2|_{R_2} \upharpoonright \hat{r}) \xrightarrow{(\hat{\xi}^+\upharpoonright\hat{r})\upharpoonright\hat{r}, 1^{\{\hat{r}\}}_{\{\hat{r}\}}\upharpoonright\hat{r}} (\textbf{if}\ \hat{\xi}|_0\ G_1|_{R_1\cup\{\hat{r}\}}\ G_2|_{R_2} \upharpoonright \hat{r})$. Then, $(G \upharpoonright r) \xrightarrow{\xi\upharpoonright r, \gamma\upharpoonright r} (G' \upharpoonright r)$.

– **Base:** [$\to$1-NIf3]. Similar to case [$\to$1-NIf2].

– **Step:** [$\to$1-Seq1], such that $G = G_1\ ;\ G_2$ and $G' = G_1'\ ;\ G_2$ and $G_1 \xrightarrow{\xi,\gamma} G_1'$, for some G_1, G_1', G_2.
- Recall $\checkmark_R(G)$. Then, $\checkmark_R(G_1\ ;\ G_2)$. Then, by Defn. 27, $\checkmark_R(G_1)$.
- Recall $G_1 \xrightarrow{\xi,\gamma} G_1'$ and $\checkmark_R(G_1)$ and $r \in R \cap \mathsf{subj}(\gamma)$. Then, by induction, $(G_1 \upharpoonright r) \xrightarrow{\xi\upharpoonright r, \gamma\upharpoonright r} (G_1' \upharpoonright r)$. Then, by [$\to$1-Seq1],

$(G_1 \restriction r) ; (G_2 \restriction r) \xrightarrow{\xi\restriction r, \gamma\restriction r} (G_1' \restriction r) ; (G_2 \restriction r)$. Then, by Defn. 26,
$(G_1 ; G_2 \restriction r) \xrightarrow{\xi\restriction r, \gamma\restriction r} (G_1' ; G_2 \restriction r)$. Then, $(G \restriction r) \xrightarrow{\xi\restriction r, \gamma\restriction r} (G' \restriction r)$.

- **Step:** [$\rightarrow$1-Seq2], such that $G = G_1 ; G_2$ and $G' = G_1 ; G_2'$ and $\mathsf{subj}(G_1) \cap (\mathsf{subj}(\xi) \cup \mathsf{subj}(\gamma)) = \emptyset$ and $G_2 \xrightarrow{\xi, \gamma} G_2'$, for some G_1, G_2, G_2'.
  - By Lem. 43, $\mathsf{subj}(G_1 \restriction r) \subseteq \mathsf{subj}(G_1)$ and $\mathsf{subj}(\xi \restriction r) \subseteq \mathsf{subj}(\xi)$ and $\mathsf{subj}(\gamma \restriction r) \subseteq \mathsf{subj}(\gamma)$.
  - Recall $\mathsf{subj}(\xi \restriction r) \subseteq \mathsf{subj}(\xi)$ and $\mathsf{subj}(\gamma \restriction r) \subseteq \mathsf{subj}(\gamma)$. Then, $\mathsf{subj}(\xi \restriction r) \cup \mathsf{subj}(\gamma \restriction r) \subseteq \mathsf{subj}(\xi) \cup \mathsf{subj}(\gamma)$.
  - Recall $\mathsf{subj}(G_1) \cap (\mathsf{subj}(\xi) \cup \mathsf{subj}(\gamma)) = \emptyset$ and $\mathsf{subj}(G_1 \restriction r) \subseteq \mathsf{subj}(G_1)$ and $\mathsf{subj}(\xi \restriction r) \cup \mathsf{subj}(\gamma \restriction r) \subseteq \mathsf{subj}(\xi) \cup \mathsf{subj}(\gamma)$. Then, $\mathsf{subj}(G_1 \restriction r) \cap (\mathsf{subj}(\xi \restriction r) \cup \mathsf{subj}(\gamma \restriction r)) = \emptyset$.
  - Recall $\checkmark_R(G)$. Then, $\checkmark_R(G_1 ; G_2)$. Then, by Defn. 27, $\checkmark_R(G_2)$.
  - Recall $G_2 \xrightarrow{\xi, \gamma} G_2'$ and $\checkmark_R(G_2)$ and $r \in R \cap \mathsf{subj}(\gamma)$. Then, by induction, $(G_2 \restriction r) \xrightarrow{\xi\restriction r, \gamma\restriction r} (G_2' \restriction r)$.
  - Recall $\mathsf{subj}(G_1 \restriction r) \cap (\mathsf{subj}(\xi \restriction r) \cup \mathsf{subj}(\gamma \restriction r)) = \emptyset$ and $(G_2 \restriction r) \xrightarrow{\xi\restriction r, \gamma\restriction r} (G_2' \restriction r)$. Then, by [$\rightarrow$1-Seq2], $(G_1 \restriction r) ; (G_2 \restriction r) \xrightarrow{\xi\restriction r, \gamma\restriction r} (G_1 \restriction r) ; (G_2' \restriction r)$. Then, by Defn. 26, $(G_1 ; G_2 \restriction r) \xrightarrow{\xi\restriction r, \gamma\restriction r} (G_1 ; G_2' \restriction r)$. Then, $(G \restriction r) \xrightarrow{\xi\restriction r, \gamma\restriction r} (G' \restriction r)$.
- **Step:** [$\rightarrow$1-Par1]. Similar to case [$\rightarrow$1-Seq1].
- **Step:** [$\rightarrow$1-Par2]. Similar to case [$\rightarrow$1-Seq1].
- **Step:** [$\rightarrow$1-NIf4], such that $G = \textbf{if}\ \hat{\xi}|_0\ G_1|_{R_1}\ G_2|_{R_2}$ and $G' = \textbf{if}\ \hat{\xi}|_0\ G_1'|_{R_1}\ G_2|_{R_2}$ and $G_1 \xrightarrow{\xi, \gamma} G_1'$ and $\mathsf{subj}(\xi) \cup \mathsf{subj}(\gamma) \subseteq R_1 \setminus R_2$, for some $R_1, R_2, G_1, G_1', G_2, \hat{\xi}$.
  - Recall $\checkmark_R(G)$. Then, $\checkmark_R(\textbf{if}\ \hat{\xi}|_0\ G_1|_{R_1}\ G_2|_{R_2})$. Then, by Defn. 27, $\checkmark_R(G_1)$.
  - Recall $G_1 \xrightarrow{\xi, \gamma} G_1'$ and $\checkmark_R(G_1)$ and $r \in R \cap \mathsf{subj}(\gamma)$. Then, by induction, $(G_1 \restriction r) \xrightarrow{\xi\restriction r, \gamma\restriction r} (G_1' \restriction r)$.
  - By Lem. 10, $\mathsf{subj}(\xi \restriction r) \subseteq \mathsf{subj}(\xi)$.
  - By Lem. 33, $\mathsf{subj}(\gamma \restriction r) \subseteq \mathsf{subj}(\gamma)$.
  - Recall $\mathsf{subj}(\xi \restriction r) \subseteq \mathsf{subj}(\xi)$ and $\mathsf{subj}(\gamma \restriction r) \subseteq \mathsf{subj}(\gamma)$. Then, $\mathsf{subj}(\xi \restriction r) \cup \mathsf{subj}(\gamma \restriction r) \subseteq \mathsf{subj}(\xi) \cup \mathsf{subj}(\gamma)$.
  - Recall $r \in R \cap \mathsf{subj}(\gamma)$. Then, $r \in \mathsf{subj}(\gamma)$. Then, by Lem. 32, $\mathsf{subj}(\gamma \restriction r) = \{r\}$.
  - Recall $(G_1 \restriction r) \xrightarrow{\xi\restriction r, \gamma\restriction r} (G_1' \restriction r)$. Then, by Lem. 53, $\mathsf{subj}(\xi \restriction r) \subseteq \mathsf{subj}(\gamma \restriction r)$. Then, $\mathsf{subj}(\xi \restriction r) \subseteq \{r\}$. Then, $\mathsf{subj}(\xi \restriction r) \cap \{r\} = \mathsf{subj}(\xi \restriction r)$.
  - Recall $\mathsf{subj}(\xi \restriction r) \cup \mathsf{subj}(\gamma \restriction r) \subseteq \mathsf{subj}(\xi) \cup \mathsf{subj}(\gamma) \subseteq R_1 \setminus R_2$. Then,
    $\mathsf{subj}(\xi \restriction r) \cup \mathsf{subj}(\gamma \restriction r) \subseteq R_1 \setminus R_2$. Then,
    $(\mathsf{subj}(\xi \restriction r) \cup \mathsf{subj}(\gamma \restriction r)) \cap \{r\} \subseteq (R_1 \setminus R_2) \cap \{r\}$. Then,
    $(\mathsf{subj}(\xi \restriction r) \cap \{r\}) \cup (\mathsf{subj}(\gamma \restriction r) \cap \{r\}) \subseteq (R_1 \cap \{r\}) \setminus (R_2 \cap \{r\})$. Then,
    $\mathsf{subj}(\xi \restriction r) \cup (\mathsf{subj}(\gamma \restriction r) \cap \{r\}) \subseteq (R_1 \cap \{r\}) \setminus (R_2 \cap \{r\})$. Then,
    $\mathsf{subj}(\xi \restriction r) \cup (\mathsf{subj}(\gamma \restriction r) \cap \mathsf{subj}(\gamma \restriction r)) \subseteq (R_1 \cap \{r\}) \setminus (R_2 \cap \{r\})$. Then,
    $\mathsf{subj}(\xi \restriction r) \cup \mathsf{subj}(\gamma \restriction r) \subseteq (R_1 \cap \{r\}) \setminus (R_2 \cap \{r\})$.
  - Recall $(G_1 \restriction r) \xrightarrow{\xi\restriction r, \gamma\restriction r} (G_1' \restriction r)$ and $\mathsf{subj}(\xi \restriction r) \cup \mathsf{subj}(\gamma \restriction r) \subseteq (R_1 \cap \{r\}) \setminus (R_2 \cap \{r\})$. Then, by [$\rightarrow$1-NIf4],
    $\textbf{if}\ (\hat{\xi} \restriction r)|_{|\mathsf{subj}(\hat{\xi}) \setminus (R_1 \cup R_2 \cup \{r\})|}\ (G_1 \restriction r)|_{R_1 \cap \{r\}}\ (G_2 \restriction r)|_{R_2 \cap \{r\}} \xrightarrow{\xi\restriction r, \gamma\restriction r}$
    $\textbf{if}\ (\hat{\xi} \restriction r)|_{|\mathsf{subj}(\hat{\xi}) \setminus (R_1 \cup R_2 \cup \{r\})|}\ (G_1' \restriction r)|_{R_1 \cap \{r\}}\ (G_2 \restriction r)|_{R_2 \cap \{r\}}$. Then, by Defn. 26,
    $(\textbf{if}\ \hat{\xi}|_0\ G_1|_{R_1}\ G_2|_{R_2} \restriction r) \xrightarrow{\xi\restriction r, \gamma\restriction r} (\textbf{if}\ \hat{\xi}|_0\ G_1'|_{R_1}\ G_2|_{R_2} \restriction r)$. Then,
    $(G \restriction r) \xrightarrow{\xi\restriction r, \gamma\restriction r} (G' \restriction r)$.
- **Step:** [$\rightarrow$1-NIf5]. Similar to case [$\rightarrow$1-NIf4].
- **Step:** [$\rightarrow$1-NWhile], such that $G = \textbf{while}\ \hat{\xi}|_0\ \{\psi\}\ \hat{G}|_\emptyset$ and $\textbf{if}\ \hat{\xi}|_0\ (\hat{G} ; \textbf{while}\ \hat{\xi}|_0\ \{\psi\}\ \hat{G}|_\emptyset)|_\emptyset\ \textbf{skip}|_\emptyset \xrightarrow{\xi, \gamma} G'$, for some $\hat{G}, \psi, \hat{\xi}$.
  - Recall $\checkmark_R(G)$. Then, $\checkmark_R(\textbf{while}\ \hat{\xi}|_0\ \{\psi\}\ \hat{G}|_\emptyset)$. Then, by Lem. 46, $\checkmark_R(\textbf{if}\ \hat{\xi}|_0\ (\hat{G} ; \textbf{while}\ \hat{\xi}|_0\ \{\psi\}\ \hat{G}|_\emptyset)|_\emptyset\ \textbf{skip}|_\emptyset)$.
  - Recall $\textbf{if}\ \hat{\xi}|_0\ (\hat{G} ; \textbf{while}\ \hat{\xi}|_0\ \{\psi\}\ \hat{G}|_\emptyset)|_\emptyset\ \textbf{skip}|_\emptyset \xrightarrow{\xi, \gamma} G'$ and $\checkmark_R(\textbf{if}\ \hat{\xi}|_0\ (\hat{G} ; \textbf{while}\ \hat{\xi}|_0\ \{\psi\}\ \hat{G}|_\emptyset)|_\emptyset\ \textbf{skip}|_\emptyset)$ and $r \in R \cap \mathsf{subj}(\gamma)$. Then, by induction, $(\textbf{if}\ \hat{\xi}|_0\ (\hat{G} ; \textbf{while}\ \hat{\xi}|_0\ \{\psi\}\ \hat{G}|_\emptyset)|_\emptyset\ \textbf{skip}|_\emptyset \restriction r) \xrightarrow{\xi\restriction r, \gamma\restriction r} (G' \restriction r)$. Then, by Defn. 26,
    $\textbf{if}\ (\hat{\xi} \restriction r)|_{|\mathsf{subj}(\hat{\xi}) \setminus (\emptyset \cup \emptyset \cup \{r\})|}\ (\hat{G} ; \textbf{while}\ \hat{\xi}|_0\ \{\psi\}\ \hat{G}|_\emptyset \restriction r)|_{\emptyset \cap \{r\}}\ (\textbf{skip} \restriction r)|_{\emptyset \cap \{r\}} \xrightarrow{\xi\restriction r, \gamma\restriction r}$
    $(G' \restriction r)$. Then, by Defn. 26,
    $\textbf{if}\ (\hat{\xi} \restriction r)|_{|\mathsf{subj}(\hat{\xi}) \setminus (\emptyset \cup \emptyset \cup \{r\})|}\ ((\hat{G} \restriction r) ; (\textbf{while}\ \hat{\xi}|_0\ \{\psi\}\ \hat{G}|_\emptyset \restriction r))|_{\emptyset \cap \{r\}}\ \textbf{skip}|_{\emptyset \cap \{r\}} \xrightarrow{\xi\restriction r, \gamma\restriction r}$

$(G' \upharpoonright r)$. Then, by Defn. 26, $\textbf{if}\ (\hat{\xi} \upharpoonright r)|_{|\mathsf{subj}(\hat{\xi}) \setminus (\emptyset \cup \emptyset \cup \{r\})|}((\hat{G} \upharpoonright r)\ ;\ \textbf{while}\ (\hat{\xi} \upharpoonright r)|_{|\mathsf{subj}(\hat{\xi}) \setminus \{r\}|}\ \{\texttt{true}\}\ (\hat{G} \upharpoonright r)|_{\emptyset})|_{\emptyset \cap \{r\}}\ \textbf{skip}|_{\emptyset \cap \{r\}} \xrightarrow{\xi \upharpoonright r, \gamma \upharpoonright r} (G' \upharpoonright r)$. Then, $\textbf{if}\ (\hat{\xi} \upharpoonright r)|_{|\mathsf{subj}(\hat{\xi}) \setminus \{r\}|}\ ((\hat{G} \upharpoonright r)\ ;\ \textbf{while}\ (\hat{\xi} \upharpoonright r)|_{|\mathsf{subj}(\hat{\xi}) \setminus \{r\}|}\ \{\texttt{true}\}\ (\hat{G} \upharpoonright r)|_{\emptyset})|_{\emptyset}\ \textbf{skip}|_{\emptyset} \xrightarrow{\xi \upharpoonright r, \gamma \upharpoonright r} (G' \upharpoonright r)$. Then, by [$\rightarrow$1-NWhile], $\textbf{while}\ (\hat{\xi} \upharpoonright r)|_{|\mathsf{subj}(\hat{\xi}) \setminus \{r\}|}\ \{\texttt{true}\}\ (\hat{G} \upharpoonright r)|_{\emptyset} \xrightarrow{\xi \upharpoonright r, \gamma \upharpoonright r} (G' \upharpoonright r)$. Then, by Defn. 26, $(\textbf{while}\ \hat{\xi}|_0\ \{\psi\}\ \hat{G}|_{\emptyset} \upharpoonright r) \xrightarrow{\xi \upharpoonright r, \gamma \upharpoonright r} (G' \upharpoonright r)$. Then, $(G \upharpoonright r) \xrightarrow{\xi \upharpoonright r, \gamma \upharpoonright r} (G' \upharpoonright r)$. □

2. Recall $G \xrightarrow{\xi, \gamma} G'$. Then, by Defn. 31:
   - **Base:** [$\rightarrow$1-Act], such that $G = \gamma$ and $G' = \textbf{skip}$ and $\xi = \texttt{true}$.
     - Recall $r \in R \setminus \mathsf{subj}(\gamma)$. Then, $r \notin \mathsf{subj}(\gamma)$. Then, by Lem. 34, $\gamma \upharpoonright r = \tau$.
     - By [$\rightarrow$1-Act], $(\gamma \upharpoonright r) \xrightarrow{\texttt{true}, \gamma \upharpoonright r} \textbf{skip}$. Then, $(\gamma \upharpoonright r) \xrightarrow{\texttt{true}, \tau} \textbf{skip}$. Then by Defn. 26, $(\gamma \upharpoonright r) \xrightarrow{\texttt{true}, \tau} (\textbf{skip} \upharpoonright r)$. Then, $(G \upharpoonright r) \xrightarrow{\texttt{true}, \tau} (G' \upharpoonright r)$.
   - **Base:** [$\rightarrow$1-If1], such that $G = \hat{R}.\textbf{if}\ \hat{\xi}\ G_1\ G_2$, for some $\hat{R}, G_1, G_2, \hat{\xi}$.
     - Recall $\checkmark_R(G)$. Then, $\hat{R}.\textbf{if}\ \hat{\xi}\ G_1\ G_2$. Then, by Defn. 27, $R = \mathsf{subj}(\hat{\xi})$.
     - Recall $r \in R \setminus \mathsf{subj}(\gamma)$. Then, $r \in R \setminus \mathsf{subj}(1^{R}_{\mathsf{subj}(\hat{\xi})})$. Then, by Defn. 16, $r \in R \setminus \mathsf{subj}(\hat{\xi})$. Then, $r \in R \setminus R$. Then, $r \in \emptyset$. Then, **false**.
   - **Base:** [$\rightarrow$1-If2]. Similar to case [$\rightarrow$1-If1].
   - **Base:** [$\rightarrow$1-While1]. Similar to case [$\rightarrow$1-If1].
   - **Base:** [$\rightarrow$1-While2]. Similar to case [$\rightarrow$1-If1].
   - **Base:** [$\rightarrow$1-NIf1], such that $0 > 0$. Then, **false**.
   - **Base:** [$\rightarrow$1-NIf2], such that $G = \textbf{if}\ \hat{\xi}|_0\ G_1|_{R_1}\ G_2|_{R_2}$ and $G' = \textbf{if}\ \hat{\xi}|_0\ G_1|_{R_1 \cup \{\hat{r}\}}\ G_2|_{R_2}$ and $\xi = \hat{\xi}^+ \upharpoonright \hat{r}$ and $\gamma = 1^{\{\hat{r}\}}_{\{\hat{r}\}}$ and $\hat{r} \in \mathsf{subj}(\hat{\xi}) \setminus (R_1 \cup R_2)$, for some $R_1, R_2, G_1, G_2, \hat{r}, \hat{\xi}$.
     - Recall $\hat{r} \in \mathsf{subj}(\hat{\xi}) \setminus (R_1 \cup R_2)$. Then, $\hat{r} \in \mathsf{subj}(\hat{\xi})$ and $\hat{r} \notin R_1 \cup R_2$.
     - Recall $r \in R \setminus \mathsf{subj}(\gamma)$. Then, $r \in R \setminus \mathsf{subj}(1^{\{\hat{r}\}}_{\{\hat{r}\}})$. Then, $r \notin \mathsf{subj}(1^{\{\hat{r}\}}_{\{\hat{r}\}})$. Then, by Defn. 16, $r \notin \{\hat{r}\}$. Then, $r \neq \hat{r}$. Then, $\hat{r} \notin \{r\}$ and $\{r\} \cap \{\hat{r}\} = \emptyset$.
     - Recall $\hat{r} \notin R_1 \cup R_2$ and $\hat{r} \notin \{r\}$. Then, $\hat{r} \notin R_1 \cup R_2 \cup \{r\}$.
     - Recall $\hat{r} \in \mathsf{subj}(\hat{\xi})$ and $\hat{r} \notin R_1 \cup R_2 \cup \{r\}$. Then, $\hat{r} \in \mathsf{subj}(\hat{\xi}) \setminus (R_1 \cup R_2 \cup \{r\})$. Then, $|(\mathsf{subj}(\hat{\xi}) \setminus (R_1 \cup R_2 \cup \{r\})) \setminus \{\hat{r}\}| = |\mathsf{subj}(\hat{\xi}) \setminus (R_1 \cup R_2 \cup \{r\})| - 1$. Then, $|\mathsf{subj}(\hat{\xi}) \setminus (R_1 \cup \{\hat{r}\} \cup R_2 \cup \{r\})| = |\mathsf{subj}(\hat{\xi}) \setminus (R_1 \cup R_2 \cup \{r\})| - 1$.
     - Recall $\hat{r} \in \mathsf{subj}(\hat{\xi})$ and $\hat{r} \notin R_1 \cup R_2 \cup \{\hat{r}\}$. Then, $\hat{r} \in \mathsf{subj}(\hat{\xi}) \setminus (R_1 \cup R_2 \cup \{r\})$. Then, $\mathsf{subj}(\hat{\xi}) \setminus (R_1 \cup R_2 \cup \{r\}) \neq \emptyset$. Then, $|\mathsf{subj}(\hat{\xi}) \setminus (R_1 \cup R_2 \cup \{r\})| > 0$. Then, by [$\rightarrow$1-NIf1], $\textbf{if}\ (\hat{\xi} \upharpoonright r)|_{|\mathsf{subj}(\hat{\xi}) \setminus (R_1 \cup R_2 \cup \{r\})|}\ (G_1 \upharpoonright r)|_{R_1 \cap \{r\}}\ (G_2 \upharpoonright r)|_{R_2 \cap \{r\}} \xrightarrow{\texttt{true}, \tau}$ $\textbf{if}\ (\hat{\xi} \upharpoonright r)|_{|\mathsf{subj}(\hat{\xi}) \setminus (R_1 \cup R_2 \cup \{r\})| - 1}\ (G_1 \upharpoonright r)|_{R_1 \cap \{r\}}\ (G_2 \upharpoonright r)|_{R_2 \cap \{r\}}$. Then, $\textbf{if}\ (\hat{\xi} \upharpoonright r)|_{|\mathsf{subj}(\hat{\xi}) \setminus (R_1 \cup R_2 \cup \{r\})|}\ (G_1 \upharpoonright r)|_{R_1 \cap \{r\}}\ (G_2 \upharpoonright r)|_{R_2 \cap \{r\}} \xrightarrow{\texttt{true}, \tau}$ $\textbf{if}\ (\hat{\xi} \upharpoonright r)|_{|\mathsf{subj}(\hat{\xi}) \setminus (R_1 \cup \{\hat{r}\} \cup R_2 \cup \{r\})|}\ (G_1 \upharpoonright r)|_{R_1 \cap \{r\}}\ (G_2 \upharpoonright r)|_{R_2 \cap \{r\}}$. Then, $\textbf{if}\ (\hat{\xi} \upharpoonright r)|_{|\mathsf{subj}(\hat{\xi}) \setminus (R_1 \cup R_2 \cup \{r\})|}\ (G_1 \upharpoonright r)|_{R_1 \cap \{r\}}\ (G_2 \upharpoonright r)|_{R_2 \cap \{r\}} \xrightarrow{\texttt{true}, \tau}$ $\textbf{if}\ (\hat{\xi} \upharpoonright r)|_{|\mathsf{subj}(\hat{\xi}) \setminus (R_1 \cup \{\hat{r}\} \cup R_2 \cup \{r\})|}\ (G_1 \upharpoonright r)|_{(R_1 \cap \{r\}) \cup \emptyset}\ (G_2 \upharpoonright r)|_{R_2 \cap \{r\}}$. Then, $\textbf{if}\ (\hat{\xi} \upharpoonright r)|_{|\mathsf{subj}(\hat{\xi}) \setminus (R_1 \cup R_2 \cup \{r\})|}\ (G_1 \upharpoonright r)|_{R_1 \cap \{r\}}\ (G_2 \upharpoonright r)|_{R_2 \cap \{r\}} \xrightarrow{\texttt{true}, \tau}$ $\textbf{if}\ (\hat{\xi} \upharpoonright r)|_{|\mathsf{subj}(\hat{\xi}) \setminus (R_1 \cup \{\hat{r}\} \cup R_2 \cup \{r\})|}\ (G_1 \upharpoonright r)|_{(R_1 \cap \{r\}) \cup (\{\hat{r}\} \cap \{r\})}\ (G_2 \upharpoonright r)|_{R_2 \cap \{r\}}$. Then, $\textbf{if}\ (\hat{\xi} \upharpoonright r)|_{|\mathsf{subj}(\hat{\xi}) \setminus (R_1 \cup R_2 \cup \{r\})|}\ (G_1 \upharpoonright r)|_{R_1 \cap \{r\}}\ (G_2 \upharpoonright r)|_{R_2 \cap \{r\}} \xrightarrow{\texttt{true}, \tau}$ $\textbf{if}\ (\hat{\xi} \upharpoonright r)|_{|\mathsf{subj}(\hat{\xi}) \setminus (R_1 \cup \{\hat{r}\} \cup R_2 \cup \{r\})|}\ (G_1 \upharpoonright r)|_{(R_1 \cup \{\hat{r}\}) \cap \{r\}}\ (G_2 \upharpoonright r)|_{R_2 \cap \{r\}}$. Then, by Defn. 26, $(\textbf{if}\ \hat{\xi}|_0\ G_1|_{R_1}\ G_2|_{R_2} \upharpoonright r) \xrightarrow{\texttt{true}, \tau} (\textbf{if}\ \hat{\xi}|_0\ G_1|_{R_1 \cup \{\hat{r}\}}\ G_2|_{R_2} \upharpoonright r)$. Then, $(G \upharpoonright r) \xrightarrow{\texttt{true}, \tau} (G' \upharpoonright r)$.
   - **Base:** [$\rightarrow$1-NIf3]. Similar to case [$\rightarrow$1-NIf2].
   - **Step:** [$\rightarrow$1-Seq1], such that $G = G_1\ ;\ G_2$ and $G' = G_1'\ ;\ G_2$ and $G_1 \xrightarrow{\xi, \gamma} G_1'$, for some G_1, G_1', G_2.
     - Recall $\checkmark_R(G)$. Then, $\checkmark_R(G_1\ ;\ G_2)$. Then, by Defn. 27, $\checkmark_R(G_1)$.
     - Recall $G_1 \xrightarrow{\xi, \gamma} G_1'$ and $\checkmark_R(G_1)$ and $r \in R \setminus \mathsf{subj}(\gamma)$. Then, by induction, $(G_1 \upharpoonright r) \xrightarrow{\texttt{true}, \tau} (G_1' \upharpoonright r)$. Then, by [$\rightarrow$1-Seq1], $(G_1 \upharpoonright r)\ ;\ (G_2 \upharpoonright r) \xrightarrow{\texttt{true}, \tau} (G_1' \upharpoonright r)\ ;\ (G_2 \upharpoonright r)$. Then, by Defn. 26, $(G_1\ ;\ G_2 \upharpoonright r) \xrightarrow{\texttt{true}, \tau} (G_1'\ ;\ G_2 \upharpoonright r)$. Then, $(G \upharpoonright r) \xrightarrow{\texttt{true}, \tau} (G' \upharpoonright r)$.
   - **Step:** [$\rightarrow$1-Seq2], such that $G = G_1\ ;\ G_2$ and $G' = G_1\ ;\ G_2'$ and $G_2 \xrightarrow{\xi, \gamma} G_2'$, for some G_1, G_2, G_2'.
     - Recall $\mathsf{subj}(G_1 \upharpoonright r) \cap (\emptyset \cup \emptyset) = \emptyset$. Then, by Lem. 4, $\mathsf{subj}(G_1 \upharpoonright r) \cap (\mathsf{subj}(\texttt{true}) \cup \emptyset) = \emptyset$. Then, by Defn. 16, $\mathsf{subj}(G_1 \upharpoonright r) \cap (\mathsf{subj}(\texttt{true}) \cup \mathsf{subj}(\tau)) = \emptyset$.

- Recall $\checkmark_R(G)$. Then, $\checkmark_R(G_1\,;G_2)$. Then, by Defn. 27, $\checkmark_R(G_2)$.
- Recall $G_2 \xrightarrow{\xi,\gamma} G_2'$ and $\checkmark_R(G_2)$ and $r \in R \setminus \mathsf{subj}(\gamma)$. Then, by induction, $(G_2 \upharpoonright r) \xrightarrow{\texttt{true},\tau} (G_2' \upharpoonright r)$.
- Recall $\mathsf{subj}(G_1 \upharpoonright r) \cap (\mathsf{subj}(\texttt{true}) \cup \mathsf{subj}(\tau)) = \emptyset$ and $(G_2 \upharpoonright r) \xrightarrow{\texttt{true},\tau} (G_2' \upharpoonright r)$. Then, by [$\to$1-SEQ2], $(G_1 \upharpoonright r)\,;(G_2 \upharpoonright r) \xrightarrow{\texttt{true},\tau} (G_1 \upharpoonright r)\,;(G_2' \upharpoonright r)$. Then, by Defn. 26, $(G_1\,;G_2 \upharpoonright r) \xrightarrow{\texttt{true},\tau} (G_1\,;G_2' \upharpoonright r)$. Then, $(G \upharpoonright r) \xrightarrow{\texttt{true},\tau} (G' \upharpoonright r)$.

– **Step:** [$\to$1-PAR1]. Similar to case [$\to$1-SEQ1].

– **Step:** [$\to$1-PAR2]. Similar to case [$\to$1-SEQ1].

– **Step:** [$\to$1-NIF4], such that $G = \textbf{if}\ \hat{\xi}|_0\ G_1|_{R_1}\ G_2|_{R_2}$ and $G' = \textbf{if}\ \hat{\xi}|_0\ G_1'|_{R_1}\ G_2|_{R_2}$ and $G_1 \xrightarrow{\xi,\gamma} G_1'$ and $\mathsf{subj}(\xi) \cup \mathsf{subj}(\gamma) \subseteq R_1 \setminus R_2$, for some $R_1, R_2, G_1, G_1', G_2, \hat{\xi}$.
- Recall $\checkmark_R(G)$. Then, $\checkmark_R(\textbf{if}\ \hat{\xi}|_0\ G_1|_{R_1}\ G_2|_{R_2})$. Then, by Defn. 27, $\checkmark_R(G_1)$.
- Recall $G_1 \xrightarrow{\xi,\gamma} G_1'$ and $\checkmark_R(G_1)$ and $r \in R \setminus \mathsf{subj}(\gamma)$. Then, by induction, $(G_1 \upharpoonright r) \xrightarrow{\texttt{true},\tau} (G_1' \upharpoonright r)$.
- Recall $\emptyset \cup \emptyset \subseteq (R_1 \cap \{r\}) \setminus (R_2 \cap \{r\})$. Then, by Lem. 4, $\mathsf{subj}(\texttt{true}) \cup \emptyset \subseteq (R_1 \cap \{r\}) \setminus (R_2 \cap \{r\})$. Then, by Defn. 16, $\mathsf{subj}(\texttt{true}) \cup \mathsf{subj}(\tau) \subseteq (R_1 \cap \{r\}) \setminus (R_2 \cap \{r\})$.
- Recall $(G_1 \upharpoonright r) \xrightarrow{\texttt{true},\tau} (G_1' \upharpoonright r)$ and $\mathsf{subj}(\texttt{true}) \cup \mathsf{subj}(\tau) \subseteq (R_1 \cap \{r\}) \setminus (R_2 \cap \{r\})$. Then, by [$\to$1-NIF4], $\textbf{if}\ (\hat{\xi} \upharpoonright r)|_{|\mathsf{subj}(\hat{\xi}) \setminus (R_1 \cup R_2 \cup \{r\})|}\ (G_1 \upharpoonright r)|_{R_1 \cap \{r\}}\ (G_2 \upharpoonright r)|_{R_2 \cap \{r\}} \xrightarrow{\texttt{true},\tau}$ $\textbf{if}\ (\hat{\xi} \upharpoonright r)|_{|\mathsf{subj}(\hat{\xi}) \setminus (R_1 \cup R_2 \cup \{r\})|}\ (G_1' \upharpoonright r)|_{R_1 \cap \{r\}}\ (G_2 \upharpoonright r)|_{R_2 \cap \{r\}}$. Then, by Defn. 26, $(\textbf{if}\ \hat{\xi}|_0\ G_1|_{R_1}\ G_2|_{R_2} \upharpoonright r) \xrightarrow{\texttt{true},\tau} (\textbf{if}\ \hat{\xi}|_0\ G_1'|_{R_1}\ G_2|_{R_2} \upharpoonright r)$. Then, $(G \upharpoonright r) \xrightarrow{\texttt{true},\tau} (G' \upharpoonright r)$.

– **Step:** [$\to$1-NIF5]. Similar to case [$\to$1-NIF4].

– **Step:** [$\to$1-NWHILE], such that $G = \textbf{while}\ \hat{\xi}|_0\ \{\psi\}\ \hat{G}|_\emptyset$ and $\textbf{if}\ \hat{\xi}|_0\ (\hat{G}\,; \textbf{while}\ \hat{\xi}|_0\ \{\psi\}\ \hat{G}|_\emptyset)|_\emptyset\ \textbf{skip}|_\emptyset \xrightarrow{\xi,\gamma} G'$, for some $\hat{G}, \psi, \hat{\xi}$.
- Recall $\checkmark_R(G)$. Then, $\checkmark_R(\textbf{while}\ \hat{\xi}|_0\ \{\psi\}\ \hat{G}|_\emptyset)$. Then, by Lem. 46, $\checkmark_R(\textbf{if}\ \hat{\xi}|_0\ (\hat{G}\,; \textbf{while}\ \hat{\xi}|_0\ \{\psi\}\ \hat{G}|_\emptyset)|_\emptyset\ \textbf{skip}|_\emptyset)$.
- Recall $\textbf{if}\ \hat{\xi}|_0\ (\hat{G}\,; \textbf{while}\ \hat{\xi}|_0\ \{\psi\}\ \hat{G}|_\emptyset)|_\emptyset\ \textbf{skip}|_\emptyset \xrightarrow{\xi,\gamma} G'$ and $\checkmark_R(\textbf{if}\ \hat{\xi}|_0\ (\hat{G}\,; \textbf{while}\ \hat{\xi}|_0\ \{\psi\}\ \hat{G}|_\emptyset)|_\emptyset\ \textbf{skip}|_\emptyset)$ and $r \in R \setminus \mathsf{subj}(\gamma)$. Then, by induction, $(\textbf{if}\ \hat{\xi}|_0\ (\hat{G}\,; \textbf{while}\ \hat{\xi}|_0\ \{\psi\}\ \hat{G}|_\emptyset)|_\emptyset\ \textbf{skip}|_\emptyset \upharpoonright r) \xrightarrow{\texttt{true},\tau} (G' \upharpoonright r)$. Then, by Defn. 26, $\textbf{if}\ (\hat{\xi} \upharpoonright r)|_{|\mathsf{subj}(\hat{\xi}) \setminus (\emptyset \cup \emptyset \cup \{r\})|}\ (\hat{G}\,; \textbf{while}\ \hat{\xi}|_0\ \{\psi\}\ \hat{G}|_\emptyset \upharpoonright r)|_{\emptyset \cap \{r\}}\ (\textbf{skip} \upharpoonright r)|_{\emptyset \cap \{r\}} \xrightarrow{\texttt{true},\tau} (G' \upharpoonright r)$. Then, by Defn. 26, $\textbf{if}\ (\hat{\xi} \upharpoonright r)|_{|\mathsf{subj}(\hat{\xi}) \setminus (\emptyset \cup \emptyset \cup \{r\})|}\ ((\hat{G} \upharpoonright r)\,; (\textbf{while}\ \hat{\xi}|_0\ \{\psi\}\ \hat{G}|_\emptyset \upharpoonright r))|_{\emptyset \cap \{r\}}\ \textbf{skip}|_{\emptyset \cap \{r\}} \xrightarrow{\texttt{true},\tau}$ $(G' \upharpoonright r)$. Then, by Defn. 26, $\textbf{if}\ (\hat{\xi} \upharpoonright r)|_{|\mathsf{subj}(\hat{\xi}) \setminus (\emptyset \cup \emptyset \cup \{r\})|}\ ((\hat{G} \upharpoonright r)\,; \textbf{while}\ (\hat{\xi} \upharpoonright r)|_{|\mathsf{subj}(\hat{\xi}) \setminus \{r\}|}\ \{\texttt{true}\}\ (\hat{G} \upharpoonright r)|_\emptyset)|_{\emptyset \cap \{r\}}\ \textbf{skip}|_{\emptyset \cap \{r\}} \xrightarrow{\texttt{true},\tau} (G' \upharpoonright r)$. Then, $\textbf{if}\ (\hat{\xi} \upharpoonright r)|_{|\mathsf{subj}(\hat{\xi}) \setminus \{r\}|}\ ((\hat{G} \upharpoonright r)\,; \textbf{while}\ (\hat{\xi} \upharpoonright r)|_{|\mathsf{subj}(\hat{\xi}) \setminus \{r\}|}\ \{\texttt{true}\}\ (\hat{G} \upharpoonright r)|_\emptyset)|_\emptyset\ \textbf{skip}|_\emptyset \xrightarrow{\texttt{true},\tau}$ $(G' \upharpoonright r)$. Then, by [$\to$1-NWHILE], $\textbf{while}\ (\hat{\xi} \upharpoonright r)|_{|\mathsf{subj}(\hat{\xi}) \setminus \{r\}|}\ \{\texttt{true}\}\ (\hat{G} \upharpoonright r)|_\emptyset \xrightarrow{\texttt{true},\tau} (G' \upharpoonright r)$. Then, by Defn. 26, $(\textbf{while}\ \hat{\xi}|_0\ \{\psi\}\ \hat{G}|_\emptyset \upharpoonright r) \xrightarrow{\texttt{true},\tau} (G' \upharpoonright r)$. Then, $(G \upharpoonright r) \xrightarrow{\texttt{true},\tau} (G' \upharpoonright r)$. □

Proof (of Lem. 60). Recall $(G \upharpoonright r) \xrightarrow{\xi_r,\lambda_r} L_r'$. Then, by Defn. 31:

– **Base:** [$\to$1-ACT], such that $G \upharpoonright r = \lambda_r$.
- Recall $G \upharpoonright r = \lambda_r$. Then, by Defn. 26, $G = \gamma$, for some γ.
- Recall $G \upharpoonright r = \lambda_r$. Then, $\gamma \upharpoonright r = \lambda_r$.
- Recall $\mathsf{subj}(G) \subseteq \mathsf{subj}(G)$. Then, $\mathsf{subj}(\gamma) \subseteq \mathsf{subj}(G)$.

– **Base:** [$\to$1-IF1], such that $G \upharpoonright r = \hat{R}.\textbf{if}\ \hat{\xi}_r\ L_{1r}\ L_{2r}$ and $\lambda_r = 1^{\hat{R}}_{\mathsf{subj}(\hat{\xi}_r)}$, for some $\hat{R}, \hat{\xi}_r, L_{1r}, L_{2r}$.
- Recall $G \upharpoonright r = \hat{R}.\textbf{if}\ \hat{\xi}_r\ L_{1r}\ L_{2r}$. Then, by Defn. 26, $G = \hat{R}.\textbf{if}\ \hat{\xi}\ G_1\ G_2$ and $\hat{\xi}_r = \hat{\xi} \upharpoonright r$, for some $G_1, G_2, \hat{\xi}$.
- Recall $\mathsf{subj}(\hat{\xi}) \subseteq \mathsf{subj}(\hat{\xi}) \cup \mathsf{subj}(G_1) \cup \mathsf{subj}(G_2)$. Then, by Defn. 16, $\mathsf{subj}(1^{\mathsf{subj}(\hat{\xi})}_{\mathsf{subj}(\hat{\xi})}) \subseteq \mathsf{subj}(\hat{\xi}) \cup \mathsf{subj}(G_1) \cup \mathsf{subj}(G_2)$. Then, by Defn. 24, $\mathsf{subj}(1^{\mathsf{subj}(\hat{\xi})}_{\mathsf{subj}(\hat{\xi})}) \subseteq \mathsf{subj}(R.\textbf{if}\ \hat{\xi}\ G_1\ G_2)$. Then, $\mathsf{subj}(1^{\mathsf{subj}(\hat{\xi})}_{\mathsf{subj}(\hat{\xi})}) \subseteq \mathsf{subj}(G)$. Then, $\gamma = 1^{\mathsf{subj}(\hat{\xi})}_{\mathsf{subj}(\hat{\xi})}$ and $\mathsf{subj}(\gamma) \subseteq \mathsf{subj}(G)$, for some γ.
- By Defn. 18:

- ∗ **Case:** $1^{\mathsf{subj}(\hat{\xi})}_{\mathsf{subj}(\hat{\xi})} \upharpoonright r = 1^{\mathsf{subj}(\hat{\xi})}_{\{r\}}$ and $r \in \mathsf{subj}(\hat{\xi})$.
  - · Recall $r \in \mathsf{subj}(\hat{\xi})$. Then, by Lem. 9, $\mathsf{subj}(\hat{\xi} \upharpoonright r) = \{r\}$.
  - · Recall $\checkmark_R(G)$. Then, $\checkmark_R(\hat{R}.\mathbf{if}\ \hat{\xi}\ G_1\ G_2)$. Then, by Defn. 27, $\hat{R} = \mathsf{subj}(\hat{\xi})$.
  - · Recall $1^{\mathsf{subj}(\hat{\xi})}_{\mathsf{subj}(\hat{\xi})} \upharpoonright r = 1^{\mathsf{subj}(\hat{\xi})}_{\{r\}}$. Then, $1^{\mathsf{subj}(\hat{\xi})}_{\mathsf{subj}(\hat{\xi})} \upharpoonright r = 1^{\mathsf{subj}(\hat{\xi})}_{\mathsf{subj}(\hat{\xi} \upharpoonright r)}$. Then, $1^{\mathsf{subj}(\hat{\xi})}_{\mathsf{subj}(\hat{\xi})} \upharpoonright r = 1^{\hat{R}}_{\mathsf{subj}(\hat{\xi}_r)}$. Then, $\gamma \upharpoonright r = \lambda_r$.
- ∗ **Case:** $r \notin \mathsf{subj}(\hat{\xi})$.
  - · Recall $r \notin \mathsf{subj}(\hat{\xi})$. Then, by Lem. 9, $\mathsf{subj}(\hat{\xi} \upharpoonright r) = \emptyset$.
  - · By Defn. 14, $\lambda_r \neq 1^{\hat{R}}_{\emptyset}$. Then, $\lambda_r \neq 1^{\hat{R}}_{\mathsf{subj}(\hat{\xi} \upharpoonright r)}$. Then, $\lambda_r \neq 1^{\hat{R}}_{\mathsf{subj}(\hat{\xi}_r)}$. Then, **false**.

– **Base:** [→1-If2]. Similar to case [→1-If1].

– **Base:** [→1-While1]. Similar to case [→1-If1].

– **Base:** [→1-While2]. Similar to case [→1-If1].

– **Base:** [→1-NIf1], such that $\lambda_r = \tau$.
Recall $\lambda_r \neq \tau$. Then, $\tau \neq \tau$. Then, **false**.

– **Base:** [→1-NIf2], such that $G \upharpoonright r = \mathbf{if}\ \hat{\xi}_r|_n\ L_{1r}|_{R_{1r}}\ L_{2r}|_{R_{2r}}$ and $L'_r = \mathbf{if}\ \hat{\xi}_r|_n\ L_{1r}|_{R_{1r} \cup \{\hat{r}\}}\ L_{2r}|_{R_{2r}}$ and $\lambda_r = 1^{\{\hat{r}\}}_{\{\hat{r}\}}$ and $\hat{r} \in \mathsf{subj}(\hat{\xi}_r) \setminus (R_{1r} \cup R_{2r})$, for some $R_{1r}, R_{2r}, L_{1r}, L_{2r}, \hat{r}, \xi_r, n$.

- Recall $G \upharpoonright r = \mathbf{if}\ \hat{\xi}_r|_n\ L_{1r}|_{R_{1r}}\ L_{2r}|_{R_{2r}}$. Then, by Defn. 26, $G = \mathbf{if}\ \hat{\xi}|_0\ G_1|_{R_1}\ G_2|_{R_2}$ and $R_{1r} = R_1 \cap \{r\}$ and $R_{2r} = R_2 \cap \{r\}$ and $\hat{\xi}_r = \hat{\xi} \upharpoonright r$, for some $R_1, R_2, G_1, G_2, \hat{\xi}$.
- Recall $\hat{r} \in \mathsf{subj}(\hat{\xi}_r) \setminus (R_{1r} \cup R_{2r})$. Then, $\hat{r} \in \mathsf{subj}(\hat{\xi} \upharpoonright r) \setminus ((R_1 \cap \{r\}) \cup (R_2 \cap \{r\}))$. Then, $\hat{r} \in \mathsf{subj}(\hat{\xi} \upharpoonright r)$ and $\hat{r} \notin (R_1 \cap \{r\}) \cup (R_2 \cap \{r\})$.
- By Lem. 10, $\mathsf{subj}(\hat{\xi} \upharpoonright r) \subseteq \mathsf{subj}(\hat{\xi})$ and $\mathsf{subj}(\hat{\xi} \upharpoonright r) \subseteq \{r\}$.
- Recall $\hat{r} \in \mathsf{subj}(\hat{\xi} \upharpoonright r) \subseteq \mathsf{subj}(\hat{\xi})$. Then, $\hat{r} \in \mathsf{subj}(\hat{\xi})$.
- Recall $\hat{r} \in \mathsf{subj}(\hat{\xi} \upharpoonright r) \subseteq \{r\}$. Then, $\hat{r} \in \{r\}$. Then, $\hat{r} = r$.
- Recall $\hat{r} \notin (R_1 \cap \{r\}) \cup (R_2 \cap \{r\})$. Then, $\hat{r} \notin (R_1 \cap \{\hat{r}\}) \cup (R_2 \cap \{\hat{r}\})$. Then, $\hat{r} \notin R_1 \cap \{\hat{r}\}$ and $\hat{r} \notin R_2 \cap \{\hat{r}\}$. Then, $\hat{r} \notin R_1$ and $\hat{r} \notin R_2$. Then, $\hat{r} \notin R_1 \cup R_2$.
- Recall $\hat{r} \in \mathsf{subj}(\hat{\xi})$ and $\hat{r} \notin R_1 \cup R_2$. then, $\hat{r} \in \mathsf{subj}(\hat{\xi}) \setminus (R_1 \cup R_2)$. Then, $\{\hat{r}\} \subseteq \mathsf{subj}(\hat{\xi}) \setminus (R_1 \cup R_2)$. Then, by Defn. 16, $\mathsf{subj}(1^{\{\hat{r}\}}_{\{\hat{r}\}}) \subseteq \mathsf{subj}(\hat{\xi}) \setminus (R_1 \cup R_2)$. Then, $\mathsf{subj}(1^{\{\hat{r}\}}_{\{\hat{r}\}}) \subseteq (\mathsf{subj}(\hat{\xi}) \setminus (R_1 \cup R_2)) \cup (\mathsf{subj}(G_1) \setminus R_2) \cup (\mathsf{subj}(G_2) \setminus R_1)$. Then, by Defn. 24, $\mathsf{subj}(1^{\{\hat{r}\}}_{\{\hat{r}\}}) \subseteq \mathsf{subj}(\mathbf{if}\ \hat{\xi}|_0\ G_1|_{R_1}\ G_2|_{R_2})$. Then, $\mathsf{subj}(1^{\{\hat{r}\}}_{\{\hat{r}\}}) \subseteq \mathsf{subj}(G)$. Then, $\gamma = 1^{\{\hat{r}\}}_{\{\hat{r}\}}$ and $\mathsf{subj}(\gamma) \subseteq \mathsf{subj}(G)$, for some G.
- Recall $r \in \{r\}$. Then, $r \in \{\hat{r}\}$. Then, by Defn. 18, $1^{\{\hat{r}\}}_{\{\hat{r}\}} \upharpoonright r = 1^{\{\hat{r}\}}_{\{\hat{r}\}}$. Then, $\gamma \upharpoonright r = \lambda_r$.

– **Base:** [→1-NIf3]. Similar to case [→1-NIf2].

– **Step:** [→1-Seq1], such that $G \upharpoonright r = L_{1r}\ ;\ L_{2r}$ and $L_{1r} \xrightarrow{\xi_r, \lambda_r} L'_{1r}$, for some L_{1r}, L'_{1r}, L_{2r}.

- Recall $G \upharpoonright r = L_{1r}\ ;\ L_{2r}$. Then, by Defn. 26, $G = G_1\ ;\ G_2$ and $L_{1r} = G_1 \upharpoonright r$, for some G_1, G_2.
- Recall $L_{1r} \xrightarrow{\xi_r, \lambda_r} L'_{1r}$. Then, $(G_1 \upharpoonright r) \xrightarrow{\xi_r, \lambda_r} L'_{1r}$.
- Recall $\checkmark_R(G)$. Then, $\checkmark_R(G_1\ ;\ G_2)$. Then, by Defn. 27, $\checkmark_R(G_1)$.
- Recall $(G_1 \upharpoonright r) \xrightarrow{\xi_r, \lambda_r} L'_{1r}$ and $\checkmark_R(G_1)$ and $\lambda_r \neq \tau$. Then, by induction, $\mathsf{subj}(\gamma) \subseteq \mathsf{subj}(G_1)$ and $\lambda_r = \gamma \upharpoonright r$, for some γ.
- Recall $\mathsf{subj}(\gamma) \subseteq \mathsf{subj}(G_1)$. Then, $\mathsf{subj}(\gamma) \subseteq \mathsf{subj}(G_1) \cup \mathsf{subj}(G_2)$. Then, by Defn. 24, $\mathsf{subj}(\gamma) \subseteq \mathsf{subj}(G_1\ ;\ G_2)$. Then, $\mathsf{subj}(\gamma) \subseteq \mathsf{subj}(G)$.

– **Step:** [→1-Seq2]. Similar to case [→1-Seq1].

– **Step:** [→1-Par1]. Similar to case [→1-Seq1].

– **Step:** [→1-Par2]. Similar to case [→1-Seq1].

– **Step:** [→1-NIf4], such that $G \upharpoonright r = \mathbf{if}\ \hat{\xi}_r|_n\ L_{1r}|_{R_{1r}}\ L_{2r}|_{R_{2r}}$ and $L'_r = \mathbf{if}\ \hat{\xi}_r|_n\ L'_{1r}|_{R_{1r}}\ L_{2r}|_{R_{2r}}$ and $L_{1r} \xrightarrow{\xi_r, \lambda_r} L'_{1r}$ and $\mathsf{subj}(\xi_r) \cup \mathsf{subj}(\lambda_r) \subseteq R_{1r} \setminus R_{2r}$, for some $R_{1r}, R_{2r}, L_{1r}, L'_{1r}, L_{2r}, \hat{\xi}_r, n$.

- Recall $G \upharpoonright r = \mathbf{if}\ \hat{\xi}_r|_n\ L_{1r}|_{R_{1r}}\ L_{2r}|_{R_{2r}}$. Then, by Defn. 26, $G = \mathbf{if}\ \hat{\xi}|_0\ G_1|_{R_1}\ G_2|_{R_2}$ and $R_{1r} = R_1 \cap \{r\}$ and $L_{1r} = G_1 \upharpoonright r$ and $\hat{\xi}_r = \hat{\xi} \upharpoonright r$, for some $R_1, R_2, G_1, G_2, \hat{\xi}$.
- Recall $L_{1r} \xrightarrow{\xi_r, \lambda_r} L'_{1r}$. Then, $(G_1 \upharpoonright r) \xrightarrow{\xi_r, \lambda_r} L'_{1r}$.
- Recall $\checkmark_R(G)$. Then, $\checkmark_R(\mathbf{if}\ \hat{\xi}|_0\ G_1|_{R_1}\ G_2|_{R_2})$. Then, by Defn. 27, $\checkmark_R(G_1)$, and $R_1 \neq \emptyset$ implies $R_2 = \emptyset$.
- Recall $(G_1 \upharpoonright r) \xrightarrow{\xi_r, \lambda_r} L'_{1r}$ and $\checkmark_R(G_1)$ and $\lambda_r \neq \tau$. Then, by induction, $\mathsf{subj}(\gamma) \subseteq \mathsf{subj}(G_1)$ and $\lambda_r = \gamma \upharpoonright r$, for some γ.

- Recall $\lambda_r \neq \tau$. Then, by Lem. 30, $\mathsf{subj}(\lambda_r) \neq \emptyset$. Then, $\mathsf{subj}(\xi_r) \cup \mathsf{subj}(\lambda_r) \neq \emptyset$.
- Recall $\emptyset \neq \mathsf{subj}(\xi_r) \cup \mathsf{subj}(\lambda_r) \subseteq R_{1r} \setminus R_{2r}$. Then, $\emptyset \neq R_{1r} \setminus R_{2r}$. Then, $\emptyset \neq R_{1r}$. Then, $\emptyset \neq R_1 \cap \{r\}$. Then, $\emptyset \neq R_1$. Then, $R_2 = \emptyset$.
- Recall $\mathsf{subj}(\gamma) \subseteq \mathsf{subj}(G_1)$. Then, $\mathsf{subj}(\gamma) \subseteq \mathsf{subj}(G_1) \setminus \emptyset$. Then, $\mathsf{subj}(\gamma) \subseteq \mathsf{subj}(G_1) \setminus R_2$. Then, $\mathsf{subj}(\gamma) \subseteq (\mathsf{subj}(\hat{\xi}) \setminus (R_1 \cup R_2)) \cup (\mathsf{subj}(G_1) \setminus R_2) \cup (\mathsf{subj}(G_2) \setminus R_1)$. Then, by Defn. 24, $\mathsf{subj}(\gamma) \subseteq \mathsf{subj}(\mathbf{if}\ \hat{\xi}|_0\ G_1|_{R_1}\ G_2|_{R_2})$. Then, $\mathsf{subj}(\gamma) \subseteq \mathsf{subj}(G)$.

– **Step:** [$\to$1-NIf5]. Similar to case [$\to$1-NIf4].

– **Step:** [$\to$1-NWhile], such that $G \upharpoonright r = \mathbf{while}\ \hat{\xi}_r|_n\ \{\mathtt{true}\}\ \hat{L}_r|_\emptyset$ and $\mathbf{if}\ \hat{\xi}_r|_n\ (\hat{L}_r\ ;\mathbf{while}\ \hat{\xi}_r|_n\ \{\mathtt{true}\}\ \hat{L}_r|_\emptyset)|_\emptyset\ \mathbf{skip}|_\emptyset \xrightarrow{\xi_r,\lambda_r} L'_r$, for some $\hat{L}_r, \hat{\xi}_r, n$.

- Recall $G \upharpoonright r = \mathbf{while}\ \hat{\xi}_r|_n\ \{\mathtt{true}\}\ \hat{L}_r|_\emptyset$. Then, by Defn. 26, $G = \mathbf{while}\ \hat{\xi}|_0\ \{\psi\}\ \hat{G}|_\emptyset$ and $\hat{L}_r = \hat{G} \upharpoonright r$ and $\hat{\xi}_r = \hat{\xi} \upharpoonright r$ and $n = |\mathsf{subj}(\hat{\xi}) \setminus \{r\}|$, for some $\hat{G}, \psi, \hat{\xi}$.
- Recall $\mathbf{if}\ \hat{\xi}_r|_n\ (\hat{L}_r\ ;\mathbf{while}\ \hat{\xi}_r|_n\ \{\mathtt{true}\}\ \hat{L}_r|_\emptyset)|_\emptyset\ \mathbf{skip}|_\emptyset \xrightarrow{\xi_r,\lambda_r} L'_r$. Then, $\mathbf{if}\ (\hat{\xi} \upharpoonright r)|_{|\mathsf{subj}(\hat{\xi})\setminus\{r\}|}\ ((\hat{G} \upharpoonright r)\ ;\mathbf{while}\ (\hat{\xi} \upharpoonright r)|_{|\mathsf{subj}(\hat{\xi})\setminus\{r\}|}\ \{\mathtt{true}\}\ (\hat{G} \upharpoonright r)|_\emptyset)|_\emptyset\ \mathbf{skip}|_\emptyset \xrightarrow{\xi_r,\lambda_r} L'_r$. Then, $\mathbf{if}\ (\hat{\xi} \upharpoonright r)|_{|\mathsf{subj}(\hat{\xi})\setminus(\emptyset\cup\emptyset\cup\{r\})|}\ ((\hat{G} \upharpoonright r)\ ;\mathbf{while}\ (\hat{\xi} \upharpoonright r)|_{|\mathsf{subj}(\hat{\xi})\setminus\{r\}|}\ \{\mathtt{true}\}\ (\hat{G} \upharpoonright r)|_\emptyset)|_{\emptyset\cap\{r\}}\ \mathbf{skip}|_{\emptyset\cap\{r\}} \xrightarrow{\xi_r,\lambda_r} L'_r$. Then, by Defn. 26, $\mathbf{if}\ (\hat{\xi} \upharpoonright r)|_{|\mathsf{subj}(\hat{\xi})\setminus(\emptyset\cup\emptyset\cup\{r\})|}\ ((\hat{G} \upharpoonright r)\ ;(\mathbf{while}\ \hat{\xi}|_0\ \{\psi\}\ \hat{G}|_\emptyset \upharpoonright r))|_{\emptyset\cap\{r\}}\ \mathbf{skip}|_{\emptyset\cap\{r\}} \xrightarrow{\xi_r,\lambda_r} L'_r$. Then, by Defn. 26, $\mathbf{if}\ (\hat{\xi} \upharpoonright r)|_{|\mathsf{subj}(\hat{\xi})\setminus(\emptyset\cup\emptyset\cup\{r\})|}\ (\hat{G}\ ;\mathbf{while}\ \hat{\xi}|_0\ \{\psi\}\ \hat{G}|_\emptyset \upharpoonright r)|_{\emptyset\cap\{r\}}\ (\mathbf{skip} \upharpoonright r)|_{\emptyset\cap\{r\}} \xrightarrow{\xi_r,\lambda_r} L'_r$. Then, by Defn. 26, $(\mathbf{if}\ \hat{\xi}|_0\ (\hat{G}\ ;\mathbf{while}\ \hat{\xi}|_0\ \{\psi\}\ \hat{G}|_\emptyset)|_\emptyset\ \mathbf{skip}|_\emptyset \upharpoonright r) \xrightarrow{\xi_r,\lambda_r} L'_r$.
- Recall $\checkmark_R(G)$. Then, $\checkmark_R(\mathbf{while}\ \hat{\xi}|_0\ \{\psi\}\ \hat{G}|_\emptyset)$. Then, by Lem. 46, $\checkmark_R(\mathbf{if}\ \hat{\xi}|_0\ (\hat{G}\ ;\mathbf{while}\ \hat{\xi}|_0\ \{\psi\}\ \hat{G}|_\emptyset)|_\emptyset\ \mathbf{skip}|_\emptyset)$.
- Recall $(\mathbf{if}\ \hat{\xi}|_0\ (\hat{G}\ ;\mathbf{while}\ \hat{\xi}|_0\ \{\psi\}\ \hat{G}|_\emptyset)|_\emptyset\ \mathbf{skip}|_\emptyset \upharpoonright r) \xrightarrow{\xi_r,\lambda_r} L'_r$ and $\checkmark_R(\mathbf{if}\ \hat{\xi}|_0\ (\hat{G}\ ;\mathbf{while}\ \hat{\xi}|_0\ \{\psi\}\ \hat{G}|_\emptyset)|_\emptyset\ \mathbf{skip}|_\emptyset)$ and $\lambda_r \neq \tau$. Then, by induction, $\mathsf{subj}(\gamma) \subseteq \mathsf{subj}(\mathbf{if}\ \hat{\xi}|_0\ (\hat{G}\ ;\mathbf{while}\ \hat{\xi}|_0\ \{\psi\}\ \hat{G}|_\emptyset)|_\emptyset\ \mathbf{skip}|_\emptyset)$ and $\lambda_r = \gamma \upharpoonright r$, for some γ.
- Recall $\mathsf{subj}(\gamma) \subseteq \mathsf{subj}(\mathbf{if}\ \hat{\xi}|_0\ (\hat{G}\ ;\mathbf{while}\ \hat{\xi}|_0\ \{\psi\}\ \hat{G}|_\emptyset)|_\emptyset\ \mathbf{skip}|_\emptyset)$. Then, by Defn. 24, $\mathsf{subj}(\gamma) \subseteq (\mathsf{subj}(\hat{\xi}) \setminus (\emptyset \cup \emptyset)) \cup (\mathsf{subj}(\hat{G}\ ;\mathbf{while}\ \hat{\xi}|_0\ \{\psi\}\ \hat{G}|_\emptyset) \setminus \emptyset) \cup (\mathsf{subj}(\mathbf{skip}) \setminus \emptyset)$. Then, $\mathsf{subj}(\gamma) \subseteq \mathsf{subj}(\hat{\xi}) \cup \mathsf{subj}(\hat{G}\ ;\mathbf{while}\ \hat{\xi}|_0\ \{\psi\}\ \hat{G}|_\emptyset) \cup \mathsf{subj}(\mathbf{skip})$. Then, by Defn. 24, $\mathsf{subj}(\gamma) \subseteq \mathsf{subj}(\hat{\xi}) \cup \mathsf{subj}(\hat{G}) \cup \mathsf{subj}(\mathbf{while}\ \hat{\xi}|_0\ \{\psi\}\ \hat{G}|_\emptyset) \cup \emptyset$. Then, by Defn. 24, $\mathsf{subj}(\gamma) \subseteq \mathsf{subj}(\mathbf{while}\ \hat{\xi}|_0\ \{\psi\}\ \hat{G}|_\emptyset) \cup \mathsf{subj}(\mathbf{while}\ \hat{\xi}|_0\ \{\psi\}\ \hat{G}|_\emptyset) \cup \emptyset$. Then, $\mathsf{subj}(\gamma) \subseteq \mathsf{subj}(\mathbf{while}\ \hat{\xi}|_0\ \{\psi\}\ \hat{G}|_\emptyset)$. Then, $\mathsf{subj}(\gamma) \subseteq \mathsf{subj}(G)$. ☐

Proof (of Lem. 61). Recall $(G \upharpoonright q) \xrightarrow{\xi_q, q.y:=e} L'_q$. Then, by Defn. 31:

– **Base:** [$\to$1-Act], such that $G \upharpoonright q = q.y{:=}e$ and $L'_q = \mathbf{skip}$ and $\xi_q = \mathtt{true}$.

- Recall $G \upharpoonright q = q.y{:=}e$. Then, by Defn. 26. $G = q.y{:=}e$.
- By [$\to$1-Act], $q.y{:=}e \xrightarrow{\mathtt{true}, q.y:=e} \mathbf{skip}$. Then, $G \xrightarrow{\mathtt{true}, q.y:=e} \mathbf{skip}$. Then, $G' = \mathbf{skip}$ and $\xi = \mathtt{true}$ and $G \xrightarrow{\xi, q.y:=e} G'$, for some G', ξ.
- Recall $L'_q = \mathbf{skip}$. Then, by Defn. 26, $L'_q = \mathbf{skip} \upharpoonright q$. Then, $L'_{q} = G' \upharpoonright q$.
- Recall $\xi_q = \mathtt{true}$. Then, by Defn. 11, $\xi_q = \mathtt{true} \upharpoonright r$. Then, $\xi_q = \xi \upharpoonright q$.

– **Base:** [$\to$1-If1], such that $q.y{:=}e = 1^{\hat{R}}_{\mathsf{subj}(\hat{\xi}_q)}$, for some $\hat{R}, \hat{\xi}_q$. Then, **false**.

– **Base:** [$\to$1-If2]. Similar to case [$\to$1-If1].

– **Base:** [$\to$1-While1]. Similar to case [$\to$1-If1].

– **Base:** [$\to$1-While2]. Similar to case [$\to$1-If1].

– **Base:** [$\to$1-NIf1]. Similar to case [$\to$1-If1].

– **Base:** [$\to$1-NIf2]. Similar to case [$\to$1-If1].

– **Base:** [$\to$1-NIf3]. Similar to case [$\to$1-If1].

– **Step:** [$\to$1-Seq1], such that $G \upharpoonright q = L_{1q}\ ; L_{2q}$ and $L'_q = L'_{1q}\ ; L_{2q}$ and $L_{1q} \xrightarrow{\xi_q, q.y:=e} L'_{1q}$, for some L_{1q}, L'_{1q}, L_{2q}.

- Recall $G \upharpoonright q = L_{1q}\ ; L_{2q}$. Then, by Defn. 26, $G = G_1\ ; G_2$ and $L_{1q} = G_1 \upharpoonright q$ and $L_{2q} = G_2 \upharpoonright q$, for some G_1, G_2.
- Recall $L_{1q} \xrightarrow{\xi_q, q.y:=e} L'_{1q}$. Then, $(G_1 \upharpoonright q) \xrightarrow{\xi_q, q.y:=e} L'_{1q}$.
- Recall $\checkmark_R(G)$. Then, $\checkmark_R(G_1\ ; G_2)$. Then, by Defn. 27, $\checkmark_R(G_1)$.
- Recall $(G_1 \upharpoonright q) \xrightarrow{\xi_q, q.y:=e} L'_{1q}$ and $\checkmark_R(G_1)$. Then, by induction, $G_1 \xrightarrow{\xi, q.y:=e} G'_1$ and $L'_{1q} = G'_1 \upharpoonright q$ and $\xi_q = \xi \upharpoonright q$, for some ξ.

- Recall $G_1 \xrightarrow{\xi,q.y:=e} G_1'$. Then, by [$\to$1-SEQ1], $G_1 \,;\, G_2 \xrightarrow{\xi,q.y:=e} G_1' \,;\, G_2$. Then, $G \xrightarrow{\xi,q.y:=e} G_1' \,;\, G_2$. Then, $G' = G_1' \,;\, G_2$ and $G \xrightarrow{\xi,q.y:=e} G'$, for some G'.
- Recall $L_{1q}' = G_1' \restriction q$ and $L_{2q} = G_2 \restriction q$. Then, $L_{1q}' \,;\, L_{2q} = (G_1' \restriction q) \,;\, (G_2 \restriction q)$. Then, by Defn. 26, $L_{1q}' \,;\, L_{2q} = (G_1' \,;\, G_2) \restriction q$. Then, $L_q' = G' \restriction q$.

– **Step:** [$\to$1-SEQ2], such that $G \restriction q = L_{1q} \,;\, L_{2q}$ and $L_q' = L_{1q} \,;\, L_{2q}'$ and $\mathsf{subj}(L_{1q}) \cap (\mathsf{subj}(\xi_q) \cup \mathsf{subj}(q.y{:=}e)) = \emptyset$ and $L_{2q} \xrightarrow{\xi_q,q.y:=e} L_{2q}'$, for some L_{1q}, L_{2q}, L_{2q}'.
- Recall $G \restriction q = L_{1q} \,;\, L_{2q}$. Then, by Defn. 26, $G = G_1 \,;\, G_2$ and $L_{1q} = G_1 \restriction q$ and $L_{2q} = G_2 \restriction q$, for some G_1, G_2.
- Recall $L_{2q} \xrightarrow{\xi_q,q.y:=e} L_{2q}'$. Then, $(G_2 \restriction q) \xrightarrow{\xi_q,q.y:=e} L_{2q}'$.
- Recall $\checkmark_R(G)$. Then, $\checkmark_R(G_1 \,;\, G_2)$. Then, by Defn. 27, $\checkmark_R(G_2)$.
- Recall $(G_2 \restriction q) \xrightarrow{\xi_q,q.y:=e} L_{2q}'$ and $\checkmark_R(G_2)$. Then, by induction, $G_2 \xrightarrow{\xi,q.y:=e} G_2'$ and $L_{2q}' = G_2' \restriction q$ and $\xi_q = \xi \restriction q$, for some ξ.
- Recall $L_{2q} \xrightarrow{\xi_q,q.y:=e} L_{2q}'$. Then, by Lem. 53, $\mathsf{subj}(\xi_q) \subseteq \mathsf{subj}(q.y{:=}e)$. Then, $\mathsf{subj}(\xi_q) \cup \mathsf{subj}(q.y{:=}e) = \mathsf{subj}(q.y{:=}e)$.
- Recall $G_2 \xrightarrow{\xi,q.y:=e} G_2'$. Then, by Lem. 53, $\mathsf{subj}(\xi) \subseteq \mathsf{subj}(q.y{:=}e)$. Then, $\mathsf{subj}(\xi) \cup \mathsf{subj}(q.y{:=}e) = \mathsf{subj}(q.y{:=}e)$.
- Recall $\mathsf{subj}(L_{1q}) \cap (\mathsf{subj}(\xi_q) \cup \mathsf{subj}(q.y{:=}e)) = \emptyset$. Then, $\mathsf{subj}(G_1 \restriction q) \cap \mathsf{subj}(q.y{:=}e) = \emptyset$. Then, by Defn. 16, $\mathsf{subj}(G_1 \restriction q) \cap \{q\} = \emptyset$. Then, $q \notin \mathsf{subj}(G_1 \restriction q)$. Then, by Lem. 44, $q \notin \mathsf{subj}(G_1)$. Then, $\mathsf{subj}(G_1) \cap \{q\} = \emptyset$. Then, by Defn. 16, $\mathsf{subj}(G_1) \cap \mathsf{subj}(q.y{:=}e) = \emptyset$. Then, $\mathsf{subj}(G_1) \cap (\mathsf{subj}(\xi) \cup \mathsf{subj}(q.y{:=}e)) = \emptyset$.
- Recall $\mathsf{subj}(G_1) \cap (\mathsf{subj}(\xi) \cup \mathsf{subj}(q.y{:=}e)) = \emptyset$ and $G_2 \xrightarrow{\xi,q.y:=e} G_2'$. Then, by [$\to$1-SEQ2], $G_1 \,;\, G_2 \xrightarrow{\xi,q.y:=e} G_1 \,;\, G_2'$. Then, $G \xrightarrow{\xi,q.y:=e} G_1 \,;\, G_2'$. Then, $G' = G_1 \,;\, G_2'$ and $G \xrightarrow{\xi,q.y:=e} G'$, for some G'.
- Recall $L_{1q}' = G_1 \restriction q$ and $L_{2q} = G_2' \restriction q$. Then, $L_{1q} \,;\, L_{2q}' = (G_1 \restriction q) \,;\, (G_2' \restriction q)$. Then, by Defn. 26, $L_{1q} \,;\, L_{2q}' = (G_1 \,;\, G_2') \restriction q$. Then, $L_q' = G' \restriction q$.

– **Step:** [$\to$1-PAR1]. Similar to case [$\to$1-SEQ1].

– **Step:** [$\to$1-PAR2]. Similar to case [$\to$1-SEQ1].

– **Step:** [$\to$1-NIF4], such that $G \restriction q = \textbf{if}\ \hat{\xi}_q|_n\ L_{1q}|_{R_{1q}}\ L_{2q}|_{R_{2q}}$ and $L_q' = \textbf{if}\ \hat{\xi}_q|_n\ L_{1q}'|_{R_{1q}}\ L_{2q}|_{R_{2q}}$ and $L_{1q} \xrightarrow{\xi_q,q.y:=e} L_{1q}'$ and $\mathsf{subj}(\xi_q) \cup \mathsf{subj}(q.y{:=}e) \subseteq R_{1q} \setminus R_{2q}$, for some $R_{1q}, R_{2q}, L_{1q}, L_{1q}', L_{2q}, \hat{\xi}_q, n$.
- Recall $G \restriction q = \textbf{if}\ \hat{\xi}_q|_n\ L_{1q}|_{R_{1q}}\ L_{2q}|_{R_{2q}}$. Then, by Defn. 26, $G = \textbf{if}\ \hat{\xi}|_0\ G_1|_{R_1}\ G_2|_{R_2}$ and $R_{1q} = R_1 \cap \{q\}$ and $R_{2q} = R_2 \cap \{q\}$ and $L_{1q} = G_1 \restriction q$ and $L_{2q} = G_2 \restriction q$ and $\hat{\xi}_q = \hat{\xi} \restriction q$ and $n = |\mathsf{subj}(\hat{\xi}) \setminus (R_1 \cup R_2 \cup \{q\})|$, for some $R_1, R_2, G_1, G_2, \hat{\xi}$.
- Recall $L_{1q} \xrightarrow{\xi_q,q.y:=e} L_{1q}'$. Then, $(G_1 \restriction q) \xrightarrow{\xi_q,q.y:=e} L_{1q}'$.
- Recall $\checkmark_R(G)$. Then, $\checkmark_R(\textbf{if}\ \hat{\xi}|_0\ G_1|_{R_1}\ G_2|_{R_2})$. Then, by Defn. 27, $\checkmark_R(G_1)$.
- Recall $(G_1 \restriction q) \xrightarrow{\xi_q,q.y:=e} L_{1q}'$ and $\checkmark_R(G_1)$. Then, by induction, $G_1 \xrightarrow{\xi,q.y:=e} G_1'$ and $L_{1q}' = G_1' \restriction q$ and $\xi_q = \xi \restriction q$, for some G_1', for some ξ.
- Recall $G_1 \xrightarrow{\xi,q.y:=e} G_1'$. Then, by Lem. 53, $\mathsf{subj}(\xi) \subseteq \mathsf{subj}(q.y{:=}e)$. Then, $\mathsf{subj}(\xi) \cup \mathsf{subj}(q.y{:=}e) = \mathsf{subj}(q.y{:=}e)$.
- Recall $\mathsf{subj}(\xi_q) \cup \mathsf{subj}(q.y{:=}e) \subseteq R_{1q} \setminus R_{2q}$. Then, $\mathsf{subj}(\xi_q) \cup \mathsf{subj}(q.y{:=}e) \subseteq (R_1 \cap \{q\}) \setminus (R_2 \cap \{q\})$. Then, $\mathsf{subj}(q.y{:=}e) \subseteq (R_1 \cap \{q\}) \setminus (R_2 \cap \{q\})$. Then, by Defn. 16, $\{q\} \subseteq (R_1 \cap \{q\}) \setminus (R_2 \cap \{q\})$. Then, $q \in (R_1 \cap \{q\}) \setminus (R_2 \cap \{q\})$. Then, $q \in R_1 \cap \{q\}$ and $q \notin R_2 \cap \{q\}$. Then, $q \in R_1$ and $q \notin R_2$. Then, $q \in R_1 \setminus R_2$. Then, $\{q\} \subseteq R_1 \setminus R_2$. Then, by Defn. 16, $\mathsf{subj}(q.y{:=}e) \subseteq R_1 \setminus R_2$. Then, $\mathsf{subj}(\xi) \cup \mathsf{subj}(q.y{:=}e) \subseteq R_1 \setminus R_2$.
- Recall $G_1 \xrightarrow{\xi,q.y:=e} G_1'$ and $\mathsf{subj}(\xi) \cup \mathsf{subj}(q.y{:=}e) \subseteq R_1 \setminus R_2$. Then, by [$\to$1-NIF4], $\textbf{if}\ \hat{\xi}|_0\ G_1|_{R_1}\ G_2|_{R_2} \xrightarrow{\xi,q.y:=e} \textbf{if}\ \hat{\xi}|_0\ G_1'|_{R_1}\ G_2|_{R_2}$. Then, $G \xrightarrow{\xi,q.y:=e} \textbf{if}\ \hat{\xi}|_0\ G_1'|_{R_1}\ G_2|_{R_2}$. Then, $G' = \textbf{if}\ \hat{\xi}|_0\ G_1'|_{R_1}\ G_2|_{R_2}$ and $G \xrightarrow{\xi,q.y:=e} G'$, for some G'.
- Recall $L_q' = \textbf{if}\ \hat{\xi}_q|_n\ L_{1q}'|_{R_{1q}}\ L_{2q}|_{R_{2q}}$. Then, $L_q' = \textbf{if}\ (\hat{\xi} \restriction q)|_{|\mathsf{subj}(\hat{\xi}) \setminus (R_1 \cup R_2 \cup \{q\})|}\ (G_1' \restriction q)|_{R_1 \cap \{q\}}\ (G_2 \restriction q)|_{R_2 \cap \{q\}}$. Then, by Defn. 26, $L_q' = \textbf{if}\ \hat{\xi}|_0\ G_1'|_{R_1}\ G_2|_{R_2} \restriction q$. Then, $L_q' = G' \restriction q$.

– **Step:** [$\to$1-NIF5]. Similar to case [$\to$1-NIF4].

– **Step:** [$\to$1-NWHILE], such that $G \restriction q = \textbf{while}\ \hat{\xi}_q|_n\ \{\texttt{true}\}\ \hat{L}_q|_\emptyset$ and $\textbf{if}\ \hat{\xi}_q|_n\ (\hat{L}_q \,;\, \textbf{while}\ \hat{\xi}_q|_n\ \{\texttt{true}\}\ \hat{L}_q|_\emptyset)|_\emptyset\ \textbf{skip}|_\emptyset \xrightarrow{\xi_q,q.y:=e} L_q'$, for some $\hat{L}_q, \hat{\xi}_q, n$.

- Recall $G \upharpoonright q = \textbf{while}\ \hat{\xi}_q|_n\ \{\texttt{true}\}\ \hat{L}_q|_\emptyset$. Then, by Defn. 26, $G = \textbf{while}\ \hat{\xi}|_0\ \{\psi\}\ \hat{G}|_\emptyset$ and $\hat{L}_q = \hat{G} \upharpoonright q$ and $\hat{\xi}_q = \hat{\xi} \upharpoonright q$ and $n = |\mathsf{subj}(\hat{\xi}) \setminus \{q\}|$, for some $\hat{G}, \psi, \hat{\xi}$.
- Recall $\textbf{if}\ \hat{\xi}_q|_n\ (\hat{L}_q\ ;\ \textbf{while}\ \hat{\xi}_q|_n\ \{\texttt{true}\}\ \hat{L}_q|_\emptyset)|_\emptyset\ \textbf{skip}|_\emptyset \xrightarrow{\xi_q, q.y:=e} L'_q$. Then, $\textbf{if}\ (\hat{\xi}\upharpoonright q)|_{|\mathsf{subj}(\hat{\xi})\setminus\{q\}|}\ ((\hat{G}\upharpoonright q)\ ;\ \textbf{while}\ (\hat{\xi}\upharpoonright q)|_{|\mathsf{subj}(\hat{\xi})\setminus\{q\}|}\ \{\texttt{true}\}\ (\hat{G}\upharpoonright q)|_\emptyset)|_\emptyset\ \textbf{skip}|_\emptyset \xrightarrow{\xi_q, q.y:=e} L'_q$. Then, $\textbf{if}\ (\hat{\xi} \upharpoonright q)|_{|\mathsf{subj}(\hat{\xi})\setminus(\emptyset\cup\emptyset\cup\{q\})|}\ ((\hat{G} \upharpoonright q)\ ;\ \textbf{while}\ (\hat{\xi} \upharpoonright q)|_{|\mathsf{subj}(\hat{\xi})\setminus\{q\}|}\ \{\texttt{true}\}\ (\hat{G} \upharpoonright q)|_\emptyset)|_{\emptyset\cap\{q\}}\ \textbf{skip}|_{\emptyset\cap\{q\}} \xrightarrow{\xi_q, q.y:=e} L'_q$. Then, by Defn. 26, $\textbf{if}\ (\hat{\xi} \upharpoonright q)|_{|\mathsf{subj}(\hat{\xi})\setminus(\emptyset\cup\emptyset\cup\{q\})|}\ ((\hat{G} \upharpoonright q)\ ;\ (\textbf{while}\ \hat{\xi}|_0\ \{\psi\}\ \hat{G}|_\emptyset \upharpoonright q))|_{\emptyset\cap\{q\}}\ \textbf{skip}|_{\emptyset\cap\{q\}} \xrightarrow{\xi_q, q.y:=e} L'_q$. Then, by Defn. 26, $\textbf{if}\ (\hat{\xi} \upharpoonright q)|_{|\mathsf{subj}(\hat{\xi})\setminus(\emptyset\cup\emptyset\cup\{q\})|}\ (\hat{G}\ ;\ \textbf{while}\ \hat{\xi}|_0\ \{\psi\}\ \hat{G}|_\emptyset \upharpoonright q)|_{\emptyset\cap\{q\}}\ (\textbf{skip} \upharpoonright q)|_{\emptyset\cap\{q\}} \xrightarrow{\xi_q, q.y:=e} L'_q$. Then, by Defn. 26, $(\textbf{if}\ \hat{\xi}|_0\ (\hat{G}\ ;\ \textbf{while}\ \hat{\xi}|_0\ \{\psi\}\ \hat{G}|_\emptyset)|_\emptyset\ \textbf{skip}|_\emptyset \upharpoonright q) \xrightarrow{\xi_q, q.y:=e} L'_q$.
- Recall $\checkmark_R(G)$. Then, $\checkmark_R(\textbf{while}\ \hat{\xi}|_0\ \{\psi\}\ \hat{G}|_\emptyset)$. Then, by Lem. 46, $\checkmark_R(\textbf{if}\ \hat{\xi}|_0\ (\hat{G}\ ;\ \textbf{while}\ \hat{\xi}|_0\ \{\psi\}\ \hat{G}|_\emptyset)|_\emptyset\ \textbf{skip}|_\emptyset)$.
- Recall $(\textbf{if}\ \hat{\xi}|_0\ (\hat{G}\ ;\ \textbf{while}\ \hat{\xi}|_0\ \{\psi\}\ \hat{G}|_\emptyset)|_\emptyset\ \textbf{skip}|_\emptyset \upharpoonright q) \xrightarrow{\xi_q, q.y:=e} L'_q$ and $\checkmark_R(\textbf{if}\ \hat{\xi}|_0\ (\hat{G}\ ;\ \textbf{while}\ \hat{\xi}|_0\ \{\psi\}\ \hat{G}|_\emptyset)|_\emptyset\ \textbf{skip}|_\emptyset)$. Then, by induction, $\textbf{if}\ \hat{\xi}|_0\ (\hat{G}\ ;\ \textbf{while}\ \hat{\xi}|_0\ \{\psi\}\ \hat{G}|_\emptyset)|_\emptyset\ \textbf{skip}|_\emptyset \xrightarrow{\xi, q.y:=e} G'$ and $L'_q = G' \upharpoonright q$ and $\xi_q = \xi \upharpoonright q$, for some G', ξ.
- Recall $\textbf{if}\ \hat{\xi}|_0\ (\hat{G}\ ;\ \textbf{while}\ \hat{\xi}|_0\ \{\psi\}\ \hat{G}|_\emptyset)|_\emptyset\ \textbf{skip}|_\emptyset \xrightarrow{\xi, q.y:=e} G'$. Then, by [→1-NWHILE], $\textbf{while}\ \hat{\xi}|_0\ \{\psi\}\ \hat{G}|_\emptyset \xrightarrow{\xi, q.y:=e} G'$. Then, $G \xrightarrow{\xi, q.y:=e} G'$. □

Proof (of Lem. 62). Recall $(G \upharpoonright p) \xrightarrow{\xi_p, pq!e} L'_p$. Then, by Defn. 31:

- **Base:** [→1-ACT], such that $G \upharpoonright p = pq!e$ and $L'_p = \textbf{skip}$ and $\xi_p = \texttt{true}$.
  - Recall $G \upharpoonright p = pq!e$. Then, by Defn. 26, $G = p.e \twoheadrightarrow q.\hat{y}$, for some $\hat{y}$.
  - Recall $(G \upharpoonright q) \xrightarrow{\xi_q, pq?y} L'_q$. Then, $(p.e \twoheadrightarrow q.\hat{y} \upharpoonright q) \xrightarrow{\xi_q, pq?y} L'_q$. Then, by Defn. 26, $pq?\hat{y} \xrightarrow{\xi_q, pq?y} L'_q$. Then, by Defn. 31, $L'_q = \textbf{skip}$ and $\xi_q = \texttt{true}$ and $\hat{y} = y$.
  - By [→1-ACT], $p.e \twoheadrightarrow q.\hat{y} \xrightarrow{\texttt{true}, p.e \twoheadrightarrow q.\hat{y}} \textbf{skip}$. Then, $G \xrightarrow{\texttt{true}, p.e \twoheadrightarrow q.y} \textbf{skip}$. Then, $G' = \textbf{skip}$ and $\xi = \texttt{true}$ and $G \xrightarrow{\xi, p.e \twoheadrightarrow q.y} G'$, for some G', ξ.
  - Recall $L'_p = \textbf{skip}$ and $L'_q = \textbf{skip}$. Then, by Defn. 26, $L'_p = \textbf{skip} \upharpoonright p$ and $L'_q = \textbf{skip} \upharpoonright q$. Then, $L'_p = G' \upharpoonright p$ and $L'_q = G' \upharpoonright q$.
  - Recall $\xi_p = \texttt{true}$ and $\xi_q = \texttt{true}$. Then, by Defn. 11, $\xi_p = \texttt{true} \upharpoonright p$ and $\xi_q = \texttt{true} \upharpoonright q$. Then, $\xi_p = \xi \upharpoonright p$ and $\xi_q = \xi \upharpoonright q$.
- **Base:** [→1-IF1], such that $pq!e = 1^{\hat{R}}_{\mathsf{subj}(\hat{\xi}_p)}$, for some $\hat{R}, \hat{\xi}_p$. Then, **false**.
- **Base:** [→1-IF2]. Similar to case [→1-IF1].
- **Base:** [→1-WHILE1]. Similar to case [→1-IF1].
- **Base:** [→1-WHILE2]. Similar to case [→1-IF1].
- **Base:** [→1-NIF1]. Similar to case [→1-IF1].
- **Base:** [→1-NIF2]. Similar to case [→1-IF1].
- **Base:** [→1-NIF3]. Similar to case [→1-IF1].
- **Step:** [→1-SEQ1], such that $G \upharpoonright p = L_{1p}\ ;\ L_{2p}$ and $L'_p = L'_{1p}\ ;\ L_{2p}$ and $L_{1p} \xrightarrow{\xi_p, pq!e} L'_{1p}$, for some L_{1p}, L'_{1p}, L_{2p}.
  - Recall $G \upharpoonright p = L_{1p}\ ;\ L_{2p}$. Then, by Defn. 26, $G = G_1\ ;\ G_2$ and $L_{1p} = G_1 \upharpoonright p$ and $L_{2p} = G_2 \upharpoonright p$, for some G_1, G_2.
  - Recall $L_{1p} \xrightarrow{\xi_p, pq!e} L'_{1p}$. Then, $(G_1 \upharpoonright p) \xrightarrow{\xi_p, pq!e} L'_{1p}$.
  - Recall $\checkmark_R(G)$. Then, $\checkmark_R(G_1\ ;\ G_2)$. Then, by Defn. 27, $\checkmark_R(G_1)$.
  - Recall $(G \upharpoonright q) \xrightarrow{\xi_q, pq?y} L'_q$. Then, $(G_1\ ;\ G_2 \upharpoonright q) \xrightarrow{\xi_q, pq?y} L'_q$. Then, by Defn. 26, $(G_1 \upharpoonright q)\ ;\ (G_2 \upharpoonright q) \xrightarrow{\xi_q, pq?y} L'_q$. Then, by Defn. 31:
    - **Case:** [→1-SEQ1], such that $L'_q = L'_{1q}\ ;\ (G_2 \upharpoonright q)$ and $(G_1 \upharpoonright q) \xrightarrow{\xi_q, pq?y} L'_{1q}$, for some L'_{1q}.
      - Recall $(G_1 \upharpoonright p) \xrightarrow{\xi_p, pq!e} L'_{1p}$ and $(G_1 \upharpoonright q) \xrightarrow{\xi_q, pq?y} L'_{1q}$ and $\checkmark_R(G_1)$. Then, by induction, $G_1 \xrightarrow{\xi, p.e \twoheadrightarrow q.y} G'_1$ and $L'_{1p} = G'_1 \upharpoonright p$ and $\xi_p = \xi \upharpoonright p$ and $L'_{1q} = G'_1 \upharpoonright q$ and $\xi_q = \xi \upharpoonright q$, for some G'_1, for some ξ.
      - Recall $G_1 \xrightarrow{\xi, p.e \twoheadrightarrow q.y} G'_1$. Then, by [→1-SEQ1], $G_1\ ;\ G_2 \xrightarrow{\xi, p.e \twoheadrightarrow q.y} G'_1\ ;\ G_2$. Then, $G \xrightarrow{\xi, p.e \twoheadrightarrow q.y} G'_1\ ;\ G_2$. Then, $G' = G'_1\ ;\ G_2$ and $G_1\ ;\ G_2 \xrightarrow{\xi, p.e \twoheadrightarrow q.y} G'$, for some G'.

· Recall $L'_p = L'_{1p} \,;\, L_{2p}$ and $L'_q = L'_{1q} \,;\, (G_2 \restriction q)$. Then, $L'_p = (G'_1 \restriction p) \,;\, (G_2 \restriction p)$ and $L'_q = (G'_1 \restriction q) \,;\, (G_2 \restriction q)$. Then, by Defn. 26, $L'_p = (G'_1 \,;\, G_2) \restriction p$ and $L'_q = (G'_1 \,;\, G_2) \restriction q$. Then, $L'_p = G' \restriction p$ and $L'_q = G' \restriction q$.

∗ **Case:** [→1-Seq2], such that $\mathsf{subj}(G_1 \restriction q) \cap (\mathsf{subj}(\xi_q) \cup \mathsf{subj}(pq?y)) = \emptyset$.

· Recall $(G_1 \restriction p) \xrightarrow{\xi_p, pq!e} L'_{1p}$ and $\checkmark_R(G_1)$ and $pq!e \neq \tau$. Then, by Lem. 60, $\mathsf{subj}(\gamma) \subseteq \mathsf{subj}(G_1)$ and $pq!e = \gamma \restriction p$, for some γ.

· Recall $pq!e = \gamma \restriction p$. Then, by Defn. 18, $\gamma = p.e \twoheadrightarrow q.\hat{y}$, for some $\hat{y}$.

· Recall $\mathsf{subj}(\gamma) \subseteq \mathsf{subj}(G_1)$. Then, $\mathsf{subj}(p.e \twoheadrightarrow q.\hat{y}) \subseteq \mathsf{subj}(G_1)$. Then, by Defn. 16, $\{p, q\} \subseteq \mathsf{subj}(G_1)$. Then, $q \in \mathsf{subj}(G_1)$. Then, by Lem. 44, $q \in \mathsf{subj}(G_1 \restriction q)$.

· Recall $q \in \{q\}$. Then, by Defn. 16, $q \in \mathsf{subj}(pq?y)$. Then, $q \in \mathsf{subj}(\xi_q) \cup \mathsf{subj}(pq?y)$.

· Recall $q \in \mathsf{subj}(G_1 \restriction q)$ and $q \in \mathsf{subj}(\xi_q) \cup \mathsf{subj}(pq?y)$. Then, $q \in \mathsf{subj}(G_1 \restriction q) \cap (\mathsf{subj}(\xi_q) \cup \mathsf{subj}(pq?y))$. Then, $q \in \emptyset$. Then, **false**.

– **Step:** [→1-Seq2], such that $G \restriction p = L_{1p} \,;\, L_{2p}$ and $L'_p = L_{1p} \,;\, L'_{2p}$ and $\mathsf{subj}(L_{1p}) \cap (\mathsf{subj}(\xi_p) \cup \mathsf{subj}(pq!e)) = \emptyset$ and $L_{2p} \xrightarrow{\xi_p, pq!e} L'_{2p}$, for some L_{1p}, L_{2p}, L'_{2p}.

• Recall $G \restriction p = L_{1p} \,;\, L_{2p}$. Then, by Defn. 26, $G = G_1 \,;\, G_2$ and $L_{1p} = G_1 \restriction p$ and $L_{2p} = G_2 \restriction p$, for some G_1, G_2.

• Recall $(G \restriction q) \xrightarrow{\xi_q, pq?y} L'_q$. Then, $(G_1 \,;\, G_2 \restriction q) \xrightarrow{\xi_q, pq?y} L'_q$. Then, by Defn. 26, $(G_1 \restriction q) \,;\, (G_2 \restriction q) \xrightarrow{\xi_q, pq?y} L'_q$. Then, by Defn. 31:

∗ **Case:** [→1-Seq1], such that $L'_q = L'_{1q} \,;\, (G_2 \restriction q)$ and $(G_1 \restriction q) \xrightarrow{\xi_q, pq?y} L'_{1q}$, for some L'_{1q}.

· Recall $\checkmark_R(G)$. Then, $\checkmark_R(G_1 \,;\, G_2)$. Then, by Defn. 27, $\checkmark_R(G_1)$.

· Recall $(G_1 \restriction q) \xrightarrow{\xi_q, pq?y} L'_{1q}$ and $\checkmark_R(G_1)$ and $pq?y \neq \tau$. Then, by Lem. 60, $\mathsf{subj}(\gamma) \subseteq \mathsf{subj}(G_1)$ and $pq?y = \gamma \restriction q$, for some γ.

· Recall $pq?y = \gamma \restriction q$. Then, by Defn. 18, $\gamma = p.\hat{e} \twoheadrightarrow q.y$, for some $\hat{e}$.

· Recall $\mathsf{subj}(\gamma) \subseteq \mathsf{subj}(G_1)$. Then, $\mathsf{subj}(p.\hat{e} \twoheadrightarrow q.y) \subseteq \mathsf{subj}(G_1)$. Then, by Defn. 16, $\{p, q\} \subseteq \mathsf{subj}(G_1)$. Then, $p \in \mathsf{subj}(G_1)$. Then, by Lem. 44, $p \in \mathsf{subj}(G_1 \restriction p)$.

· Recall $p \in \{p\}$. Then, by Defn. 16, $p \in \mathsf{subj}(pq!e)$. Then, $p \in \mathsf{subj}(\xi_p) \cup \mathsf{subj}(pq!e)$.

· Recall $p \in \mathsf{subj}(G_1 \restriction p)$ and $p \in \mathsf{subj}(\xi_p) \cup \mathsf{subj}(pq!e)$. Then, $p \in \mathsf{subj}(G_1 \restriction p) \cap (\mathsf{subj}(\xi_p) \cup \mathsf{subj}(pq!e))$. Then, $p \in \mathsf{subj}(L_{1p}) \cap (\mathsf{subj}(\xi_p) \cup \mathsf{subj}(pq!e))$. Then, $p \in \emptyset$. Then, **false**.

∗ **Case:** [→1-Seq2], such that $L'_q = (G_1 \restriction q) \,;\, L'_{2q}$ and $\mathsf{subj}(G_1 \restriction q) \cap (\mathsf{subj}(\xi_q) \cup \mathsf{subj}(pq?y)) = \emptyset$ and $(G_2 \restriction q) \xrightarrow{\xi_q, pq?y} L'_{2q}$, for some L'_{1q}, L'_{2q}.

· Recall $L_{2p} \xrightarrow{\xi_p, pq!e} L'_{2p}$. Then, $(G_2 \restriction p) \xrightarrow{\xi_p, pq!e} L'_{2p}$.

· Recall $\checkmark_R(G)$. Then, $\checkmark_R(G_1 \,;\, G_2)$. Then, by Defn. 27, $\checkmark_R(G_2)$.

· Recall $(G_2 \restriction p) \xrightarrow{\xi_p, pq!e} L'_{2p}$ and $(G_2 \restriction q) \xrightarrow{\xi_q, pq?y} L'_{2q}$ and $\checkmark_R(G_2)$. Then, by induction, $G_2 \xrightarrow{\xi, p.e \twoheadrightarrow q.y} G'_2$ and $L'_{2p} = G'_2 \restriction p$ and $\xi_p = \xi \restriction p$ and $L'_{2q} = G'_2 \restriction q$ and $\xi_q = \xi \restriction q$, for some G'_2, for some ξ.

· Recall $\mathsf{subj}(L_{1p}) \cap (\mathsf{subj}(\xi_p) \cup \mathsf{subj}(pq!e)) = \emptyset$. Then, $\mathsf{subj}(G_1 \restriction p) \cap (\mathsf{subj}(\xi_p) \cup \mathsf{subj}(pq!e)) = \emptyset$.

· Recall $G_2 \xrightarrow{\xi, p.e \twoheadrightarrow q.y} G'_2$. Then, by Lem. 53, $\mathsf{subj}(\xi) \subseteq \mathsf{subj}(p.e \twoheadrightarrow q.y)$. Then, $\mathsf{subj}(\xi) \cup \mathsf{subj}(p.e \twoheadrightarrow q.y) = \mathsf{subj}(p.e \twoheadrightarrow q.y)$.

· Recall $\mathsf{subj}(G_1 \restriction p) \cap (\mathsf{subj}(\xi_p) \cup \mathsf{subj}(pq!e)) = \emptyset$ and $\mathsf{subj}(G_1 \restriction q) \cap (\mathsf{subj}(\xi_q) \cup \mathsf{subj}(pq?y)) = \emptyset$. Then, $\mathsf{subj}(G_1 \restriction p) \cap \mathsf{subj}(pq!e) = \emptyset$ and $\mathsf{subj}(G_1 \restriction q) \cap \mathsf{subj}(pq?y) = \emptyset$. Then, by Defn. 16, $\mathsf{subj}(G_1 \restriction p) \cap \{p\} = \emptyset$ and $\mathsf{subj}(G_1 \restriction q) \cap \{q\} = \emptyset$. Then, $p \notin \mathsf{subj}(G_1 \restriction p)$ and $q \notin \mathsf{subj}(G_1 \restriction q)$. Then, by Lem. 44, $p, q \notin \mathsf{subj}(G_1)$. Then, $\mathsf{subj}(G_1) \cap \{p, q\} = \emptyset$. Then, by Defn. 16, $\mathsf{subj}(G_1) \cap \mathsf{subj}(p.e \twoheadrightarrow q.y) = \emptyset$. Then, $\mathsf{subj}(G_1) \cap (\mathsf{subj}(\xi) \cup \mathsf{subj}(p.e \twoheadrightarrow q.y)) = \emptyset$.

· Recall $\mathsf{subj}(G_1) \cap (\mathsf{subj}(\xi) \cup \mathsf{subj}(p.e \twoheadrightarrow q.y)) = \emptyset$ and $G_2 \xrightarrow{\xi, p.e \twoheadrightarrow q.y} G'_2$. Then, by [→1-Seq2], $G_1 \,;\, G_2 \xrightarrow{\xi, p.e \twoheadrightarrow q.y} G_1 \,;\, G'_2$. Then, $G \xrightarrow{\xi, p.e \twoheadrightarrow q.y} G_1 \,;\, G'_2$. Then, $G' = G_1 \,;\, G'_2$ and $G_1 \,;\, G_2 \xrightarrow{\xi, p.e \twoheadrightarrow q.y} G'$, for some G'.

· Recall $L'_p = L_{1p} \,;\, L'_{2p}$ and $L'_q = (G_1 \restriction q) \,;\, L'_{2q}$. Then, $L'_p = (G_1 \restriction p) \,;\, (G'_2 \restriction p)$ and $L'_q = (G_1 \restriction q) \,;\, (G'_2 \restriction q)$. Then, by Defn. 26, $L'_p = (G_1 \,;\, G'_2) \restriction p$ and $L'_q = (G_1 \,;\, G'_2) \restriction q$. Then, $L'_p = G' \restriction p$ and $L'_q = G' \restriction q$.

- **Step:** [$\to$1-Par1], such that $G \upharpoonright p = L_{1p} \parallel L_{2p}$ and $L'_p = L'_{1p} \parallel L_{2p}$ and $L_{1p} \xrightarrow{\xi_p, pq!e} L'_{1p}$, for some L_{1p}, L'_{1p}, L_{2p}.
  - Recall $G \upharpoonright p = L_{1p} \parallel L_{2p}$. Then, by Defn. 26, $G = G_1 \parallel G_2$ and $L_{1p} = G_1 \upharpoonright p$ and $L_{2p} = G_2 \upharpoonright p$, for some G_1, G_2.
  - Recall $L_{1p} \xrightarrow{\xi_p, pq!e} L'_{1p}$. Then, $(G_1 \upharpoonright p) \xrightarrow{\xi_p, pq!e} L'_{1p}$.
  - Recall $(G \upharpoonright q) \xrightarrow{\xi_q, pq?y} L'_q$. Then, $(G_1 \parallel G_2 \upharpoonright q) \xrightarrow{\xi_q, pq?y} L'_q$. Then, by Defn. 26, $(G_1 \upharpoonright q) \parallel (G_2 \upharpoonright q) \xrightarrow{\xi_q, pq?y} L'_q$. Then, by Defn. 31:
    - ∗ **Case:** [$\to$1-Par1], such that $L'_q = L'_{1q} \parallel (G_2 \upharpoonright q)$ and $(G_1 \upharpoonright q) \xrightarrow{\xi_q, pq?y} L'_{1q}$, for some L'_{1q}.
      - · Recall $\checkmark_R(G)$. Then, $\checkmark_R(G_1 \parallel G_2)$. Then, by Defn. 27, $\checkmark_R(G_1)$.
      - · Recall $(G_1 \upharpoonright p) \xrightarrow{\xi_p, pq!e} L'_{1p}$ and $(G_1 \upharpoonright q) \xrightarrow{\xi_q, pq?y} L'_{1q}$ and $\checkmark_R(G_1)$. Then, by induction, $G_1 \xrightarrow{\xi, p.e \twoheadrightarrow q.y} G'_1$ and $L'_{1p} = G'_1 \upharpoonright p$ and $\xi_p = \xi \upharpoonright p$ and $L'_{1q} = G'_1 \upharpoonright q$ and $\xi_q = \xi \upharpoonright q$, for some G'_1, for some ξ.
      - · Recall $G_1 \xrightarrow{\xi, p.e \twoheadrightarrow q.y} G'_1$. Then, by [$\to$1-Par1], $G_1 \parallel G_2 \xrightarrow{\xi, p.e \twoheadrightarrow q.y} G'_1 \parallel G_2$. Then, $G \xrightarrow{\xi, p.e \twoheadrightarrow q.y} G'_1 \parallel G_2$. Then, $G' = G'_1 \parallel G_2$ and $G_1 \parallel G_2 \xrightarrow{\xi, p.e \twoheadrightarrow q.y} G'$, for some G'.
      - · Recall $L'_p = L'_{1p} \parallel L_{2p}$ and $L'_q = L'_{1q} \parallel (G_2 \upharpoonright q)$. Then, $L'_p = (G'_1 \upharpoonright p) \parallel (G_2 \upharpoonright p)$ and $L'_q = (G'_1 \upharpoonright q) \parallel (G_2 \upharpoonright q)$. Then, by Defn. 26, $L'_p = (G'_1 \parallel G_2) \upharpoonright p$ and $L'_q = (G'_1 \parallel G_2) \upharpoonright q$. Then, $L'_p = G' \upharpoonright p$ and $L'_q = G' \upharpoonright q$.
    - ∗ **Case:** [$\to$1-Par2], such that $(G_2 \upharpoonright q) \xrightarrow{\xi_q, pq?y} L'_{2q}$, for some L'_{2q}.
      - · Recall $(G_1 \upharpoonright p) \xrightarrow{\xi_p, pq!e} L'_{1p}$ and $(G_2 \upharpoonright q) \xrightarrow{\xi_q, pq?y} L'_{2q}$. Then, by Lem. 54, $\mathsf{chan}(pq!e) \subseteq \mathsf{chan}(G_1 \upharpoonright p)$ and $\mathsf{chan}(pq?y) \subseteq \mathsf{chan}(G_2 \upharpoonright q)$. Then, by Defn. 17, $\{pq\} \subseteq \mathsf{chan}(G_1 \upharpoonright p)$ and $\{pq\} \subseteq \mathsf{chan}(G_2 \upharpoonright q)$. Then, $\{pq\} \subseteq \mathsf{chan}(G_1 \upharpoonright p) \cap \mathsf{chan}(G_2 \upharpoonright q)$.
      - · By Lem. 45, $\mathsf{chan}(G_1 \upharpoonright p) \subseteq \mathsf{chan}(G_1)$ and $\mathsf{chan}(G_2 \upharpoonright q) \subseteq \mathsf{chan}(G_2)$. Then, $\mathsf{chan}(G_1 \upharpoonright p) \cap \mathsf{chan}(G_2 \upharpoonright q) \subseteq \mathsf{chan}(G_1) \cap \mathsf{chan}(G_2)$.
      - · Recall $\checkmark_R(G)$. Then, $\checkmark_R(G_1 \parallel G_2)$. Then, by Defn. 27, $\mathsf{chan}(G_1) \cap \mathsf{chan}(G_2) = \emptyset$.
      - · Recall $\{pq\} \subseteq \mathsf{chan}(G_1 \upharpoonright p) \cap \mathsf{chan}(G_2 \upharpoonright q) \subseteq \mathsf{chan}(G_1) \cap \mathsf{chan}(G_2) = \emptyset$. Then, $\{pq\} \subseteq \emptyset$. Then, **false**.
- **Step:** [$\to$1-Par2]. Similar to [$\to$1-Par1].
- **Step:** [$\to$1-NIf4], such that $G \upharpoonright p = \textbf{if } \hat{\xi}_p|_n \; L_{1p}|_{R_{1p}} \; L_{2p}|_{R_{2p}}$ and $L'_p = \textbf{if } \hat{\xi}_p|_n \; L'_{1p}|_{R_{1p}} \; L_{2p}|_{R_{2p}}$ and $L_{1p} \xrightarrow{\xi_p, pq!e} L'_{1p}$ and $\mathsf{subj}(\xi_p) \cup \mathsf{subj}(pq!e) \subseteq R_{1p} \setminus R_{2p}$, for some $R_{1p}, R_{2p}, L_{1p}, L'_{1p}, L_{2p}, \hat{\xi}_p, n$.
  - Recall $G \upharpoonright p = \textbf{if } \hat{\xi}_p|_n \; L_{1p}|_{R_{1p}} \; L_{2p}|_{R_{2p}}$. Then, by Defn. 26, $G = \textbf{if } \hat{\xi}|_0 \; G_1|_{R_1} \; G_2|_{R_2}$ and $R_{1p} = R_1 \cap \{p\}$ and $R_{2p} = R_2 \cap \{p\}$ and $L_{1p} = G_1 \upharpoonright p$ and $L_{2p} = G_2 \upharpoonright p$ and $\hat{\xi}_p = \hat{\xi} \upharpoonright p$ and $n = |\mathsf{subj}(\hat{\xi}) \setminus (R_1 \cup R_2 \cup \{p\})|$, for some R_1, R_2, G_1, G_2, ξ.
  - Recall $(G \upharpoonright q) \xrightarrow{\xi_q, pq?y} L'_q$. Then, $(\textbf{if } \hat{\xi}|_0 \; G_1|_{R_1} \; G_2|_{R_2} \upharpoonright q) \xrightarrow{\xi_q, pq?y} L'_q$. Then, by Defn. 26, $\textbf{if } (\hat{\xi} \upharpoonright q)|_{|\mathsf{subj}(\hat{\xi}) \setminus (R_1 \cup R_2 \cup \{q\})|} \; (G_1 \upharpoonright q)|_{R_1 \cap \{q\}} \; (G_2 \upharpoonright q)|_{R_2 \cap \{q\}} \xrightarrow{\xi_q, pq?y} L'_q$ Then, by Defn. 31:
    - ∗ **Case:** [$\to$1-NIf1], such that $pq?q = \tau$. Then, **false**.
    - ∗ **Case:** [$\to$1-NIf2]. Similar to case [$\to$1-NIf1].
    - ∗ **Case:** [$\to$1-NIf3]. Similar to case [$\to$1-NIf1].
    - ∗ **Case:** [$\to$1-NIf4], such that $L'_q = \textbf{if } (\hat{\xi} \upharpoonright q)|_{|\mathsf{subj}(\hat{\xi}) \setminus (R_1 \cup R_2 \cup \{q\})|} \; L'_{1q}|_{R_1 \cap \{q\}} \; (G_2 \upharpoonright q)|_{R_2 \cap \{q\}}$ and $(G_1 \upharpoonright q) \xrightarrow{\xi_q, pq?y} L'_{1q}$ and $\mathsf{subj}(\xi_q) \cup \mathsf{subj}(pq?y) \subseteq (R_1 \cap \{q\}) \setminus (R_2 \cap \{q\})$, for some L'_{1q}.
      - · Recall $L_{1p} \xrightarrow{\xi_p, pq!e} L'_{1p}$. Then, $(G_1 \upharpoonright p) \xrightarrow{\xi_p, pq!e} L'_{1p}$.
      - · Recall $\checkmark_R(G)$. Then, $\checkmark_R(\textbf{if } \hat{\xi}|_0 \; G_1|_{R_1} \; G_2|_{R_2})$. Then, by Defn. 27, $\checkmark_R(G_1)$.
      - · Recall $(G_1 \upharpoonright p) \xrightarrow{\xi_p, pq!e} L'_{1p}$ and $(G_1 \upharpoonright q) \xrightarrow{\xi_q, pq?y} L'_{1q}$ and $\checkmark_R(G_1)$. Then, by induction, $G_1 \xrightarrow{\xi, p.e \twoheadrightarrow q.y} G'_1$ and $L'_{1p} = G'_1 \upharpoonright p$ and $\xi_p = \xi \upharpoonright p$ and $L'_{1q} = G'_1 \upharpoonright q$ and $\xi_q = \xi \upharpoonright q$, for some G'_1, for some ξ.
      - · Recall $G_1 \xrightarrow{\xi, p.e \twoheadrightarrow q.y} G'_1$. Then, by Lem. 53, $\mathsf{subj}(\xi) \subseteq \mathsf{subj}(p.e \twoheadrightarrow q.y)$. Then, $\mathsf{subj}(\xi) \cup \mathsf{subj}(p.e \twoheadrightarrow q.y) = \mathsf{subj}(p.e \twoheadrightarrow q.y)$.
      - · Recall $\mathsf{subj}(\xi_p) \cup \mathsf{subj}(pq!e) \subseteq R_{1p} \setminus R_{2p}$. Then, $\mathsf{subj}(\xi_p) \cup \mathsf{subj}(pq!e) \subseteq (R_1 \cap \{p\}) \setminus (R_2 \cap \{p\})$. Then,

$\mathsf{subj}(pq!e) \subseteq (R_1 \cap \{p\}) \setminus (R_2 \cap \{p\})$. Then, by Defn. 16, $\{p\} \subseteq (R_1 \cap \{p\}) \setminus (R_2 \cap \{p\})$. Then, $p \in (R_1 \cap \{p\}) \setminus (R_2 \cap \{p\})$. Then, $p \in R_1 \cap \{p\}$ and $p \notin R_2 \cap \{p\}$. Then, $p \in R_1$ and $p \notin R_2$. Then, $p \in R_1 \setminus R_2$.

- Recall $\mathsf{subj}(\xi_q) \cup \mathsf{subj}(pq?y) \subseteq (R_1 \cap \{q\}) \setminus (R_2 \cap \{q\})$. Then, $\mathsf{subj}(pq?y) \subseteq (R_1 \cap \{q\}) \setminus (R_2 \cap \{q\})$. Then, by Defn. 16, $\{q\} \subseteq (R_1 \cap \{q\}) \setminus (R_2 \cap \{q\})$. Then, $q \in (R_1 \cap \{q\}) \setminus (R_2 \cap \{q\})$. Then, $q \in R_1 \cap \{q\}$ and $q \notin R_2 \cap \{q\}$. Then, $q \in R_1$ and $q \notin R_2$. Then, $q \in R_1 \setminus R_2$.
- Recall $p, q \in R_1 \setminus R_2$. Then, $\{p, q\} \subseteq R_1 \setminus R_2$. Then, by Defn. 16, $\mathsf{subj}(p.e \twoheadrightarrow q.y) \subseteq R_1 \setminus R_2$. Then, $\mathsf{subj}(\xi) \cup \mathsf{subj}(p.e \twoheadrightarrow q.y) \subseteq R_1 \setminus R_2$.
- Recall $G_1 \xrightarrow{\xi, p.e \twoheadrightarrow q.y} G_1'$ and $\mathsf{subj}(\xi) \cup \mathsf{subj}(p.e \twoheadrightarrow q.y) \subseteq R_1 \setminus R_2$. Then, by [→1-NIf4], $\textbf{if}\ \hat{\xi}|_0\ G_1|_{R_1}\ G_2|_{R_2} \xrightarrow{\xi, p.e \twoheadrightarrow q.y} \textbf{if}\ \hat{\xi}|_0\ G_1'|_{R_1}\ G_2|_{R_2}$. Then, $G \xrightarrow{\xi, p.e \twoheadrightarrow q.y} \textbf{if}\ \hat{\xi}|_0\ G_1'|_{R_1}\ G_2|_{R_2}$. Then, $G' = \textbf{if}\ \hat{\xi}|_0\ G_1'|_{R_1}\ G_2|_{R_2}$ and $G \xrightarrow{\xi, p.e \twoheadrightarrow q.y} G'$, for some G'.
- Recall $L_p' = \textbf{if}\ \hat{\xi}_p|_n\ L_{1p}'|_{R_{1p}}\ L_{2p}|_{R_{2p}}$. Then, $L_p' = \textbf{if}\ (\hat{\xi} \upharpoonright p)|_{|\mathsf{subj}(\hat{\xi}) \setminus (R_1 \cup R_2 \cup \{p\})|}\ (G_1' \upharpoonright p)|_{R_1 \cap \{p\}}\ (G_2 \upharpoonright p)|_{R_2 \cap \{p\}}$. Then, by Defn. 26, $L_p' = \textbf{if}\ \hat{\xi}|_0\ G_1'|_{R_1}\ G_2|_{R_2} \upharpoonright p$. Then, $L_p' = G' \upharpoonright p$.
- Recall $L_q' = \textbf{if}\ \hat{\xi}_q|_{|\mathsf{subj}(\hat{\xi}) \setminus (R_1 \cup R_2 \cup \{q\})|}\ L_{1q}'|_{R_{1q}}\ L_{2q}|_{R_{2q}}$. Then, $L_q' = \textbf{if}\ (\hat{\xi} \upharpoonright q)|_{|\mathsf{subj}(\hat{\xi}) \setminus (R_1 \cup R_2 \cup \{q\})|}\ (G_1' \upharpoonright q)|_{R_1 \cap \{q\}}\ (G_2 \upharpoonright q)|_{R_2 \cap \{q\}}$. Then, by Defn. 26, $L_q' = \textbf{if}\ \hat{\xi}|_0\ G_1'|_{R_1}\ G_2|_{R_2} \upharpoonright q$. Then, $L_q' = G' \upharpoonright q$.

∗ **Case:** [→1-NIf5], such that $\mathsf{subj}(\xi_q) \cup \mathsf{subj}(pq?y) \subseteq (R_2 \cap \{q\}) \setminus (R_1 \cap \{q\})$.

- Recall $\checkmark_R(G)$. Then, $\checkmark_R(\textbf{if}\ \hat{\xi}|_0\ G_1|_{R_1}\ G_2|_{R_2})$. Then, by Defn. 27, $R_1 \neq \emptyset$ implies $R_2 = \emptyset$.
- Recall $\mathsf{subj}(\xi_p) \cup \mathsf{subj}(pq!e) \subseteq R_{1p} \setminus R_{2p}$. Then, $\mathsf{subj}(\xi_p) \cup \mathsf{subj}(pq!e) \subseteq (R_1 \cap \{p\}) \setminus (R_2 \cap \{p\})$. Then, $\mathsf{subj}(pq!e) \subseteq (R_1 \cap \{p\}) \setminus (R_2 \cap \{p\})$. Then, by Defn. 16, $\{p\} \subseteq (R_1 \cap \{p\}) \setminus (R_2 \cap \{p\})$. Then, $p \in (R_1 \cap \{p\}) \setminus (R_2 \cap \{p\})$. Then, $p \in R_1 \cap \{p\}$. Then, $p \in R_1$. Then, $R_1 \neq \emptyset$. Then, $R_2 = \emptyset$.
- Recall $\mathsf{subj}(\xi_q) \cup \mathsf{subj}(pq?y) \subseteq (R_2 \cap \{q\}) \setminus (R_1 \cap \{q\})$. Then, $\mathsf{subj}(\xi_q) \cup \mathsf{subj}(pq?y) \subseteq (\emptyset \cap \{q\}) \setminus (R_1 \cap \{q\})$. Then, $\mathsf{subj}(pq?y) \subseteq (\emptyset \cap \{q\}) \setminus (R_1 \cap \{q\})$. Then, by Defn. 16, $\{q\} \subseteq (\emptyset \cap \{q\}) \setminus (R_1 \cap \{q\})$. Then, $q \in (\emptyset \cap \{q\}) \setminus (R_1 \cap \{q\})$. Then, $q \in \emptyset \cap \{q\}$. Then, $q \in \emptyset$. Then, **false**.

– **Step:** [→1-NIf5]. Similar to case [→1-NIf4].

– **Step:** [→1-NWhile], such that $G \upharpoonright p = \textbf{while}\ \hat{\xi}_p|_n\ \{\texttt{true}\}\ \hat{L}_p|_\emptyset$ and $\textbf{if}\ \hat{\xi}_p|_n\ (\hat{L}_p\ ;\textbf{while}\ \hat{\xi}_p|_n\ \{\texttt{true}\}\ \hat{L}_p|_\emptyset)|_\emptyset\ \textbf{skip}|_\emptyset \xrightarrow{\xi_p, pq!e} L_p'$, for some $\hat{L}_p, \hat{\xi}_p, n$.

- Recall $G \upharpoonright p = \textbf{while}\ \hat{\xi}_p|_n\ \{\texttt{true}\}\ \hat{L}_p|_\emptyset$. Then, by Defn. 26, $G = \textbf{while}\ \hat{\xi}|_0\ \{\psi\}\ \hat{G}|_\emptyset$ and $\hat{L}_p = \hat{G} \upharpoonright p$ and $\hat{\xi}_p = \hat{\xi} \upharpoonright p$ and $n = |\mathsf{subj}(\hat{\xi}) \setminus \{p\}|$, for some $\hat{G}, \psi, \hat{\xi}$.
- Recall $\textbf{if}\ \hat{\xi}_p|_n\ (\hat{L}_p\ ;\textbf{while}\ \hat{\xi}_p|_n\ \{\texttt{true}\}\ \hat{L}_p|_\emptyset)|_\emptyset\ \textbf{skip}|_\emptyset \xrightarrow{\xi_p, pq!e} L_p'$. Then, $\textbf{if}\ (\hat{\xi} \upharpoonright p)|_{|\mathsf{subj}(\hat{\xi}) \setminus \{p\}|}\ ((\hat{G} \upharpoonright p)\ ;\textbf{while}\ (\hat{\xi} \upharpoonright p)|_{|\mathsf{subj}(\hat{\xi}) \setminus \{p\}|}\ \{\texttt{true}\}\ (\hat{G} \upharpoonright p)|_\emptyset)|_\emptyset\ \textbf{skip}|_\emptyset \xrightarrow{\xi_p, pq!e} L_p'$. Then, $\textbf{if}\ (\hat{\xi} \upharpoonright p)|_{|\mathsf{subj}(\hat{\xi}) \setminus (\emptyset \cup \emptyset \cup \{p\})|}\ ((\hat{G} \upharpoonright p)\ ;\textbf{while}\ (\hat{\xi} \upharpoonright p)|_{|\mathsf{subj}(\hat{\xi}) \setminus \{p\}|}\ \{\texttt{true}\}\ (\hat{G} \upharpoonright p)|_\emptyset)|_{\emptyset \cap \{p\}}\ \textbf{skip}|_{\emptyset \cap \{p\}} \xrightarrow{\xi_p, pq!e} L_p'$. Then, by Defn. 26, $\textbf{if}\ (\hat{\xi} \upharpoonright p)|_{|\mathsf{subj}(\hat{\xi}) \setminus (\emptyset \cup \emptyset \cup \{p\})|}\ ((\hat{G} \upharpoonright p)\ ;(\textbf{while}\ \hat{\xi}|_0\ \{\psi\}\ \hat{G}|_\emptyset \upharpoonright p))|_{\emptyset \cap \{p\}}\ \textbf{skip}|_{\emptyset \cap \{p\}} \xrightarrow{\xi_p, pq!e} L_p'$. Then, by Defn. 26, $\textbf{if}\ (\hat{\xi} \upharpoonright p)|_{|\mathsf{subj}(\hat{\xi}) \setminus (\emptyset \cup \emptyset \cup \{p\})|}\ (\hat{G}\ ;\textbf{while}\ \hat{\xi}|_0\ \{\psi\}\ \hat{G}|_\emptyset \upharpoonright p)|_{\emptyset \cap \{p\}}\ (\textbf{skip} \upharpoonright p)|_{\emptyset \cap \{p\}} \xrightarrow{\xi_p, pq!e} L_p'$. Then, by Defn. 26, $(\textbf{if}\ \hat{\xi}|_0\ (\hat{G}\ ;\textbf{while}\ \hat{\xi}|_0\ \{\psi\}\ \hat{G}|_\emptyset)|_\emptyset\ \textbf{skip}|_\emptyset \upharpoonright p) \xrightarrow{\xi_p, pq!e} L_p'$.
- Recall $(G \upharpoonright q) \xrightarrow{\xi_q, pq?y} L_q'$. Then, $(\textbf{while}\ \hat{\xi}|_0\ \{\psi\}\ \hat{G}|_\emptyset \upharpoonright q) \xrightarrow{\xi_q, pq?y} L_q'$. Then, by Defn. 26, $\textbf{while}\ (\hat{\xi} \upharpoonright q)|_{|\mathsf{subj}(\hat{\xi}) \setminus \{q\}|}\ \{\texttt{true}\}\ (\hat{G} \upharpoonright q)|_\emptyset \xrightarrow{\xi_q, pq?y} L_q'$. Then, by Defn. 31, $\textbf{if}\ (\hat{\xi} \upharpoonright q)|_{|\mathsf{subj}(\hat{\xi}) \setminus \{q\}|}\ ((\hat{G} \upharpoonright q)\ ;\textbf{while}\ (\hat{\xi} \upharpoonright q)|_{|\mathsf{subj}(\hat{\xi}) \setminus \{q\}|}\ \{\texttt{true}\}\ (\hat{G} \upharpoonright q)|_\emptyset)|_\emptyset\ \textbf{skip}|_\emptyset \xrightarrow{\xi_q, pq?y} L_q'$. Then, $\textbf{if}\ (\hat{\xi} \upharpoonright q)|_{|\mathsf{subj}(\hat{\xi}) \setminus (\emptyset \cup \emptyset \cup \{q\})|}\ ((\hat{G} \upharpoonright q)\ ;\textbf{while}\ (\hat{\xi} \upharpoonright q)|_{|\mathsf{subj}(\hat{\xi}) \setminus \{q\}|}\ \{\texttt{true}\}\ (\hat{G} \upharpoonright q)|_\emptyset)|_{\emptyset \cap \{q\}}\ \textbf{skip}|_{\emptyset \cap \{q\}} \xrightarrow{\xi_q, pq?y} L_q'$. Then, by Defn. 26, $\textbf{if}\ (\hat{\xi} \upharpoonright q)|_{|\mathsf{subj}(\hat{\xi}) \setminus (\emptyset \cup \emptyset \cup \{q\})|}\ ((\hat{G} \upharpoonright q)\ ;(\textbf{while}\ \hat{\xi}|_0\ \{\psi\}\ \hat{G}|_\emptyset \upharpoonright q))|_{\emptyset \cap \{q\}}\ \textbf{skip}|_{\emptyset \cap \{q\}} \xrightarrow{\xi_q, pq?y} L_q'$. Then, by Defn. 26, $\textbf{if}\ (\hat{\xi} \upharpoonright q)|_{|\mathsf{subj}(\hat{\xi}) \setminus (\emptyset \cup \emptyset \cup \{q\})|}\ (\hat{G}\ ;\textbf{while}\ \hat{\xi}|_0\ \{\psi\}\ \hat{G}|_\emptyset \upharpoonright q)|_{\emptyset \cap \{q\}}\ (\textbf{skip} \upharpoonright q)|_{\emptyset \cap \{q\}} \xrightarrow{\xi_q, pq?y} L_q'$. Then, by Defn. 26, $(\textbf{if}\ \hat{\xi}|_0\ (\hat{G}\ ;\textbf{while}\ \hat{\xi}|_0\ \{\psi\}\ \hat{G}|_\emptyset)|_\emptyset\ \textbf{skip}|_\emptyset \upharpoonright q) \xrightarrow{\xi_q, pq?y} L_q'$.

- Recall $\checkmark_R(G)$. Then, $\checkmark_R(\mathbf{while}\ \hat{\xi}|_0\ \{\psi\}\ \hat{G}|_\emptyset)$. Then, by Lem. 46, $\checkmark_R(\mathbf{if}\ \hat{\xi}|_0\ (\hat{G}\,;\mathbf{while}\ \hat{\xi}|_0\ \{\psi\}\ \hat{G}|_\emptyset)|_\emptyset\ \mathbf{skip}|_\emptyset)$.
- Recall $(\mathbf{if}\ \hat{\xi}|_0\ (\hat{G}\,;\mathbf{while}\ \hat{\xi}|_0\ \{\psi\}\ \hat{G}|_\emptyset)|_\emptyset\ \mathbf{skip}|_\emptyset \upharpoonright p) \xrightarrow{\xi_p, pq!e} L'_p$ and $(\mathbf{if}\ \hat{\xi}|_0\ (\hat{G}\,;\mathbf{while}\ \hat{\xi}|_0\ \{\psi\}\ \hat{G}|_\emptyset)|_\emptyset\ \mathbf{skip}|_\emptyset \upharpoonright q) \xrightarrow{\xi_q, pq?y} L'_q$ and $\checkmark_R(\mathbf{if}\ \hat{\xi}|_0\ (\hat{G}\,;\mathbf{while}\ \hat{\xi}|_0\ \{\psi\}\ \hat{G}|_\emptyset)|_\emptyset\ \mathbf{skip}|_\emptyset)$. Then, by induction, $\mathbf{if}\ \hat{\xi}|_0\ (\hat{G}\,;\mathbf{while}\ \hat{\xi}|_0\ \{\psi\}\ \hat{G}|_\emptyset)|_\emptyset\ \mathbf{skip}|_\emptyset \xrightarrow{\xi, p.e \twoheadrightarrow q.y} G'$ and $L'_p = G' \upharpoonright p$ and $\xi_p = \xi \upharpoonright p$ and $L'_q = G' \upharpoonright q$ and $\xi_q = \xi \upharpoonright q$, for some G', ξ.
- Recall $\mathbf{if}\ \hat{\xi}|_0\ (\hat{G}\,;\mathbf{while}\ \hat{\xi}|_0\ \{\psi\}\ \hat{G}|_\emptyset)|_\emptyset\ \mathbf{skip}|_\emptyset \xrightarrow{\xi, p.e \twoheadrightarrow q.y} G'$. Then, by [→1-NWHILE], $\mathbf{while}\ \hat{\xi}|_0\ \{\psi\}\ \hat{G}|_\emptyset \xrightarrow{\xi, p.e \twoheadrightarrow q.y} G'$. Then, $G \xrightarrow{\xi, p.e \twoheadrightarrow q.y} G'$. □

Proof (of Lem. 63). Recall $(G \upharpoonright r) \xrightarrow{\xi_r, i^{\hat{R}}_{\{r\}}} L'_r$. Then, by Defn. 31:

- **Base:** [→1-ACT], such that $G \upharpoonright r = i^{\hat{R}}_{\{r\}}$ and $L'_r = \mathbf{skip}$ and $\xi_r = \mathtt{true}$.
  - Recall $G \upharpoonright r = i^{\hat{R}}_{\{r\}}$. Then, by Defn. 26, $G = i^{\hat{R}}_{R}$.
  - By Defn. 27, not $\checkmark_R(i^{\hat{R}}_R)$. Then, not $\checkmark_R(G)$. Then, **false**.
- **Base:** [→1-IF1], such that $G \upharpoonright r = \hat{\hat{R}}.\mathbf{if}\ \hat{\xi}_r\ L_{1r}\ L_{2r}$ and $\hat{\hat{R}} = \hat{R}$ and $i = 1$, for some $\hat{\hat{R}}, \hat{\xi}_r, L_{1r}, L_{2r}$.
  - Recall $G \upharpoonright r = \hat{\hat{R}}.\mathbf{if}\ \hat{\xi}_r\ L_{1r}\ L_{2r}$. Then, by Defn. 26, $G = \hat{\hat{R}}.\mathbf{if}\ \hat{\xi}\ G_1\ G_2$, for some $G_1, G_2, \hat{\xi}$.
  - By [→1-IF1], $\hat{\hat{R}}.\mathbf{if}\ \hat{\xi}\ G_1\ G_2 \xrightarrow{\hat{\xi}^+, 1^{\hat{R}}_{\hat{R}}} G_1$. Then, $G \xrightarrow{\hat{\xi}^+, i^{\hat{R}}_{\hat{R}}} G_1$. Then, $G' = G_1$ and $\xi = \hat{\xi}^+$ and $G \xrightarrow{\xi, i^{\hat{R}}_{\hat{R}}} G'$, for some G', ξ.
  - Recall $(G \upharpoonright \hat{r}) \xrightarrow{\xi_{\hat{r}}, i^{\hat{R}}_{\{\hat{r}\}}} L'_{\hat{r}}$ for every $\hat{r} \in \hat{R}$. Then, $(\hat{\hat{R}}.\mathbf{if}\ \hat{\xi}\ G_1\ G_2 \upharpoonright \hat{r}) \xrightarrow{\xi_{\hat{r}}, 1^{\hat{R}}_{\{\hat{r}\}}} L'_{\hat{r}}$ for every $\hat{r} \in \hat{R}$. Then, by Defn. 26, $\hat{\hat{R}}.\mathbf{if}\ (\hat{\xi} \upharpoonright \hat{r})\ (G_1 \upharpoonright \hat{r})\ (G_2 \upharpoonright \hat{r}) \xrightarrow{\xi_{\hat{r}}, 1^{\hat{R}}_{\{\hat{r}\}}} L'_{\hat{r}}$ for every $\hat{r} \in \hat{R}$. Then, by Defn. 31, $L'_{\hat{r}} = G_1 \upharpoonright \hat{r}$ for every $\hat{r} \in \hat{R}$. Then, $L'_{\hat{r}} = G' \upharpoonright \hat{r}$ for every $r \in \hat{R}$.
  - Recall $(G \upharpoonright \hat{r}) \xrightarrow{\xi_{\hat{r}}, i^{\hat{R}}_{\{\hat{r}\}}} L'_{\hat{r}}$ for every $\hat{r} \in \hat{R}$. Then, $(\hat{\hat{R}}.\mathbf{if}\ \hat{\xi}\ G_1\ G_2 \upharpoonright \hat{r}) \xrightarrow{\xi_{\hat{r}}, 1^{\hat{R}}_{\{\hat{r}\}}} L'_{\hat{r}}$ for every $\hat{r} \in \hat{R}$. Then, by Defn. 26, $\hat{\hat{R}}.\mathbf{if}\ (\hat{\xi} \upharpoonright \hat{r})\ (G_1 \upharpoonright \hat{r})\ (G_2 \upharpoonright \hat{r}) \xrightarrow{\xi_{\hat{r}}, 1^{\hat{R}}_{\{\hat{r}\}}} L'_{\hat{r}}$ for every $\hat{r} \in \hat{R}$. Then, by Defn. 31, $\xi_{\hat{r}} = (\hat{\xi} \upharpoonright \hat{r})^+$ for every $\hat{r} \in \hat{R}$. Then, by Lem. 11, $\xi_{\hat{r}} = \hat{\xi}^+ \upharpoonright \hat{r}$ for every $\hat{r} \in \hat{R}$. Then, $\xi_{\hat{r}} = \xi \upharpoonright \hat{r}$ for every $r \in \hat{R}$.
- **Base:** [→1-IF2]. Similar to case [→1-IF1].
- **Base:** [→1-WHILE1], such that $G \upharpoonright r = \hat{\hat{R}}.\mathbf{while}\ \hat{\xi}_r\ \{\psi_r\}\ L_r$ and $\hat{\hat{R}} = \hat{R}$ and $i = 1$, for some $\hat{\hat{R}}, \psi_r, \hat{\xi}_r, L_r$.
  - Recall $G \upharpoonright r = \hat{\hat{R}}.\mathbf{while}\ \hat{\xi}_r\ \{\psi_r\}\ L_r$. Then, by Defn. 26, $G = \hat{\hat{R}}.\mathbf{while}\ \hat{\xi}\ \{\psi\}\ \hat{G}$, for some $\hat{G}, \psi, \hat{\xi}$.
  - By [→1-WHILE1], $\hat{\hat{R}}.\mathbf{while}\ \hat{\xi}\ \{\psi\}\ \hat{G} \xrightarrow{\hat{\xi}^+, 1^{\hat{R}}_{\hat{R}}} \hat{G}\,; \hat{\hat{R}}.\mathbf{while}\ \hat{\xi}\ \{\psi\}\ \hat{G}$. Then, $G \xrightarrow{\hat{\xi}^+, i^{\hat{R}}_{\hat{R}}} \hat{G}\,; \hat{\hat{R}}.\mathbf{while}\ \hat{\xi}\ \{\psi\}\ \hat{G}$. Then, $G' = \hat{G}\,; \hat{\hat{R}}.\mathbf{while}\ \hat{\xi}\ \{\psi\}\ \hat{G}$ and $\xi = \hat{\xi}^+$ and $G \xrightarrow{\xi, i^{\hat{R}}_{\hat{R}}} G'$, for some G', ξ.
  - Recall $(G \upharpoonright \hat{r}) \xrightarrow{\xi_{\hat{r}}, i^{\hat{R}}_{\{\hat{r}\}}} L'_{\hat{r}}$ for every $\hat{r} \in \hat{R}$. Then, $(\hat{\hat{R}}.\mathbf{while}\ \hat{\xi}\ \{\psi\}\ \hat{G} \upharpoonright \hat{r}) \xrightarrow{\xi_{\hat{r}}, 1^{\hat{R}}_{\{\hat{r}\}}} L'_{\hat{r}}$ for every $\hat{r} \in \hat{R}$. Then, by Defn. 26, $\hat{\hat{R}}.\mathbf{while}\ (\hat{\xi} \upharpoonright \hat{r})\ \{\mathtt{true}\}\ (\hat{G} \upharpoonright \hat{r}) \xrightarrow{\xi_{\hat{r}}, 1^{\hat{R}}_{\{\hat{r}\}}} L'_{\hat{r}}$ for every $\hat{r} \in \hat{R}$. Then, by Defn. 31, $L'_{\hat{r}} = (\hat{G} \upharpoonright \hat{r})\,; \hat{\hat{R}}.\mathbf{while}\ (\hat{\xi} \upharpoonright \hat{r})\ \{\mathtt{true}\}\ (\hat{G} \upharpoonright \hat{r})$ for every $\hat{r} \in \hat{R}$. Then, by Defn. 26, $L'_{\hat{r}} = (\hat{G} \upharpoonright \hat{r})\,; (\hat{\hat{R}}.\mathbf{while}\ \hat{\xi}\ \{\psi\}\ \hat{G} \upharpoonright \hat{r})$ for every $\hat{r} \in \hat{R}$. Then, by Defn. 26, $L'_{\hat{r}} = \hat{G}\,; \hat{\hat{R}}.\mathbf{while}\ \hat{\xi}\ \{\psi\}\ \hat{G} \upharpoonright \hat{r}$ for every $\hat{r} \in \hat{R}$. Then, $L'_{\hat{r}} = G' \upharpoonright \hat{r}$ for every $r \in \hat{R}$.
  - Recall $(G \upharpoonright \hat{r}) \xrightarrow{\xi_{\hat{r}}, i^{\hat{R}}_{\{\hat{r}\}}} L'_{\hat{r}}$ for every $\hat{r} \in \hat{R}$. Then, $(\hat{\hat{R}}.\mathbf{while}\ \hat{\xi}\ \{\psi\}\ \hat{G} \upharpoonright \hat{r}) \xrightarrow{\xi_{\hat{r}}, 1^{\hat{R}}_{\{\hat{r}\}}} L'_{\hat{r}}$ for every $\hat{r} \in \hat{R}$. Then, by Defn. 26, $\hat{\hat{R}}.\mathbf{while}\ (\hat{\xi} \upharpoonright \hat{r})\ \{\mathtt{true}\}\ (\hat{G} \upharpoonright \hat{r}) \xrightarrow{\xi_{\hat{r}}, 1^{\hat{R}}_{\{\hat{r}\}}} L'_{\hat{r}}$ for every $\hat{r} \in \hat{R}$. Then, by Defn. 31, $\xi_{\hat{r}} = (\hat{\xi} \upharpoonright \hat{r})^+$ for every $\hat{r} \in \hat{R}$. Then, by Lem. 11, $\xi_{\hat{r}} = \hat{\xi}^+ \upharpoonright \hat{r}$ for every $\hat{r} \in \hat{R}$. Then, $\xi_{\hat{r}} = \xi \upharpoonright \hat{r}$ for every $r \in \hat{R}$.
- **Base:** [→1-WHILE2], such that $G \upharpoonright r = \hat{\hat{R}}.\mathbf{while}\ \hat{\xi}_r\ \{\psi_r\}\ L_r$ and $\hat{\hat{R}} = \hat{R}$ and $i = 2$, for some $\hat{\hat{R}}, \psi_r, \hat{\xi}_r, L_r$.
  - Recall $G \upharpoonright r = \hat{\hat{R}}.\mathbf{while}\ \hat{\xi}_r\ \{\psi_r\}\ L_r$. Then, by Defn. 26, $G = \hat{\hat{R}}.\mathbf{while}\ \hat{\xi}\ \{\psi\}\ \hat{G}$, for some $\hat{G}, \psi, \hat{\xi}$.
  - By [→1-WHILE2], $\hat{\hat{R}}.\mathbf{while}\ \hat{\xi}\ \{\psi\}\ \hat{G} \xrightarrow{\hat{\xi}^-, 2^{\hat{R}}_{\hat{R}}} \mathbf{skip}$. Then, $G \xrightarrow{\hat{\xi}^-, i^{\hat{R}}_{\hat{R}}} \mathbf{skip}$. Then, $G' = \mathbf{skip}$ and $\xi = \hat{\xi}^-$ and $G \xrightarrow{\xi, i^{\hat{R}}_{\hat{R}}} G'$, for some G', ξ.
  - Recall $(G \upharpoonright \hat{r}) \xrightarrow{\xi_{\hat{r}}, i^{\hat{R}}_{\{\hat{r}\}}} L'_{\hat{r}}$ for every $\hat{r} \in \hat{R}$. Then, $(\hat{\hat{R}}.\mathbf{while}\ \hat{\xi}\ \{\psi\}\ \hat{G} \upharpoonright \hat{r}) \xrightarrow{\xi_{\hat{r}}, 2^{\hat{R}}_{\{\hat{r}\}}} L'_{\hat{r}}$ for every $\hat{r} \in \hat{R}$. Then, by Defn. 26, $\hat{\hat{R}}.\mathbf{while}\ (\hat{\xi} \upharpoonright \hat{r})\ \{\mathtt{true}\}\ (\hat{G} \upharpoonright \hat{r}) \xrightarrow{\xi_{\hat{r}}, 2^{\hat{R}}_{\{\hat{r}\}}} L'_{\hat{r}}$ for every $\hat{r} \in \hat{R}$. Then, by Defn. 31, $L'_{\hat{r}} = \mathbf{skip}$ for every $\hat{r} \in \hat{R}$. Then, by Defn. 26, $L'_{\hat{r}} = \mathbf{skip} \upharpoonright \hat{r}$ for every $\hat{r} \in \hat{R}$. Then, $L'_{\hat{r}} = G' \upharpoonright \hat{r}$ for every $r \in \hat{R}$.

- Recall $(G \upharpoonright \hat{r}) \xrightarrow{\xi_{\hat{r}}, i^{\hat{R}}_{\{\hat{r}\}}} L'_{\hat{r}}$ for every $\hat{r} \in \hat{R}$. Then, $(\hat{R}.\mathbf{while}\ \hat{\xi}\ \{\psi\}\ \hat{G} \upharpoonright \hat{r}) \xrightarrow{\xi_{\hat{r}}, 2^{\hat{R}}_{\{\hat{r}\}}} L'_{\hat{r}}$ for every $\hat{r} \in \hat{R}$. Then, by Defn. 26, $\hat{R}.\mathbf{while}\ (\hat{\xi} \upharpoonright \hat{r})\ \{\mathtt{true}\}\ (\hat{G} \upharpoonright \hat{r}) \xrightarrow{\xi_{\hat{r}}, 2^{\hat{R}}_{\{\hat{r}\}}} L'_{\hat{r}}$ for every $\hat{r} \in \hat{R}$. Then, by Defn. 31, $\xi_{\hat{r}} = (\hat{\xi} \upharpoonright \hat{r})^-$ for every $\hat{r} \in \hat{R}$. Then, by Lem. 11, $\xi_{\hat{r}} = \hat{\xi}^- \upharpoonright \hat{r}$ for every $\hat{r} \in \hat{R}$. Then, $\xi_{\hat{r}} = \xi \upharpoonright \hat{r}$ for every $r \in \hat{R}$.

– **Base:** [→1-NIf1], such that $i^{\hat{R}}_{\{r\}} = \tau$. Then, **false**.

– **Base:** [→1-NIf2], such that $G \upharpoonright r = \mathbf{if}\ \hat{\xi}_r|_n\ L_{1r}|_{R_{1r}}\ L_{2r}|_{R_{2r}}$ and $L'_r = \mathbf{if}\ \hat{\xi}_r|_n\ L_{1r}|_{R_{1r} \cup \{r\}}\ L_{2r}|_{R_{2r}}$ and $\xi_r = (\hat{\xi}_r \upharpoonright r)^+$ and $\hat{R} = \{r\}$ and $i = 1$ and $r \in \mathsf{subj}(\hat{\xi}_r) \setminus (R_{1r} \cup R_{2r})$, for some $R_{1r}, R_{2r}, L_{1r}, L_{2r}, \hat{\xi}_r, n$.
  - Recall $G \upharpoonright r = \mathbf{if}\ \hat{\xi}_r|_n\ L_{1r}|_{R_{1r}}\ L_{2r}|_{R_{2r}}$. Then, by Defn. 26, $G = \mathbf{if}\ \hat{\xi}|_0\ G_1|_{R_1}\ G_2|_{R_2}$ and $R_{1r} = R_1 \cap \{r\}$ and $R_{2r} = R_2 \cap \{r\}$ and $L_{1r} = G_1 \upharpoonright r$ and $L_{2r} = G_2 \upharpoonright r$ and $\hat{\xi}_r = \hat{\xi} \upharpoonright r$ and $n = |\mathsf{subj}(\hat{\xi}) \setminus (R_1 \cup R_2 \cup \{r\})|$, for some $R_1, R_2, G_1, G_2, \hat{\xi}$.
  - Recall $r \in \mathsf{subj}(\hat{\xi}_r) \setminus (R_{1r} \cup R_{2r})$. Then, $r \in \mathsf{subj}(\hat{\xi} \upharpoonright r) \setminus ((R_1 \cap \{r\}) \cup (R_2 \cap \{r\}))$. Then, $r \in \mathsf{subj}(\hat{\xi} \upharpoonright r)$ and $r \notin (R_1 \cap \{r\}) \cup (R_2 \cap \{r\})$.
  - By Lem. 10, $\mathsf{subj}(\hat{\xi} \upharpoonright r) \subseteq \mathsf{subj}(\hat{\xi})$.
  - Recall $r \in \mathsf{subj}(\hat{\xi} \upharpoonright r) \subseteq \mathsf{subj}(\hat{\xi})$. Then, $r \in \mathsf{subj}(\hat{\xi})$.
  - Recall $r \notin (R_1 \cap \{r\}) \cup (R_2 \cap \{r\})$. Then, $r \notin R_1 \cap \{r\}$ and $r \notin R_2 \cap \{r\}$. Then, $r \notin R_1$ and $r \notin R_2$. Then, $r \notin R_1 \cup R_2$.
  - Recall $r \in \mathsf{subj}(\hat{\xi})$ and $r \notin R_1 \cup R_2$. Then, $r \in \mathsf{subj}(\hat{\xi}) \setminus (R_1 \cup R_2)$. Then, by [→1-NIf2], $\mathbf{if}\ \hat{\xi}|_0\ G_1|_{R_1}\ G_2|_{R_2} \xrightarrow{\hat{\xi}^+ \upharpoonright r, 1^{\hat{R}}_{\{r\}}} \mathbf{if}\ \hat{\xi}|_0\ G_1|_{R_1 \cup \{r\}}\ G_2|_{R_2}$. Then, $G \xrightarrow{\hat{\xi}^+ \upharpoonright r, i^{\hat{R}}_{\hat{R}}} \mathbf{if}\ \hat{\xi}|_0\ G_1|_{R_1 \cup \{r\}}\ G_2|_{R_2}$. Then, $G' = \mathbf{if}\ \hat{\xi}|_0\ G_1|_{R_1 \cup \{r\}}\ G_2|_{R_2}$ and $\xi = \hat{\xi}^+ \upharpoonright r$ and $G \xrightarrow{\xi, i^{\hat{R}}_{\hat{R}}} G'$, for some G', ξ.
  - Recall $L'_r = \mathbf{if}\ \hat{\xi}_r|_n\ L_{1r}|_{R_{1r} \cup \{r\}}\ L_{2r}|_{R_{2r}}$. Then, $L'_r = \mathbf{if}\ (\hat{\xi} \upharpoonright r)|_{|\mathsf{subj}(\hat{\xi}) \setminus (R_1 \cup R_2 \cup \{r\})|}\ (G_1 \upharpoonright r)|_{(R_1 \cap \{r\}) \cup \{r\}}\ (G_2 \upharpoonright r)|_{R_2 \cap \{r\}}$. Then, $L'_r = \mathbf{if}\ (\hat{\xi} \upharpoonright r)|_{|\mathsf{subj}(\hat{\xi}) \setminus (R_1 \cup \{r\} \cup R_2 \cup \{r\})|}\ (G_1 \upharpoonright r)|_{(R_1 \cap \{r\}) \cup \{r\}}\ (G_2 \upharpoonright r)|_{R_2 \cap \{r\}}$. Then, $L'_r = \mathbf{if}\ (\hat{\xi} \upharpoonright r)|_{|\mathsf{subj}(\hat{\xi}) \setminus (R_1 \cup \{r\} \cup R_2 \cup \{r\})|}\ (G_1 \upharpoonright r)|_{(R_1 \cup \{r\}) \cap (\{r\} \cup \{r\})}\ (G_2 \upharpoonright r)|_{R_2 \cap \{r\}}$. Then, $L'_r = \mathbf{if}\ (\hat{\xi} \upharpoonright r)|_{|\mathsf{subj}(\hat{\xi}) \setminus (R_1 \cup \{r\} \cup R_2 \cup \{r\})|}\ (G_1 \upharpoonright r)|_{(R_1 \cup \{r\}) \cap \{r\}}\ (G_2 \upharpoonright r)|_{R_2 \cap \{r\}}$. Then, by Defn. 26, $L'_r = \mathbf{if}\ \hat{\xi}|_0\ G_1|_{R_1 \cup \{r\}}\ G_2|_{R_2} \upharpoonright r$. Then, $L'_r = G' \upharpoonright r$. Then, $L'_{\hat{r}} = G' \upharpoonright \hat{r}$ for every $\hat{r} \in \{r\}$. Then, $L'_{\hat{r}} = G' \upharpoonright \hat{r}$ for every $\hat{r} \in \hat{R}$.
  - Recall $\xi_r = (\hat{\xi}_r \upharpoonright r)^+$. Then, $\xi_r = (\hat{\xi} \upharpoonright r)^+ \upharpoonright r$. Then, by Lem. 11, $\xi_r = (\hat{\xi}^+ \upharpoonright r) \upharpoonright r$. Then, $\xi_r = \xi \upharpoonright r$. Then, $\xi_{\hat{r}} = \xi \upharpoonright \hat{r}$ for every $\hat{r} \in \{r\}$. Then, $\xi_{\hat{r}} = \xi \upharpoonright \hat{r}$ for every $\hat{r} \in \hat{R}$.

– **Base:** [→1-NIf3]. Similar to case [→1-NIf2].

– **Step:** [→1-Seq1], such that $G \upharpoonright r = L_{1r} ; L_{2r}$ and $L_{1r} \xrightarrow{\xi_r, i^{\hat{R}}_{\{r\}}} L'_{1r}$, for some L_{1r}, L'_{1r}, L_{2r}.
  - Recall $G \upharpoonright r = L_{1r} ; L_{2r}$. Then, by Defn. 26, $G = G_1 ; G_2$ and $L_{1r} = G_1 \upharpoonright r$, for some G_1, G_2.
  - Recall $L_{1r} \xrightarrow{\xi_r, i^{\hat{R}}_{\{r\}}} L'_{1r}$. Then, $(G_1 \upharpoonright r) \xrightarrow{\xi_r, i^{\hat{R}}_{\{r\}}} L'_{1r}$.
  - Recall $\checkmark_R(G)$. Then, $\checkmark_R(G_1 ; G_2)$. Then, by Defn. 27, $\checkmark_R(G_1)$.
  - Recall $(G \upharpoonright \hat{r}) \xrightarrow{\xi_{\hat{r}}, i^{\hat{R}}_{\{\hat{r}\}}} L'_{\hat{r}}$ for every $\hat{r} \in \hat{R}$. Then, $(G_1 ; G_2 \upharpoonright \hat{r}) \xrightarrow{\xi_{\hat{r}}, i^{\hat{R}}_{\{\hat{r}\}}} L'_{\hat{r}}$ for every $\hat{r} \in \hat{R}$. Then, by Defn. 26, $(G_1 \upharpoonright \hat{r}) ; (G_2 \upharpoonright \hat{r}) \xrightarrow{\xi_{\hat{r}}, i^{\hat{R}}_{\{\hat{r}\}}} L'_{\hat{r}}$ for every $\hat{r} \in \hat{R}$. Then, by Defn. 31:
    * **Case:** $L'_{\hat{r}} = L'_{1\hat{r}} ; (G_2 \upharpoonright \hat{r})$ and $(G_1 \upharpoonright \hat{r}) \xrightarrow{\xi_{\hat{r}}, i^{\hat{R}}_{\{\hat{r}\}}} L'_{1\hat{r}}$, for every $\hat{r} \in \hat{R}$, for some $\{L'_{1\hat{r}}\}_{\hat{r} \in \hat{R}}$.
      · Recall $(G_1 \upharpoonright r) \xrightarrow{\xi_r, i^{\hat{R}}_{\{r\}}} L'_{1r}$ and $r \in \hat{R}$, and $(G_1 \upharpoonright \hat{r}) \xrightarrow{\xi_{\hat{r}}, i^{\hat{R}}_{\{\hat{r}\}}} L'_{1\hat{r}}$ for every $\hat{r} \in \hat{R}$, and $\checkmark_R(G_1)$. Then, by induction, $G_1 \xrightarrow{\xi, i^{\hat{R}}_{\hat{R}}} G'_1$, and $L'_{1\hat{r}} = G'_1 \upharpoonright \hat{r}$ for every $\hat{r} \in \hat{R}$, and $\xi_{\hat{r}} = \xi \upharpoonright \hat{r}$ for every $\hat{r} \in \hat{R}$, for some G'_1, for some ξ.
      · Recall $G_1 \xrightarrow{\xi, i^{\hat{R}}_{\hat{R}}} G'_1$. Then, by [→1-Seq1], $G_1 ; G_2 \xrightarrow{\xi, i^{\hat{R}}_{\hat{R}}} G'_1 ; G_2$. Then, $G \xrightarrow{\xi, i^{\hat{R}}_{\hat{R}}} G'_1 ; G_2$. Then, $G' = G'_1 ; G_2$ and $G \xrightarrow{\xi, i^{\hat{R}}_{\hat{R}}} G'$, for some G'.
      · Recall $L'_{\hat{r}} = L'_{1\hat{r}} ; (G_2 \upharpoonright \hat{r})$ for every $\hat{r} \in \hat{R}$. Then, $L'_{\hat{r}} = (G'_1 \upharpoonright \hat{r}) ; (G_2 \upharpoonright \hat{r})$ for every $\hat{r} \in \hat{R}$. Then, by Defn. 26, $L'_{\hat{r}} = (G'_1 ; G_2) \upharpoonright \hat{r}$ for every $\hat{r} \in \hat{R}$. Then, $L'_{\hat{r}} = G' \upharpoonright \hat{r}$ for every $\hat{r} \in \hat{R}$.
    * **Case:** $\mathsf{subj}(G_1 \upharpoonright \hat{r}) \cap (\mathsf{subj}(\xi_{\hat{r}}) \cup \mathsf{subj}(i^{\hat{R}}_{\{\hat{r}\}})) = \emptyset$ and $\hat{r} \in \hat{R}$, for some $\hat{r}$.
      · Recall $(G_1 \upharpoonright r) \xrightarrow{\xi_r, i^{\hat{R}}_{\{r\}}} L'_{1r}$ and $\checkmark_R(G_1)$ and $i^{\hat{R}}_{\{r\}} \neq \tau$. Then, by Lem. 60, $\mathsf{subj}(\gamma) \subseteq \mathsf{subj}(G_1)$ and $i^{\hat{R}}_{\{r\}} = \gamma \upharpoonright r$, for some γ.
      · Recall $i^{\hat{R}}_{\{r\}} = \gamma \upharpoonright r$. Then, by Defn. 18, $\gamma = i^{\hat{R}}_{\hat{R}}$.
      · Recall $\mathsf{subj}(\gamma) \subseteq \mathsf{subj}(G_1)$. Then, $\mathsf{subj}(i^{\hat{R}}_{\hat{R}}) \subseteq \mathsf{subj}(G_1)$. Then, by Defn. 16, $\hat{R} \subseteq \mathsf{subj}(G_1)$.

- · Recall $\hat{r} \in \hat{R} \subseteq \mathsf{subj}(G_1)$. Then, $\hat{r} \in \mathsf{subj}(G_1)$. Then, by Lem. 44, $\hat{r} \in \mathsf{subj}(G_1 \upharpoonright \hat{r})$.
- · Recall $\hat{r} \in \{\hat{r}\}$. Then, by Defn. 16, $\hat{r} \in \mathsf{subj}(i^{\hat{R}}_{\{\hat{r}\}})$. Then, $\hat{r} \in \mathsf{subj}(\xi_{\hat{r}}) \cup \mathsf{subj}(i^{\hat{R}}_{\{\hat{r}\}})$.
- · Recall $\hat{r} \in \mathsf{subj}(G_1 \upharpoonright \hat{r})$ and $\hat{r} \in \mathsf{subj}(\xi_{\hat{r}}) \cup \mathsf{subj}(i^{\hat{R}}_{\{\hat{r}\}})$. Then, $\hat{r} \in \mathsf{subj}(G_1 \upharpoonright \hat{r}) \cap (\mathsf{subj}(\xi_{\hat{r}}) \cup \mathsf{subj}(i^{\hat{R}}_{\{\hat{r}\}}))$. Then, $\hat{r} \in \emptyset$. Then, **false**.

– **Step:** [$\to$1-Seq2], such that $G \upharpoonright r = L_{1r} ; L_{2r}$ and $\mathsf{subj}(G_1 \upharpoonright r) \cap (\mathsf{subj}(\xi_r) \cup \mathsf{subj}(i^{\hat{R}}_{\{r\}})) = \emptyset$ and $L_{2r} \xrightarrow{\xi_r, i^{\hat{R}}_{\{r\}}} L'_{2r}$, for some L_{1r}, L_{2r}, L'_{2r}.

- • Recall $G \upharpoonright r = L_{1r} ; L_{2r}$. Then, by Defn. 26, $G = G_1 ; G_2$ and $L_{2r} = G_2 \upharpoonright r$, for some G_1, G_2.
- • Recall $(G \upharpoonright \hat{r}) \xrightarrow{\xi_{\hat{r}}, i^{\hat{R}}_{\{\hat{r}\}}} L'_{\hat{r}}$ for every $\hat{r} \in \hat{R}$. Then, $(G_1 ; G_2 \upharpoonright \hat{r}) \xrightarrow{\xi_{\hat{r}}, i^{\hat{R}}_{\{\hat{r}\}}} L'_{\hat{r}}$ for every $\hat{r} \in \hat{R}$. Then, by Defn. 26, $(G_1 \upharpoonright \hat{r}) ; (G_2 \upharpoonright \hat{r}) \xrightarrow{\xi_{\hat{r}}, i^{\hat{R}}_{\{\hat{r}\}}} L'_{\hat{r}}$ for every $\hat{r} \in \hat{R}$. Then, by Defn. 31:
  - ∗ **Case:** $L'_{\hat{r}} = L'_{1\hat{r}} ; (G_2 \upharpoonright \hat{r})$ and $(G_1 \upharpoonright \hat{r}) \xrightarrow{\xi_{\hat{r}}, i^{\hat{R}}_{\{\hat{r}\}}} L'_{1\hat{r}}$ and $\hat{r} \in \hat{R}$, for some $L'_{1\hat{r}}, \hat{r}$.
    - · Recall $\checkmark_R(G)$. Then, $\checkmark_R(G_1 ; G_2)$. Then, by Defn. 27, $\checkmark_R(G_1)$.
    - · Recall $(G_1 \upharpoonright \hat{r}) \xrightarrow{\xi_{\hat{r}}, i^{\hat{R}}_{\{\hat{r}\}}} L'_{1\hat{r}}$ and $\checkmark_R(G_1)$ and $i^{\hat{R}}_{\{\hat{r}\}} \neq \tau$. Then, by Lem. 60, $\mathsf{subj}(\gamma) \subseteq \mathsf{subj}(G_1)$ and $i^{\hat{R}}_{\{\hat{r}\}} = \gamma \upharpoonright \hat{r}$, for some γ.
    - · Recall $i^{\hat{R}}_{\{\hat{r}\}} = \gamma \upharpoonright \hat{r}$. Then, by Defn. 18, $\gamma = i^{\hat{R}}_{\hat{R}}$.
    - · Recall $\mathsf{subj}(\gamma) \subseteq \mathsf{subj}(G_1)$. Then, $\mathsf{subj}(i^{\hat{R}}_{\hat{R}}) \subseteq \mathsf{subj}(G_1)$. Then, by Defn. 16, $\hat{R} \subseteq \mathsf{subj}(G_1)$.
    - · Recall $r \in \hat{R} \subseteq \mathsf{subj}(G_1)$. Then, $r \in \mathsf{subj}(G_1)$. Then, by Lem. 44, $r \in \mathsf{subj}(G_1 \upharpoonright r)$.
    - · Recall $r \in \{r\}$. Then, by Defn. 16, $r \in \mathsf{subj}(i^{\hat{R}}_{\{r\}})$. Then, $r \in \mathsf{subj}(\xi_r) \cup \mathsf{subj}(i^{\hat{R}}_{\{r\}})$.
    - · Recall $r \in \mathsf{subj}(G_1 \upharpoonright r)$ and $r \in \mathsf{subj}(\xi_r) \cup \mathsf{subj}(i^{\hat{R}}_{\{r\}})$. Then, $r \in \mathsf{subj}(G_1 \upharpoonright r) \cap (\mathsf{subj}(\xi_r) \cup \mathsf{subj}(i^{\hat{R}}_{\{r\}}))$. Then, $r \in \emptyset$. Then, **false**.
  - ∗ **Case:** $L'_{\hat{r}} = (G_1 \upharpoonright \hat{r}) ; L'_{2\hat{r}}$ and $\mathsf{subj}(G_1 \upharpoonright \hat{r}) \cap (\mathsf{subj}(\xi_{\hat{r}}) \cup \mathsf{subj}(i^{\hat{R}}_{\{\hat{r}\}})) = \emptyset$ and $(G_2 \upharpoonright \hat{r}) \xrightarrow{\xi_{\hat{r}}, i^{\hat{R}}_{\{\hat{r}\}}} L'_{2\hat{r}}$, for every $\hat{r} \in \hat{R}$, for some $\{L'_{2\hat{r}}\}_{\hat{r} \in \hat{R}}$.
    - · Recall $L_{2r} \xrightarrow{\xi_r, i^{\hat{R}}_{\{r\}}} L'_{2r}$. Then, $(G_2 \upharpoonright r) \xrightarrow{\xi_r, i^{\hat{R}}_{\{r\}}} L'_{2r}$.
    - · Recall $\checkmark_R(G)$. Then, $\checkmark_R(G_1 ; G_2)$. Then, by Defn. 27, $\checkmark_R(G_2)$.
    - · Recall $(G_2 \upharpoonright r) \xrightarrow{\xi_r, i^{\hat{R}}_{\{r\}}} L'_{2r}$ and $r \in \hat{R}$, and $(G_2 \upharpoonright \hat{r}) \xrightarrow{\xi_{\hat{r}}, i^{\hat{R}}_{\{\hat{r}\}}} L'_{2\hat{r}}$ for every $\hat{r} \in \hat{R}$, and $\checkmark_R(G_2)$. Then, by induction, $G_2 \xrightarrow{\xi, i^{\hat{R}}_{\hat{R}}} G'_2$, and $L'_{2\hat{r}} = G'_2 \upharpoonright \hat{r}$ for every $\hat{r} \in \hat{R}$, and $\xi_{\hat{r}} = \xi \upharpoonright \hat{r}$ for every $\hat{r} \in \hat{R}$, for some G'_2, for some ξ.
    - · Recall $\checkmark_R(G_2)$ and $G_2 \xrightarrow{\xi, i^{\hat{R}}_{\hat{R}}} G'_2$. Then, by Lem. 53, $\mathsf{subj}(\xi) \subseteq \mathsf{subj}(i^{\hat{R}}_{\hat{R}})$. Then, $\mathsf{subj}(\xi) \cup \mathsf{subj}(i^{\hat{R}}_{\hat{R}}) = \mathsf{subj}(i^{\hat{R}}_{\hat{R}})$.
    - · Recall $\mathsf{subj}(G_1 \upharpoonright \hat{r}) \cap (\mathsf{subj}(\xi_{\hat{r}}) \cup \mathsf{subj}(i^{\hat{R}}_{\{\hat{r}\}})) = \emptyset$ for every $\hat{r} \in \hat{R}$. Then, $\mathsf{subj}(G_1 \upharpoonright \hat{r}) \cap \mathsf{subj}(i^{\hat{R}}_{\{\hat{r}\}}) = \emptyset$ for every $\hat{r} \in \hat{R}$. Then, by Defn. 16, $\mathsf{subj}(G_1 \upharpoonright \hat{r}) \cap \{\hat{r}\} = \emptyset$ for every $\hat{r} \in \hat{R}$. Then, $\hat{r} \notin \mathsf{subj}(G_1 \upharpoonright \hat{r})$ for every $\hat{r} \in \hat{R}$. Then, by Lem. 44, $\hat{r} \notin \mathsf{subj}(G_1)$ for every $\hat{r} \in \hat{R}$. Then, $\mathsf{subj}(G_1) \cap \hat{R} = \emptyset$. Then, by Defn. 16, $\mathsf{subj}(G_1) \cap \mathsf{subj}(i^{\hat{R}}_{\hat{R}}) = \emptyset$. Then, $\mathsf{subj}(G_1) \cap (\mathsf{subj}(\xi) \cup \mathsf{subj}(i^{\hat{R}}_{\hat{R}})) = \emptyset$.
    - · Recall $\mathsf{subj}(G_1) \cap (\mathsf{subj}(\xi) \cup \mathsf{subj}(i^{\hat{R}}_{\hat{R}})) = \emptyset$ and $G_2 \xrightarrow{\xi, i^{\hat{R}}_{\hat{R}}} G'_2$. Then, by [$\to$1-Seq2], $G_1 ; G_2 \xrightarrow{\xi, i^{\hat{R}}_{\hat{R}}} G_1 ; G'_2$. Then, $G \xrightarrow{\xi, i^{\hat{R}}_{\hat{R}}} G_1 ; G'_2$. Then, $G' = G_1 ; G'_2$ and $G \xrightarrow{\xi, i^{\hat{R}}_{\hat{R}}} G'$, for some G'.
    - · Recall $L'_{\hat{r}} = (G_1 \upharpoonright \hat{r}) ; L'_{2\hat{r}}$ for every $\hat{r} \in \hat{R}$. Then, $L'_{\hat{r}} = (G_1 \upharpoonright \hat{r}) ; (G'_2 \upharpoonright \hat{r})$ for every $\hat{r} \in \hat{R}$. Then, by Defn. 26, $L'_{\hat{r}} = (G_1 ; G'_2) \upharpoonright \hat{r}$ for every $\hat{r} \in \hat{R}$. Then, $L'_{\hat{r}} = G' \upharpoonright \hat{r}$ for every $\hat{r} \in \hat{R}$.

– **Step:** [$\to$1-Par1], such that $G \upharpoonright r = L_{1r} \parallel L_{2r}$ and $L_{1r} \xrightarrow{\xi_r, i^{\hat{R}}_{\{r\}}} L'_{1r}$, for some L_{1r}, L'_{1r}, L_{2r}.

- • Recall $G \upharpoonright r = L_{1r} \parallel L_{2r}$. Then, by Defn. 26, $G = G_1 \parallel G_2$ and $L_{1r} = G_1 \upharpoonright r$, for some G_1, G_2.
- • Recall $L_{1r} \xrightarrow{\xi_r, i^{\hat{R}}_{\{r\}}} L'_{1r}$. Then, $(G_1 \upharpoonright r) \xrightarrow{\xi_r, i^{\hat{R}}_{\{r\}}} L'_{1r}$.
- • Recall $(G \upharpoonright \hat{r}) \xrightarrow{\xi_{\hat{r}}, i^{\hat{R}}_{\{\hat{r}\}}} L'_{\hat{r}}$ for every $\hat{r} \in \hat{R}$. Then, $(G_1 \parallel G_2 \upharpoonright \hat{r}) \xrightarrow{\xi_{\hat{r}}, i^{\hat{R}}_{\{\hat{r}\}}} L'_{\hat{r}}$ for every $\hat{r} \in \hat{R}$. Then, by Defn. 26, $(G_1 \upharpoonright \hat{r}) \parallel (G_2 \upharpoonright \hat{r}) \xrightarrow{\xi_{\hat{r}}, i^{\hat{R}}_{\{\hat{r}\}}} L'_{\hat{r}}$ for every $\hat{r} \in \hat{R}$. Then, by Defn. 31:
  - ∗ **Case:** $L'_{\hat{r}} = L'_{1\hat{r}} \parallel (G_2 \upharpoonright \hat{r})$ and $(G_1 \upharpoonright \hat{r}) \xrightarrow{\xi_{\hat{r}}, i^{\hat{R}}_{\{\hat{r}\}}} L'_{1\hat{r}}$, for every $\hat{r} \in \hat{R}$, for some $\{L'_{1\hat{r}}\}_{\hat{r} \in \hat{R}}$.
    - · Recall $\checkmark_R(G)$. Then, $\checkmark_R(G_1 \parallel G_2)$. Then, by Defn. 27, $\checkmark_R(G_1)$.
    - · Recall $(G_1 \upharpoonright r) \xrightarrow{\xi_r, i^{\hat{R}}_{\{r\}}} L'_{1r}$ and $r \in \hat{R}$, and $(G_1 \upharpoonright \hat{r}) \xrightarrow{\xi_{\hat{r}}, i^{\hat{R}}_{\{\hat{r}\}}} L'_{1\hat{r}}$ for every $\hat{r} \in \hat{R}$, and $\checkmark_R(G_1)$. Then, by induction, $G_1 \xrightarrow{\xi, i^{\hat{R}}_{\hat{R}}} G'_1$, and $L'_{1\hat{r}} = G'_1 \upharpoonright \hat{r}$ for every $\hat{r} \in \hat{R}$, and $\xi_{\hat{r}} = \xi \upharpoonright \hat{r}$ for every $\hat{r} \in \hat{R}$, for some G'_1, for some ξ.

- Recall $G_1 \xrightarrow{\xi, i^{\hat{R}}_{\hat{R}}} G'_1$. Then, by [→1-PAR1], $G_1 \parallel G_2 \xrightarrow{\xi, i^{\hat{R}}_{\hat{R}}} G'_1 \parallel G_2$. Then, $G \xrightarrow{\xi, i^{\hat{R}}_{\hat{R}}} G'_1 \parallel G_2$. Then, $G' = G'_1 \parallel G_2$ and $G \xrightarrow{\xi, i^{\hat{R}}_{\hat{R}}} G'$, for some G'.
- Recall $L'_{\hat{r}} = L'_{1\hat{r}} \parallel (G_2 \upharpoonright \hat{r})$ for every $\hat{r} \in \hat{R}$. Then, $L'_{\hat{r}} = (G'_1 \upharpoonright \hat{r}) \parallel (G_2 \upharpoonright \hat{r})$ for every $\hat{r} \in \hat{R}$. Then, by Defn. 26, $L'_{\hat{r}} = (G'_1 \parallel G_2) \upharpoonright \hat{r}$ for every $\hat{r} \in \hat{R}$. Then, $L'_{\hat{r}} = G' \upharpoonright \hat{r}$ for every $\hat{r} \in \hat{R}$.

  * **Case:** $(G_2 \upharpoonright \hat{r}) \xrightarrow{\xi_{\hat{r}}, i^{\hat{R}}_{\{\hat{r}\}}} L'_{2\hat{r}}$ and $\hat{r} \in \hat{R}$, for some $L_{2\hat{r}}, \hat{r}$.
    - Recall $(G_1 \upharpoonright r) \xrightarrow{\xi_r, i^{\hat{R}}_{\{r\}}} L'_{1r}$ and $(G_2 \upharpoonright \hat{r}) \xrightarrow{\xi_{\hat{r}}, i^{\hat{R}}_{\{\hat{r}\}}} L'_{2\hat{r}}$. Then, by Lem. 54, $\mathsf{chan}(i^{\hat{R}}_{\{\hat{r}\}}) \subseteq \mathsf{chan}(G_1 \upharpoonright r)$ and $\mathsf{chan}(i^{\hat{R}}_{\{\hat{r}\}}) \subseteq \mathsf{chan}(G_2 \upharpoonright \hat{r})$. Then, by Defn. 17, $\{\hat{R}\} \subseteq \mathsf{chan}(G_1 \upharpoonright r)$ and $\{\hat{R}\} \subseteq \mathsf{chan}(G_2 \upharpoonright \hat{r})$. Then, $\{\hat{R}\} \subseteq \mathsf{chan}(G_1 \upharpoonright r) \cap \mathsf{chan}(G_2 \upharpoonright \hat{r})$.
    - By Lem. 45, $\mathsf{chan}(G_1 \upharpoonright r) \subseteq \mathsf{chan}(G_1)$ and $\mathsf{chan}(G_2 \upharpoonright \hat{r}) \subseteq \mathsf{chan}(G_2)$. Then, $\mathsf{chan}(G_1 \upharpoonright r) \cap \mathsf{chan}(G_2 \upharpoonright \hat{r}) \subseteq \mathsf{chan}(G_1) \cap \mathsf{chan}(G_2)$.
    - Recall $\checkmark_R(G)$. Then, $\checkmark_R(G_1 \parallel G_2)$. Then, by Defn. 27, $\mathsf{chan}(G_1) \cap \mathsf{chan}(G_2) = \emptyset$.
    - Recall $\{\hat{R}\} \subseteq \mathsf{chan}(G_1 \upharpoonright r) \cap \mathsf{chan}(G_2 \upharpoonright \hat{r}) \subseteq \mathsf{chan}(G_1) \cap \mathsf{chan}(G_2) = \emptyset$. Then, $\{\hat{R}\} \subseteq \emptyset$. Then, **false**.

– **Step:** [→1-PAR2]. Similar to case [→1-PAR1].

– **Step:** [→1-NIF4], such that $G \upharpoonright r = \mathbf{if}\ \hat{\xi}_r|_n\ L_{1r}|_{R_{1r}}\ L_{2r}|_{R_{2r}}$ and $L'_r = \mathbf{if}\ \hat{\xi}_r|_n\ L'_{1r}|_{R_{1r}}\ L_{2r}|_{R_{2r}}$ and $L_{1r} \xrightarrow{\xi_r, i^{\hat{R}}_{\{r\}}} L'_{1r}$ and $\mathsf{subj}(\xi_r) \cup \mathsf{subj}(i^{\hat{R}}_{\{r\}}) \subseteq R_{1r} \setminus R_{2r}$, for some $R_{1r}, R_{2r}, L_{1r}, L'_{1r}, L_{2r}, \hat{\xi}_r, n$.

- Recall $G \upharpoonright r = \mathbf{if}\ \hat{\xi}_r|_n\ L_{1r}|_{R_{1r}}\ L_{2r}|_{R_{2r}}$. Then, by Defn. 26, $G = \mathbf{if}\ \hat{\xi}|_0\ G_1|_{R_1}\ G_2|_{R_2}$ and $R_{1r} = R_1 \cap \{r\}$ and $R_{2r} = R_2 \cap \{r\}$ and $L_{1r} = G_1 \upharpoonright r$ and $L_{2r} = G_2 \upharpoonright r$ and $\hat{\xi}_r = \hat{\xi} \upharpoonright r$ and $n = |\mathsf{subj}(\hat{\xi}) \setminus (R_1 \cup R_2 \cup \{r\})|$, for some $R_1, R_2, G_1, G_2, \hat{\xi}$.
- Recall $(G \upharpoonright \hat{r}) \xrightarrow{\xi_{\hat{r}}, i^{\hat{R}}_{\{\hat{r}\}}} L'_{\hat{r}}$ for every $\hat{r} \in \hat{R}$. Then, $(\mathbf{if}\ \hat{\xi}|_0\ G_1|_{R_1}\ G_2|_{R_2} \upharpoonright \hat{r}) \xrightarrow{\xi_{\hat{r}}, i^{\hat{R}}_{\{\hat{r}\}}} L'_{\hat{r}}$ for every $\hat{r} \in \hat{R}$. Then, by Defn. 26, $\mathbf{if}\ (\hat{\xi} \upharpoonright \hat{r})|_{|\mathsf{subj}(\hat{\xi}) \setminus (R_1 \cup R_2 \cup \{\hat{r}\})|}\ (G_1 \upharpoonright \hat{r})|_{R_1 \cap \{\hat{r}\}}\ (G_2 \upharpoonright \hat{r})|_{R_2 \cap \{\hat{r}\}} \xrightarrow{\xi_{\hat{r}}, i^{\hat{R}}_{\{\hat{r}\}}} L'_{\hat{r}}$ for every $\hat{r} \in \hat{R}$. Then, by Defn. 31:

  * **Case:** $i^{\hat{R}}_{\{r\}} = \tau$. Then, **false**.
  * **Case:** $\hat{R} = \{\hat{r}\}$ and $\hat{r} \in \mathsf{subj}(\hat{\xi} \upharpoonright \hat{r}) \setminus ((R_1 \cap \{\hat{r}\}) \cup (R_2 \cap \{\hat{r}\}))$, for some $\hat{r}$.
    - Recall $\mathsf{subj}(\xi_r) \cup \mathsf{subj}(i^{\hat{R}}_{\{r\}}) \subseteq R_{1r} \setminus R_{2r}$. Then, $\mathsf{subj}(\xi_r) \cup \mathsf{subj}(i^{\hat{R}}_{\{r\}}) \subseteq (R_1 \cap \{r\}) \setminus (R_2 \cap \{r\})$. Then, $\mathsf{subj}(i^{\hat{R}}_{\{r\}}) \subseteq (R_1 \cap \{r\}) \setminus (R_2 \cap \{r\})$. Then, by Defn. 16, $\{r\} \subseteq (R_1 \cap \{r\}) \setminus (R_2 \cap \{r\})$. Then, $r \in (R_1 \cap \{r\}) \setminus (R_2 \cap \{r\})$. Then, $r \in R_1 \cap \{r\}$. Then, $r \in R_1$.
    - Recall $r \in \hat{R}$. Then, $r \in \{\hat{r}\}$. Then, $r = \hat{r}$.
    - Recall $\hat{r} \in \mathsf{subj}(\hat{\xi} \upharpoonright \hat{r}) \setminus ((R_1 \cap \{\hat{r}\}) \cup (R_2 \cap \{\hat{r}\}))$. Then, $r \in \mathsf{subj}(\hat{\xi} \upharpoonright r) \setminus ((R_1 \cap \{r\}) \cup (R_2 \cap \{r\}))$. Then, $r \notin (R_1 \cap \{r\}) \cup (R_2 \cap \{r\})$. Then, $r \notin R_1 \cap \{r\}$. Then, $r \notin R_1$. Then, **false**.
  * **Case:** $\mathsf{subj}(\xi_{\hat{r}}) \cup \mathsf{subj}(i^{\hat{R}}_{\{\hat{r}\}}) \subseteq (R_2 \cap \{\hat{r}\}) \setminus (R_1 \cap \{\hat{r}\})$, for some $\hat{r}$.
    - Recall $\checkmark_R(G)$. Then, $\checkmark_R(\mathbf{if}\ \hat{\xi}|_0\ G_1|_{R_1}\ G_2|_{R_2})$. Then, by Defn. 27, $R_1 \neq \emptyset$ implies $R_2 = \emptyset$.
    - Recall $\mathsf{subj}(\xi_r) \cup \mathsf{subj}(i^{\hat{R}}_{\{r\}}) \subseteq R_{1r} \setminus R_{2r}$. Then, $\mathsf{subj}(\xi_r) \cup \mathsf{subj}(i^{\hat{R}}_{\{r\}}) \subseteq (R_1 \cap \{r\}) \setminus (R_2 \cap \{r\})$. Then, $\mathsf{subj}(i^{\hat{R}}_{\{r\}}) \subseteq (R_1 \cap \{r\}) \setminus (R_2 \cap \{r\})$. Then, by Defn. 16, $\{r\} \subseteq (R_1 \cap \{r\}) \setminus (R_2 \cap \{r\})$. Then, $r \in (R_1 \cap \{r\}) \setminus (R_2 \cap \{r\})$. Then, $r \in R_1 \cap \{r\}$. Then, $r \in R_1$. Then, $R_1 \neq \emptyset$. Then, $R_2 = \emptyset$.
    - Recall $\mathsf{subj}(\xi_{\hat{r}}) \cup \mathsf{subj}(i^{\hat{R}}_{\{\hat{r}\}}) \subseteq (R_2 \cap \{\hat{r}\}) \setminus (R_1 \cap \{\hat{r}\})$. Then, $\mathsf{subj}(\xi_{\hat{r}}) \cup \mathsf{subj}(i^{\hat{R}}_{\{\hat{r}\}}) \subseteq (\emptyset \cap \{\hat{r}\}) \setminus (R_1 \cap \{\hat{r}\})$. Then, $\mathsf{subj}(i^{\hat{R}}_{\{\hat{r}\}}) \subseteq (\emptyset \cap \{\hat{r}\}) \setminus (R_1 \cap \{\hat{r}\})$. Then, by Defn. 16, $\{\hat{r}\} \subseteq (\emptyset \cap \{\hat{r}\}) \setminus (R_1 \cap \{\hat{r}\})$. Then, $\hat{r} \in (\emptyset \cap \{\hat{r}\}) \setminus (R_1 \cap \{\hat{r}\})$. Then, $\hat{r} \in \emptyset \cap \{\hat{r}\}$. Then, $\hat{r} \in \emptyset$. Then, **false**.
  * **Case:** $L'_{\hat{r}} = \mathbf{if}\ (\hat{\xi} \upharpoonright \hat{r})|_{|\mathsf{subj}(\hat{\xi}) \setminus (R_1 \cup R_2 \cup \{\hat{r}\})|}\ L'_{1\hat{r}}|_{R_1 \cap \{\hat{r}\}}\ (G_2 \upharpoonright \hat{r})|_{R_2 \cap \{\hat{r}\}}$ and $(G_1 \upharpoonright \hat{r}) \xrightarrow{\xi_{\hat{r}}, i^{\hat{R}}_{\{\hat{r}\}}} L'_{1\hat{r}}$ and $\mathsf{subj}(\xi_{\hat{r}}) \cup \mathsf{subj}(i^{\hat{R}}_{\{\hat{r}\}}) \subseteq (R_1 \cap \{\hat{r}\}) \setminus (R_2 \cap \{\hat{r}\})$, for every $\hat{r} \in \hat{R}$, for some $\{L'_{1\hat{r}}\}_{\hat{r} \in \hat{R}}$.
    - Recall $L_{1r} \xrightarrow{\xi_r, i^{\hat{R}}_{\{r\}}} L'_{1r}$. Then, $(G_1 \upharpoonright r) \xrightarrow{\xi_r, i^{\hat{R}}_{\{r\}}} L'_{1r}$.

- Recall $\checkmark_R(G)$. $\checkmark_R(\mathbf{if}\ \hat{\xi}|_0\ G_1|_{R_1}\ G_2|_{R_2})$. Then, by Defn. 27, $\checkmark_R(G_1)$.
- Recall $(G_1 \restriction r) \xrightarrow{\xi_r, i^{\hat{R}}_{\{r\}}} L'_{1r}$ and $r \in R$, and $(G_1 \restriction \hat{r}) \xrightarrow{\xi_{\hat{r}}, i^{\hat{R}}_{\{\hat{r}\}}} L'_{1\hat{r}}$ for every $\hat{r} \in \hat{R}$, and $\checkmark_R(G_1)$. Then, by induction, $G_1 \xrightarrow{\xi, i^{\hat{R}}_{\hat{R}}} G'_1$, and $L'_{1\hat{r}} = G'_1 \restriction \hat{r}$ for every $\hat{r} \in \hat{R}$, and $\xi_{\hat{r}} = \xi \restriction \hat{r}$ for every $\hat{r} \in \hat{R}$, for some G'_1, for some ξ.
- Recall $G_1 \xrightarrow{\xi, i^{\hat{R}}_{\hat{R}}} G'_1$. Then, by Lem. 53, $\mathsf{subj}(\xi) \subseteq \mathsf{subj}(i^{\hat{R}}_{\hat{R}})$. Then, $\mathsf{subj}(\xi) \cup \mathsf{subj}(i^{\hat{R}}_{\hat{R}}) = \mathsf{subj}(i^{\hat{R}}_{\hat{R}})$.
- Recall $\mathsf{subj}(\xi_{\hat{r}}) \cup \mathsf{subj}(i^{\hat{R}}_{\{\hat{r}\}}) \subseteq (R_1 \cap \{\hat{r}\}) \setminus (R_2 \cap \{\hat{r}\})$ for every $\hat{r} \in \hat{R}$. Then, $\mathsf{subj}(i^{\hat{R}}_{\{\hat{r}\}}) \subseteq (R_1 \cap \{\hat{r}\}) \setminus (R_2 \cap \{\hat{r}\})$ for every $\hat{r} \in \hat{R}$. Then, by Defn. 16, $\{\hat{r}\} \subseteq (R_1 \cap \{\hat{r}\}) \setminus (R_2 \cap \{\hat{r}\})$ for every $\hat{r} \in \hat{R}$. Then, $\hat{r} \in (R_1 \cap \{\hat{r}\}) \setminus (R_2 \cap \{\hat{r}\})$ for every $\hat{r} \in \hat{R}$. Then, $\hat{r} \in R_1 \cap \{\hat{r}\}$ and $\hat{r} \notin R_2 \cap \{\hat{r}\}$, for every $\hat{r} \in \hat{R}$. Then, $\hat{r} \in R_1$ and $\hat{r} \notin R_2$, for every $\hat{r} \in \hat{R}$. Then, $\hat{r} \in R_1 \setminus R_2$ for every $\hat{r} \in \hat{R}$. Then, $\hat{R} \subseteq R_1 \setminus R_2$. Then, by Defn. 16, $\mathsf{subj}(i^{\hat{R}}_{\hat{R}}) \subseteq R_1 \setminus R_2$. Then, $\mathsf{subj}(\xi) \cup \mathsf{subj}(i^{\hat{R}}_{\hat{R}}) \subseteq R_1 \setminus R_2$.
- Recall $G_1 \xrightarrow{\xi, i^{\hat{R}}_{\hat{R}}} G'_1$ and $\mathsf{subj}(\xi) \cup \mathsf{subj}(i^{\hat{R}}_{\hat{R}}) \subseteq R_1 \setminus R_2$. Then, by [$\rightarrow$1-NIf4], $\mathbf{if}\ \hat{\xi}|_0\ G_1|_{R_1}\ G_2|_{R_2} \xrightarrow{\xi, i^{\hat{R}}_{\hat{R}}} \mathbf{if}\ \hat{\xi}|_0\ G'_1|_{R_1}\ G_2|_{R_2}$. Then, $G \xrightarrow{\xi, i^{\hat{R}}_{\hat{R}}} \mathbf{if}\ \hat{\xi}|_0\ G'_1|_{R_1}\ G_2|_{R_2}$. Then, $G' = \mathbf{if}\ \hat{\xi}|_0\ G'_1|_{R_1}\ G_2|_{R_2}$ and $G \xrightarrow{\xi, i^{\hat{R}}_{\hat{R}}} G'$, for some G'.
- Recall $L'_{\hat{r}} = \mathbf{if}\ (\hat{\xi} \restriction \hat{r})|_{|\mathsf{subj}(\hat{\xi}) \setminus (R_1 \cup R_2 \cup \{\hat{r}\})|}\ L'_{1\hat{r}}|_{R_1 \cap \{\hat{r}\}}\ (G_2 \restriction \hat{r})|_{R_2 \cap \{\hat{r}\}}$ for every $\hat{r} \in \hat{R}$. Then, $L'_{\hat{r}} = \mathbf{if}\ (\hat{\xi} \restriction \hat{r})|_{|\mathsf{subj}(\hat{\xi}) \setminus (R_1 \cup R_2 \cup \{\hat{r}\})|}\ (G'_1 \restriction \hat{r})|_{R_1 \cap \{\hat{r}\}}\ (G_2 \restriction \hat{r})|_{R_2 \cap \{\hat{r}\}}$ for every $\hat{r} \in \hat{R}$. Then, by Defn. 26, $L'_{\hat{r}} = \mathbf{if}\ \hat{\xi}|_0\ G'_1|_{R_1}\ G_2|_{R_2} \restriction \hat{r}$ for every $\hat{r} \in \hat{R}$. Then, $L'_{\hat{r}} = G' \restriction \hat{r}$ for every $\hat{r} \in \hat{R}$.

– **Step:** [$\rightarrow$1-NIf5]. Similar to case [$\rightarrow$1-NIf4].

– **Step:** [$\rightarrow$1-NWhile], such that $G \restriction r = \mathbf{while}\ \hat{\xi}_r|_n\ \{\mathtt{true}\}\ \hat{L}_r|_\emptyset$ and $\mathbf{if}\ \hat{\xi}_r|_n\ (\hat{L}_r\,;\mathbf{while}\ \hat{\xi}_r|_n\ \{\mathtt{true}\}\ \hat{L}_r|_\emptyset)|_\emptyset\ \mathbf{skip}|_\emptyset \xrightarrow{\xi_r, i^{\hat{R}}_{\{r\}}} L'_r$, for some $\hat{L}_r, \hat{\xi}_r, n$.

- Recall $G \restriction r = \mathbf{while}\ \hat{\xi}_r|_n\ \{\mathtt{true}\}\ \hat{L}_r|_\emptyset$. Then, by Defn. 26, $G = \mathbf{while}\ \hat{\xi}|_0\ \{\psi\}\ \hat{G}|_\emptyset$ and $\hat{L}_r = \hat{G} \restriction r$ and $\hat{\xi}_r = \hat{\xi} \restriction r$ and $n = |\mathsf{subj}(\hat{\xi}) \setminus \{r\}|$, for some $\hat{G}, \psi, \hat{\xi}$.
- Recall $\mathbf{if}\ \hat{\xi}_r|_n\ (\hat{L}_r\,;\mathbf{while}\ \hat{\xi}_r|_n\ \{\mathtt{true}\}\ \hat{L}_r|_\emptyset)|_\emptyset\ \mathbf{skip}|_\emptyset \xrightarrow{\xi_r, i^{\hat{R}}_{\{r\}}} L'_r$. Then, $\mathbf{if}\ (\hat{\xi} \restriction r)|_{|\mathsf{subj}(\hat{\xi}) \setminus \{r\}|}\ ((\hat{G} \restriction r)\,;\mathbf{while}\ (\hat{\xi} \restriction r)|_{|\mathsf{subj}(\hat{\xi}) \setminus \{r\}|}\ \{\mathtt{true}\}\ (\hat{G} \restriction r)|_\emptyset)|_\emptyset\ \mathbf{skip}|_\emptyset \xrightarrow{\xi_r, i^{\hat{R}}_{\{r\}}} L'_r$. Then, $\mathbf{if}\ (\hat{\xi} \restriction r)|_{|\mathsf{subj}(\hat{\xi}) \setminus (\emptyset \cup \emptyset \cup \{r\})|}\ ((\hat{G} \restriction r)\,;\mathbf{while}\ (\hat{\xi} \restriction r)|_{|\mathsf{subj}(\hat{\xi}) \setminus \{r\}|}\ \{\mathtt{true}\}\ (\hat{G} \restriction r)|_\emptyset)|_{\emptyset \cap \{r\}}\ \mathbf{skip}|_{\emptyset \cap \{r\}} \xrightarrow{\xi_r, i^{\hat{R}}_{\{r\}}} L'_r$. Then, by Defn. 26, $\mathbf{if}\ (\hat{\xi} \restriction r)|_{|\mathsf{subj}(\hat{\xi}) \setminus (\emptyset \cup \emptyset \cup \{r\})|}\ ((\hat{G} \restriction r)\,;(\mathbf{while}\ \hat{\xi}|_0\ \{\psi\}\ \hat{G}|_\emptyset \restriction r))|_{\emptyset \cap \{r\}}\ \mathbf{skip}|_{\emptyset \cap \{r\}} \xrightarrow{\xi_r, i^{\hat{R}}_{\{r\}}} L'_r$. Then, by Defn. 26, $\mathbf{if}\ (\hat{\xi} \restriction r)|_{|\mathsf{subj}(\hat{\xi}) \setminus (\emptyset \cup \emptyset \cup \{r\})|}\ (\hat{G}\,;\mathbf{while}\ \hat{\xi}|_0\ \{\psi\}\ \hat{G}|_\emptyset \restriction r)|_{\emptyset \cap \{r\}}\ (\mathbf{skip} \restriction r)|_{\emptyset \cap \{r\}} \xrightarrow{\xi_r, i^{\hat{R}}_{\{r\}}} L'_r$. Then, by Defn. 26, $(\mathbf{if}\ \hat{\xi}|_0\ (\hat{G}\,;\mathbf{while}\ \hat{\xi}|_0\ \{\psi\}\ \hat{G}|_\emptyset)|_\emptyset\ \mathbf{skip}|_\emptyset \restriction r) \xrightarrow{\xi_r, i^{\hat{R}}_{\{r\}}} L'_r$.
- Recall $(G \restriction \hat{r}) \xrightarrow{\xi_{\hat{r}}, i^{\hat{R}}_{\{\hat{r}\}}} L'_{\hat{r}}$ for every $\hat{r} \in \hat{R}$. Then, $(\mathbf{while}\ \hat{\xi}|_0\ \{\psi\}\ \hat{G}|_\emptyset \restriction \hat{r}) \xrightarrow{\xi_{\hat{r}}, i^{\hat{R}}_{\{\hat{r}\}}} L'_{\hat{r}}$ for every $\hat{r} \in \hat{R}$. Then, by Defn. 26, $\mathbf{while}\ (\hat{\xi} \restriction \hat{r})|_{|\mathsf{subj}(\hat{\xi}) \setminus \{\hat{r}\}|}\ \{\mathtt{true}\}\ (\hat{G} \restriction \hat{r})|_\emptyset \xrightarrow{\xi_{\hat{r}}, i^{\hat{R}}_{\{\hat{r}\}}} L'_{\hat{r}}$ for every $\hat{r} \in \hat{R}$. Then, by Defn. 31, $\mathbf{if}\ (\hat{\xi} \restriction \hat{r})|_{|\mathsf{subj}(\hat{\xi}) \setminus \{\hat{r}\}|}\ ((\hat{G} \restriction \hat{r})\,;\mathbf{while}\ (\hat{\xi} \restriction \hat{r})|_{|\mathsf{subj}(\hat{\xi}) \setminus \{\hat{r}\}|}\ \{\mathtt{true}\}\ (\hat{G} \restriction \hat{r})|_\emptyset)|_\emptyset\ \mathbf{skip}|_\emptyset \xrightarrow{\xi_{\hat{r}}, i^{\hat{R}}_{\{\hat{r}\}}} L'_{\hat{r}}$ for every $\hat{r} \in \hat{R}$. Then, $\mathbf{if}\ (\hat{\xi} \restriction \hat{r})|_{|\mathsf{subj}(\hat{\xi}) \setminus (\emptyset \cup \emptyset \cup \{\hat{r}\})|}\ ((\hat{G} \restriction \hat{r})\,;\mathbf{while}\ (\hat{\xi} \restriction \hat{r})|_{|\mathsf{subj}(\hat{\xi}) \setminus \{\hat{r}\}|}\ \{\mathtt{true}\}\ (\hat{G} \restriction \hat{r})|_\emptyset)|_{\emptyset \cap \{\hat{r}\}}\ \mathbf{skip}|_{\emptyset \cap \{\hat{r}\}} \xrightarrow{\xi_{\hat{r}}, i^{\hat{R}}_{\{\hat{r}\}}} L'_{\hat{r}}$ for every $\hat{r} \in \hat{R}$. Then, by Defn. 26, $\mathbf{if}\ (\hat{\xi} \restriction \hat{r})|_{|\mathsf{subj}(\hat{\xi}) \setminus (\emptyset \cup \emptyset \cup \{\hat{r}\})|}\ ((\hat{G} \restriction \hat{r})\,;(\mathbf{while}\ \hat{\xi}|_0\ \{\psi\}\ \hat{G}|_\emptyset \restriction \hat{r}))|_{\emptyset \cap \{\hat{r}\}}\ \mathbf{skip}|_{\emptyset \cap \{\hat{r}\}} \xrightarrow{\xi_{\hat{r}}, i^{\hat{R}}_{\{\hat{r}\}}} L'_{\hat{r}}$ for every $\hat{r} \in \hat{R}$. Then, by Defn. 26, $\mathbf{if}\ (\hat{\xi} \restriction \hat{r})|_{|\mathsf{subj}(\hat{\xi}) \setminus (\emptyset \cup \emptyset \cup \{\hat{r}\})|}\ (\hat{G}\,;\mathbf{while}\ \hat{\xi}|_0\ \{\psi\}\ \hat{G}|_\emptyset \restriction \hat{r})|_{\emptyset \cap \{\hat{r}\}}\ (\mathbf{skip} \restriction \hat{r})|_{\emptyset \cap \{\hat{r}\}} \xrightarrow{\xi_{\hat{r}}, i^{\hat{R}}_{\{\hat{r}\}}} L'_{\hat{r}}$ for every $\hat{r} \in \hat{R}$. Then, by Defn. 26, $(\mathbf{if}\ \hat{\xi}|_0\ (\hat{G}\,;\mathbf{while}\ \hat{\xi}|_0\ \{\psi\}\ \hat{G}|_\emptyset)|_\emptyset\ \mathbf{skip}|_\emptyset \restriction \hat{r}) \xrightarrow{\xi_{\hat{r}}, i^{\hat{R}}_{\{\hat{r}\}}} L'_{\hat{r}}$ for every $\hat{r} \in \hat{R}$.
- Recall $\checkmark_R(G)$. Then, $\checkmark_R(\mathbf{while}\ \hat{\xi}|_0\ \{\psi\}\ \hat{G}|_\emptyset)$. Then, by Lem. 46, $\checkmark_R(\mathbf{if}\ \hat{\xi}|_0\ (\hat{G}\,;\mathbf{while}\ \hat{\xi}|_0\ \{\psi\}\ \hat{G}|_\emptyset)|_\emptyset\ \mathbf{skip}|_\emptyset)$.
- Recall $(\mathbf{if}\ \hat{\xi}|_0\ (\hat{G}\,;\mathbf{while}\ \hat{\xi}|_0\ \{\psi\}\ \hat{G}|_\emptyset)|_\emptyset\ \mathbf{skip}|_\emptyset \restriction r) \xrightarrow{\xi_r, i^{\hat{R}}_{\{r\}}} L'_r$ and $r \in \hat{R}$, and $(\mathbf{if}\ \hat{\xi}|_0\ (\hat{G}\,;\mathbf{while}\ \hat{\xi}|_0\ \{\psi\}\ \hat{G}|_\emptyset)|_\emptyset\ \mathbf{skip}|_\emptyset \restriction \hat{r}) \xrightarrow{\xi_{\hat{r}}, i^{\hat{R}}_{\{\hat{r}\}}} L'_{\hat{r}}$ for every $\hat{r} \in \hat{R}$, and $\checkmark_R(\mathbf{if}\ \hat{\xi}|_0\ (\hat{G}\,;\mathbf{while}\ \hat{\xi}|_0\ \{\psi\}\ \hat{G}|_\emptyset)|_\emptyset\ \mathbf{skip}|_\emptyset)$. Then, by induction, $\mathbf{if}\ \hat{\xi}|_0\ (\hat{G}\,;\mathbf{while}\ \hat{\xi}|_0\ \{\psi\}\ \hat{G}|_\emptyset)|_\emptyset\ \mathbf{skip}|_\emptyset \xrightarrow{\xi, i^{\hat{R}}_{\hat{R}}} G'$ and $L'_{\hat{r}} = G' \restriction \hat{r}$ for every $\hat{r} \in \hat{R}$ and $\xi_{\hat{r}} = \xi \restriction \hat{r}$ for every $\hat{r} \in \hat{R}$, for some G', ξ.
- Recall $\mathbf{if}\ \hat{\xi}|_0\ (\hat{G}\,;\mathbf{while}\ \hat{\xi}|_0\ \{\psi\}\ \hat{G}|_\emptyset)|_\emptyset\ \mathbf{skip}|_\emptyset \xrightarrow{\xi, i^{\hat{R}}_{\hat{R}}} G'$. Then, by [$\rightarrow$1-NWhile], $\mathbf{while}\ \hat{\xi}|_0\ \{\psi\}\ \hat{G}|_\emptyset \xrightarrow{\xi, i^{\hat{R}}_{\hat{R}}} G'$. Then, $G \xrightarrow{\xi, i^{\hat{R}}_{\hat{R}}} G'$. □

Proof (of Lem. 64). By Defn. 14:

- **Case:** $\gamma = \tau$. Then, **false**.
- **Case:** $\gamma = q.y{:=}e$, for some q, y, e.
  - Recall $(G \upharpoonright r) \xrightarrow{\xi_r,\gamma\upharpoonright r} L'_r$ for every $r \in \mathsf{subj}(\gamma)$. Then, $(G \upharpoonright r) \xrightarrow{\xi_r,q.y:=e\upharpoonright r} L'_r$ for every $r \in \mathsf{subj}(q.y{:=}e)$. Then, by Defn. 16, $(G \upharpoonright r) \xrightarrow{\xi_r,q.y:=e\upharpoonright r} L'_r$ for every $r \in \{q\}$. Then, $(G \upharpoonright q) \xrightarrow{\xi_q,q.y:=e\upharpoonright q} L'_q$. Then, by Defn. 18, $(G \upharpoonright q) \xrightarrow{\xi_q,q.y:=e} L'_q$. Then, by Lem. 61, $G \xrightarrow{\xi,q.y:=e} G'$ and $L'_q = G' \upharpoonright q$ and $\xi_q = \xi \upharpoonright q$, for some G', ξ.
  - Recall $G \xrightarrow{\xi,q.y:=e} G'$. Then, $G \xrightarrow{\xi,\gamma} G'$.
  - Recall $L'_q = G' \upharpoonright q$. Then, $L'_r = G' \upharpoonright r$ for every $r \in \{q\}$. Then, by Defn. 16, $L'_r = G' \upharpoonright r$ for every $r \in \mathsf{subj}(q.y{:=}e)$. Then, $L'_r = G' \upharpoonright r$ for every $r \in \mathsf{subj}(\gamma)$.
  - Recall $\xi_q = \xi \upharpoonright q$. Then, $\xi_r = \xi \upharpoonright r$ for every $r \in \{q\}$. Then, by Defn. 16, $\xi_r = \xi \upharpoonright r$ for every $r \in \mathsf{subj}(q.y{:=}e)$. Then, $\xi_r = \xi \upharpoonright r$ for every $r \in \mathsf{subj}(\gamma)$.
- **Case:** $\gamma = p.e \twoheadrightarrow q.y$, for some p, q, y, e.
  - Recall $(G \upharpoonright r) \xrightarrow{\xi_r,\gamma\upharpoonright r} L'_r$ for every $r \in \mathsf{subj}(\gamma)$. Then, $(G \upharpoonright r) \xrightarrow{\xi_r,p.e\twoheadrightarrow q.y\upharpoonright r} L'_r$ for every $r \in \mathsf{subj}(p.e \twoheadrightarrow q.y)$. Then, by Defn. 16, $(G \upharpoonright r) \xrightarrow{\xi_r,p.e\twoheadrightarrow q.y\upharpoonright r} L'_r$ for every $r \in \{p, q\}$. Then, $(G \upharpoonright p) \xrightarrow{\xi_p,p.e\twoheadrightarrow q.y\upharpoonright p} L'_p$ and $(G \upharpoonright q) \xrightarrow{\xi_q,p.e\twoheadrightarrow q.y\upharpoonright q} L'_q$. Then, by Defn. 18, $(G \upharpoonright p) \xrightarrow{\xi_p,pq!e} L'_p$ and $(G \upharpoonright q) \xrightarrow{\xi_q,pq?y} L'_q$.
  - Recall $(G \upharpoonright p) \xrightarrow{\xi_p,pq!e} L'_p$ and $(G \upharpoonright q) \xrightarrow{\xi_q,pq?y} L'_q$ and $\checkmark_R(G)$. Then, by Lem. 62, $G \xrightarrow{\xi,p.e\twoheadrightarrow q.y} G'$ and $L'_p = G' \upharpoonright p$ and $\xi_p = \xi \upharpoonright p$ and $L'_q = G' \upharpoonright q$ and $\xi_q = \xi \upharpoonright q$, for some G', ξ.
  - Recall $G \xrightarrow{\xi,p.e\twoheadrightarrow q.y} G'$. Then, $G \xrightarrow{\xi,\gamma} G'$.
  - Recall $L'_p = G' \upharpoonright p$ and $L'_q = G' \upharpoonright q$. Then, $L'_r = G' \upharpoonright r$ for every $r \in \{p, q\}$. Then, by Defn. 16, $L'_r = G' \upharpoonright r$ for every $r \in \mathsf{subj}(p.e \twoheadrightarrow q.y)$. Then, $L'_r = G' \upharpoonright r$ for every $r \in \mathsf{subj}(\gamma)$.
  - Recall $\xi_p = \xi \upharpoonright p$ and $\xi_q = \xi \upharpoonright q$. Then, $\xi_r = \xi \upharpoonright r$ for every $r \in \{p, q\}$. Then, by Defn. 16, $\xi_r = \xi \upharpoonright r$ for every $r \in \mathsf{subj}(p.e \twoheadrightarrow q.y)$. Then, $\xi_r = \xi \upharpoonright r$ for every $r \in \mathsf{subj}(\gamma)$.
- **Case:** $\gamma = i^{\hat{R}}_R$, for some $\hat{R}$.
  - Recall $(G \upharpoonright \hat{r}) \xrightarrow{\xi_{\hat{r}},\gamma\upharpoonright \hat{r}} L'_{\hat{r}}$ for every $\hat{r} \in \mathsf{subj}(\gamma)$. Then, $(G \upharpoonright \hat{r}) \xrightarrow{\xi_{\hat{r}},i^{\hat{R}}_R\upharpoonright \hat{r}} L'_{\hat{r}}$ for every $\hat{r} \in \mathsf{subj}(i^{\hat{R}}_R)$. Then, by Defn. 16, $(G \upharpoonright \hat{r}) \xrightarrow{\xi_{\hat{r}},i^{\hat{R}}_R\upharpoonright \hat{r}} L'_{\hat{r}}$ for every $\hat{r} \in \hat{R}$. Then, by Defn. 18, $(G \upharpoonright \hat{r}) \xrightarrow{\xi_{\hat{r}},i^{\hat{R}}_{\{\hat{r}\}}} L'_{\hat{r}}$ for every $\hat{r} \in \hat{R}$.
  - Recall $\gamma = i^{\hat{R}}_R$. Then, by Rem. 2, $\hat{R} \neq \emptyset$. Then, $r \in \hat{R}$, for some r.
  - Recall $r \in \hat{R}$. Then, $(G \upharpoonright r) \xrightarrow{\xi_r,i^{\hat{R}}_{\{r\}}} L'_r$.
  - Recall $(G \upharpoonright r) \xrightarrow{\xi_r,i^{\hat{R}}_{\{r\}}} L'_r$ and $r \in \hat{R}$, and $(G \upharpoonright \hat{r}) \xrightarrow{\xi_{\hat{r}},i^{\hat{R}}_{\{\hat{r}\}}} L'_{\hat{r}}$ for every $\hat{r} \in \hat{R}$, and $\checkmark_R(G)$. Then, by Lem. 63, $G \xrightarrow{\xi,i^{\hat{R}}_R} G'$, and $L'_{\hat{r}} = G' \upharpoonright \hat{r}$ for every $\hat{r} \in \hat{R}$, and $\xi_{\hat{r}} = \xi \upharpoonright \hat{r}$ for every $\hat{r} \in \hat{R}$, for some G', ξ.
  - Recall $G \xrightarrow{\xi,i^{\hat{R}}_R} G'$. Then, $G \xrightarrow{\xi,\gamma} G'$.
  - Recall $L'_{\hat{r}} = G' \upharpoonright \hat{r}$ for every $\hat{r} \in \hat{R}$. Then, by Defn. 16, $L'_{\hat{r}} = G' \upharpoonright \hat{r}$ for every $\hat{r} \in \mathsf{subj}(i^{\hat{R}}_R)$. Then, $L'_{\hat{r}} = G' \upharpoonright \hat{r}$ for every $\hat{r} \in \mathsf{subj}(\gamma)$.
  - Recall $\xi_{\hat{r}} = \xi \upharpoonright \hat{r}$ for every $\hat{r} \in \hat{R}$. Then, by Defn. 16, $\xi_{\hat{r}} = \xi \upharpoonright \hat{r}$ for every $\hat{r} \in \mathsf{subj}(i^{\hat{R}}_R)$. Then, $\xi_{\hat{r}} = \xi \upharpoonright \hat{r}$ for every $\hat{r} \in \mathsf{subj}(\gamma)$. □

Proof (of Lem. 65). Recall $\mathcal{A} \xrightarrow{\xi,\gamma} \mathcal{A}'$. Then, by Defn. 32:

- **Case:** [→2-Glob], such that $G \xrightarrow{\xi,\gamma} G'$, for some G, G'.
  Recall $G \xrightarrow{\xi,\gamma} G'$ and $(\xi, \gamma) \neq (\mathtt{true}, \tau)$. Then, by Lem. 50, $\gamma \neq \tau$.
- **Case:** [→2-Locs1], such that $\gamma = \tau$ and $L_{\hat{r}} \xrightarrow{\xi,\tau} L'_{\hat{r}}$, for some $L_{\hat{r}}, L'_{\hat{r}}$.
  - Recall $(\xi, \gamma) \neq (\mathtt{true}, \tau)$. Then, $(\xi, \tau) \neq (\mathtt{true}, \tau)$.
  - Recall $L_{\hat{r}} \xrightarrow{\xi,\tau} L'_{\hat{r}}$ and $(\xi, \tau) \neq (\mathtt{true}, \tau)$. Then, by Lem. 50, $\tau \neq \tau$. Then, **false**.
- **Case:** [→2-Locs2], such that $\gamma \neq \tau$.
- **Case:** [→2-Locs3]. Similar to case [→2-Locs3].
- **Case:** [→2-Locs4]. Similar to case [→2-Locs3]. □

Proof (of Lem. 66). Recall $\{L_r\}_{r\in R} \xrightarrow{\xi,\gamma} \{L'_r\}_{r\in R}$. Then, by Defn. 32:

- **Case:** [$\rightarrow$2-Locs1], such that $\gamma = \tau$.
  Recall $\emptyset \subseteq R$. Then, by Defn. 16, $\mathsf{subj}(\tau) \subseteq R$. Then, $\mathsf{subj}(\gamma) \subseteq R$.
- **Case:** [$\rightarrow$2-Locs2], such that $\gamma = q.y{:=}e$ and $q \in R$, for some q, y, e.
  Recall $q \in R$. Then, $\{q\} \subseteq R$. Then, by Defn. 16, $\mathsf{subj}(q.y{:=}e) \subseteq R$. Then, $\mathsf{subj}(\gamma) \subseteq R$.
- **Case:** [$\rightarrow$2-Locs3]. Similar to case [$\rightarrow$2-Locs2].
- **Case:** [$\rightarrow$2-Locs4]. Similar to case [$\rightarrow$2-Locs2]. □

Proof (of Lem. 67).

1. - Recall $\{L_r\}_{r\in R} \xrightarrow{\xi,\tau} \{L'_r\}_{r\in R}$. Then, by Defn. 32, $\{L_r\}_{r\in R} = \{L_r\}_{r\in R\setminus\{\hat{r}\}} \cup \{L_r\}_{r\in\{\hat{r}\}}$ and $L_{\hat{r}} \xrightarrow{\xi,\tau} L'_{\hat{r}}$ and $\hat{r} \in R$, for some $\hat{r}$.
   - Recall $\hat{r} \in R$. Then, $(R \setminus \{\hat{r}\}) \cup \{\hat{r}\} = R$.
   - Recall $\{L'_r\}_{r\in R} = \{L_r\}_{r\in R\setminus\{\hat{r}\}} \cup \{L'_r\}_{r\in\{\hat{r}\}}$. Then, $\{L'_r\}_{r\in(R\setminus\{\hat{r}\})\cup\{\hat{r}\}} = \{L_r\}_{r\in R\setminus\{\hat{r}\}} \cup \{L'_r\}_{r\in\{\hat{r}\}}$. Then, $\{L'_r\}_{r\in R\setminus\{\hat{r}\}} \cup \{L'_r\}_{r\in\{\hat{r}\}} = \{L_r\}_{r\in R\setminus\{\hat{r}\}} \cup \{L'_r\}_{r\in\{\hat{r}\}}$. Then, $\{L'_r\}_{r\in R\setminus\{\hat{r}\}} = \{L_r\}_{r\in R\setminus\{\hat{r}\}}$. Then, $L_r = L'_r$ for every $r \in R \setminus \{\hat{r}\}$. □
2. By Defn. 14:
   - **Case:** $\gamma = \tau$. Then, **false**.
   - **Case:** $\gamma = q.y{:=}e$, for some q, y, e.
     - Recall $\{L_r\}_{r\in R} \xrightarrow{\xi,\gamma} \{L'_r\}_{r\in R}$. Then, $\{L_r\}_{r\in R} \xrightarrow{\xi,q.y:=e} \{L'_r\}_{r\in R}$. Then, by Defn. 32, $\{L'_r\}_{r\in R} = \{L_r\}_{r\in R\setminus\{q\}} \cup \{L'_r\}_{r\in\{q\}}$ and $L_q \xrightarrow{\xi,q.y:=e} L'_q$ and $q \in R$.
     - Recall $L_q \xrightarrow{\xi,q.y:=e} L'_q$. Then, by Defn. 18, $L_q \xrightarrow{\xi,q.y:=e\upharpoonright q} L'_q$. Then, $L_r \xrightarrow{\xi,q.y:=e\upharpoonright r} L'_r$ for every $r \in \{q\}$. Then, by Defn. 16, $L_r \xrightarrow{\xi,q.y:=e\upharpoonright r} L'_r$ for every $r \in \mathsf{subj}(q.y{:=}e)$. Then, $L_r \xrightarrow{\xi,\gamma\upharpoonright r} L'_r$ for every $r \in \mathsf{subj}(\gamma)$.
     - Recall $\xi = \bigwedge\{\xi\}_{r\in\{q\}}$. Then, by Defn. 16, $\xi = \bigwedge\{\xi\}_{r\in\mathsf{subj}(q.y:=e)}$. Then, $\xi = \bigwedge\{\xi\}_{r\in\mathsf{subj}(\gamma)}$. Then, $\bigwedge\{\xi_r\}_{r\in\mathsf{subj}(\gamma)} = \bigwedge\{\xi\}_{r\in\mathsf{subj}(\gamma)}$ and $\xi = \bigwedge\{\xi_r\}_{r\in\mathsf{subj}(\gamma)}$, for some $\{\xi_r\}_{r\in\mathsf{subj}(\gamma)}$.
     - Recall $q \in R$. Then, $(R \setminus \{q\}) \cup \{q\} = R$.
     - Recall $\{L'_r\}_{r\in R} = \{L_r\}_{r\in R\setminus\{q\}} \cup \{L'_r\}_{r\in\{q\}}$. Then, $\{L'_r\}_{r\in(R\setminus\{q\})\cup\{q\}} = \{L_r\}_{r\in R\setminus\{q\}} \cup \{L'_r\}_{r\in\{q\}}$. Then, $\{L'_r\}_{r\in R\setminus\{q\}} \cup \{L'_r\}_{r\in\{q\}} = \{L_r\}_{r\in R\setminus\{q\}} \cup \{L'_r\}_{r\in\{q\}}$. Then, $\{L'_r\}_{r\in R\setminus\{q\}} = \{L_r\}_{r\in R\setminus\{q\}}$. Then, by Defn. 16, $\{L'_r\}_{r\in R\setminus\mathsf{subj}(q.y:=e)} = \{L_r\}_{r\in R\setminus\mathsf{subj}(q.y:=e)}$. Then, $\{L'_r\}_{r\in R\setminus\mathsf{subj}(\gamma)} = \{L_r\}_{r\in R\setminus\mathsf{subj}(\gamma)}$. Then, $L_r = L'_r$ for every $r \in R \setminus \mathsf{subj}(\gamma)$.
   - **Case:** $\gamma = p.e \twoheadrightarrow q.y$, for some p, q, y, e. Similar to case "$\gamma = q.y{:=}e$, for some q, y, e".
   - **Case:** $\gamma = i^{\hat{R}}_{R}$, for some $\hat{R}$. Similar to case "$\gamma = q.y{:=}e$, for some q, y, e". □

Proof (of Lem. 68). By Defn. 14:

- **Case:** $\gamma = \tau$. Then, **false**.
- **Case:** $\gamma = q.y{:=}e$, for some q, y, e.
  - By Defn. 16, $\mathsf{subj}(q.y{:=}e) = \{q\}$. Then, $\mathsf{subj}(\gamma) = \{q\}$.
  - Recall $L_r \xrightarrow{\xi_r,\gamma\upharpoonright r} L'_r$ for every $r \in \mathsf{subj}(\gamma)$. Then, $L_r \xrightarrow{\xi_r,q.y:=e\upharpoonright r} L'_r$ for every $r \in \mathsf{subj}(\gamma)$. Then, $L_r \xrightarrow{\xi_r,q.y:=e\upharpoonright r} L'_r$ for every $r \in \{q\}$. Then, $L_q \xrightarrow{\xi_q,q.y:=e\upharpoonright q} L'_q$. Then, by Defn. 18, $L_q \xrightarrow{\xi_q,q.y:=e} L'_q$.
  - Recall $\mathsf{subj}(\gamma) \subseteq R$. Then, $\{q\} \subseteq R$. Then, $q \in R$.
  - Recall $\mathsf{subj}(\gamma) \subseteq R$. Then, $R = (R \setminus \mathsf{subj}(\gamma)) \cup \mathsf{subj}(\gamma)$.

- Recall $L_q \xrightarrow{\xi_q, q.y:=e} L'_q$ and $q \in R$. Then, by [→2-LOCS2], $\{L_r\}_{r \in R} \xrightarrow{\xi_q, q.y:=e} \{L_r\}_{r \in R \setminus \{q\}} \cup \{L'_r\}_{r \in \{q\}}$. Then, $\{L_r\}_{r \in R} \xrightarrow{\bigwedge\{\xi_r\}_{r \in \{q\}}, q.y:=e} \{L_r\}_{r \in R \setminus \{q\}} \cup \{L'_r\}_{r \in \{q\}}$. Then, $\{L_r\}_{r \in R} \xrightarrow{\bigwedge\{\xi_r\}_{r \in \mathsf{subj}(\gamma)}, q.y:=e} \{L_r\}_{r \in R \setminus \mathsf{subj}(\gamma)} \cup \{L'_r\}_{r \in \mathsf{subj}(\gamma)}$. Then, $\{L_r\}_{r \in R} \xrightarrow{\bigwedge\{\xi_r\}_{r \in \mathsf{subj}(\gamma)}, q.y:=e} \{L'_r\}_{r \in R \setminus \mathsf{subj}(\gamma)} \cup \{L'_r\}_{r \in \mathsf{subj}(\gamma)}$. Then, $\{L_r\}_{r \in R} \xrightarrow{\bigwedge\{\xi_r\}_{r \in \mathsf{subj}(\gamma)}, q.y:=e} \{L'_r\}_{r \in (R \setminus \mathsf{subj}(\gamma)) \cup \mathsf{subj}(\gamma)}$. Then, $\{L_r\}_{r \in R} \xrightarrow{\bigwedge\{\xi_r\}_{r \in \mathsf{subj}(\gamma)}, \gamma} \{L'_r\}_{r \in R}$.

– **Case:** $\gamma = p.e \twoheadrightarrow q.y$, for some p, q, y, e. Similar to case "$\gamma = q.y := e$, for some q, y, e".
– **Case:** $\gamma = i_R^{\hat{R}}$, for some $\hat{R}$. Similar to case "$\gamma = q.y := e$, for some q, y, e". □

G Reachability

Proof (of Lem. 69).

1. Recall $A \xrightarrow{\{(\mathtt{true},\tau)\}}{}^{*} A^{\dagger}$. Then, by Defn. 34:
   - **Base:** [$\rightarrow^{*}$1-Base], such that $A = A^{\dagger}$.
     Recall $A\downarrow$. then, $A^{\dagger}\downarrow$.
   - **Step:** [$\rightarrow^{*}$1-Step], such that $A \xrightarrow{\{(\mathtt{true},\tau)\}}{}^{*} A^{*} \xrightarrow{\pi} A^{\dagger}$, for some A^{*}, π.
     - Recall $A \xrightarrow{\{(\mathtt{true},\tau)\}}{}^{*} A^{*}$ and $A\downarrow$. Then, by induction, $A^{*}\downarrow$.
     - Recall $A^{*}\downarrow$ and $A^{*} \xrightarrow{\pi} A^{\dagger}$. Then, by Lem. 55, $A^{\dagger}\downarrow$. □
2. Recall $A \xrightarrow{\{(\mathtt{true},\tau)\}}{}^{*} A_1^{\dagger}$. Then, by Defn. 34:
   - **Base:** [$\rightarrow^{*}$1-Base], such that $A = A_1^{\dagger}$.
     - Recall $A \xrightarrow{\xi,\alpha} A_2'$. Then, $A_1^{\dagger} \xrightarrow{\xi,\alpha} A_2'$. Then, $A^{\ddagger} = A_2'$ and $A_1^{\dagger} \xrightarrow{\xi,\alpha} A^{\ddagger}$, for some $A^{\ddagger}$.
     - By [$\rightarrow^{*}$1-Base], $A_2' \xrightarrow{\{(\mathtt{true},\tau)\}}{}^{*} A_2'$. Then, $A_2' \xrightarrow{\{(\mathtt{true},\tau)\}}{}^{*} A^{\ddagger}$.
   - **Step:** [$\rightarrow^{*}$1-Step], such that $A \xrightarrow{\{(\mathtt{true},\tau)\}}{}^{*} A_1^{*} \xrightarrow{\pi} A_1^{\dagger}$ and $\pi \in \{(\mathtt{true},\tau)\}$, for some A_1^{*}, π.
     - Recall $\pi \in \{(\mathtt{true},\tau)\}$. Then, $\pi = (\mathtt{true},\tau)$.
     - Recall $A_1^{*} \xrightarrow{\pi} A_1^{\dagger}$. Then, $A_1^{*} \xrightarrow{\mathtt{true},\tau} A_1^{\dagger}$.
     - Recall $A \xrightarrow{\{(\mathtt{true},\tau)\}}{}^{*} A_1^{*}$ and $A \xrightarrow{\xi,\alpha} A_2'$. Then, by induction, $A_1^{*} \xrightarrow{\xi,\alpha} A^{\dagger}$ and $A_2' \xrightarrow{\{(\mathtt{true},\tau)\}}{}^{*} A^{\dagger}$, for some $A^{\dagger}$.
     - Recall $A_1^{*} \xrightarrow{\mathtt{true},\tau} A_1^{\dagger}$ and $A_1^{*} \xrightarrow{\xi,\alpha} A^{\dagger}$. Then, by Lem. 56:
       * **Case:** $A_1^{\dagger} \xrightarrow{\xi,\alpha} A^{\ddagger}$ and $A^{\dagger} \xrightarrow{\mathtt{true},\tau} A^{\ddagger}$, for some $A^{\ddagger}$.
         Recall $A_2' \xrightarrow{\{(\mathtt{true},\tau)\}}{}^{*} A^{\dagger} \xrightarrow{\mathtt{true},\tau} A^{\ddagger}$ and $(\mathtt{true},\tau) \in \{(\mathtt{true},\tau)\}$. Then, by [$\rightarrow^{*}$1-Step], $A_2' \xrightarrow{\{(\mathtt{true},\tau)\}}{}^{*} A^{\ddagger}$.
       * **Case:** $\xi = \mathtt{true}$ and $\alpha = \tau$ and $A_1^{\dagger} = A^{\dagger}$.
         · By [$\rightarrow^{*}$1-Base], $A_1^{\dagger} \xrightarrow{\{(\mathtt{true},\tau)\}}{}^{*} A_1^{\dagger}$. Then, $A_1^{\dagger} \xrightarrow{\{(\mathtt{true},\tau)\}}{}^{*} A^{\dagger}$. Then, $A^{\ddagger} = A^{\dagger}$ and $A_1^{\dagger} \xrightarrow{\{(\mathtt{true},\tau)\}}{}^{*} A^{\ddagger}$, for some $A^{\ddagger}$.
         · Recall $A_2' \xrightarrow{\{(\mathtt{true},\tau)\}}{}^{*} A^{\dagger}$. Then, $A_2' \xrightarrow{\{(\mathtt{true},\tau)\}}{}^{*} A^{\ddagger}$. □
3. Recall $A \xrightarrow{\{(\mathtt{true},\tau)\}}{}^{*} A_1^{\dagger}$. Then, by Defn. 34:
   - **Base:** [$\rightarrow^{*}$1-Base], such that $A = A_1^{\dagger}$.
     - Recall $A_1^{\dagger} \xrightarrow{\xi,\alpha} A^{\ddagger}$. Then, $A \xrightarrow{\xi,\alpha} A^{\ddagger}$. Then, $A_2^{\dagger} = A^{\ddagger}$ and $A \xrightarrow{\xi,\alpha} A_2^{\dagger}$, for some $A_2^{\dagger}$.
     - By [$\rightarrow^{*}$1-Base], $A^{\ddagger} \xrightarrow{\{(\mathtt{true},\tau)\}}{}^{*} A^{\ddagger}$. Then, $A_2^{\dagger} \xrightarrow{\{(\mathtt{true},\tau)\}}{}^{*} A^{\ddagger}$.
   - **Step:** [$\rightarrow^{*}$1-Step], such that $A \xrightarrow{\{(\mathtt{true},\tau)\}}{}^{*} A_1^{*} \xrightarrow{\pi} A_1^{\dagger}$ and $\pi \in \{(\mathtt{true},\tau)\}$, for some A_1^{*}, π.
     - Recall $\pi \in \{(\mathtt{true},\tau)\}$. Then, $\pi = (\mathtt{true},\tau)$.
     - Recall $A_1^{*} \xrightarrow{\pi} A_1^{\dagger}$. Then, $A_1^{*} \xrightarrow{\mathtt{true},\tau} A_1^{\dagger}$.
     - Recall $A_1^{*} \xrightarrow{\mathtt{true},\tau} A_1^{\dagger} \xrightarrow{\xi,\alpha} A^{\ddagger}$. Then, by Lem. 56, $A_1^{*} \xrightarrow{\xi,\alpha} A_2^{\dagger} \xrightarrow{\mathtt{true},\tau} A^{\ddagger}$, for some $A_2^{\dagger}$.
     - Recall $A \xrightarrow{\{(\mathtt{true},\tau)\}}{}^{*} A_1^{*} \xrightarrow{\xi,\alpha} A_2^{\dagger}$. Then, by induction, $A \xrightarrow{\xi,\alpha} A_2'$ and $A_2' \xrightarrow{\{(\mathtt{true},\tau)\}}{}^{*} A_2^{\dagger}$, for some A_2'.
     - Recall $A_2' \xrightarrow{\{(\mathtt{true},\tau)\}}{}^{*} A_2^{\dagger} \xrightarrow{\mathtt{true},\tau} A^{\ddagger}$ and $(\mathtt{true},\tau) \in \{(\mathtt{true},\tau)\}$. Then, by [$\rightarrow^{*}$1-Step], $A_2' \xrightarrow{\{(\mathtt{true},\tau)\}}{}^{*} A^{\ddagger}$. □

Proof (of Lem. 70).

1. – Recall $A \Longrightarrow A^{\dagger}$. Then, by Defn. 34, $A \xrightarrow{\{(\mathtt{true},\tau)\}}{}^{*} A^{\dagger}$.
   – Recall $A \xrightarrow{\{(\mathtt{true},\tau)\}}{}^{*} A^{\dagger}$ and $A\downarrow$. Then, by Lem. 69, $A^{\dagger}\downarrow$. □
2. – Recall $A_2 \Longrightarrow A^{\dagger}$. Then, by Defn. 34, $A_2 \xrightarrow{\{(\mathtt{true},\tau)\}}{}^{*} A^{\dagger}$.
   – Recall $A_2 \xrightarrow{\{(\mathtt{true},\tau)\}}{}^{*} A^{\dagger}$ and $A_2 \xrightarrow{\mathtt{true},\tau} A_2'$. Then, by Lem. 69, $A^{\dagger} \xrightarrow{\mathtt{true},\tau} A^{\ddagger}$ and $A_2' \xrightarrow{\{(\mathtt{true},\tau)\}}{}^{*} A^{\ddagger}$, for some $A^{\ddagger}$.
   – Recall $A_1 \Longrightarrow A^{\dagger}$. Then, by Defn. 34, $A_1 \xrightarrow{\{(\mathtt{true},\tau)\}}{}^{*} A^{\dagger}$.
   – Recall $A_1 \xrightarrow{\{(\mathtt{true},\tau)\}}{}^{*} A^{\dagger} \xrightarrow{\mathtt{true},\tau} A^{\ddagger}$ and $(\mathtt{true},\tau) \in \{(\mathtt{true},\tau)\}$. Then, by [$\rightarrow^{*}$1-Step], $A_1 \xrightarrow{\{(\mathtt{true},\tau)\}}{}^{*} A^{\ddagger}$. Then, by [$\Rightarrow$1], $A_1 \Longrightarrow A^{\ddagger}$.
   – Recall $A_2' \xrightarrow{\{(\mathtt{true},\tau)\}}{}^{*} A^{\ddagger}$. Then, by [$\Rightarrow$1], $A_2' \Longrightarrow A^{\ddagger}$. □
3. – Recall $A_2 \Longrightarrow A^{\dagger}$. Then, by Defn. 34, $A_2 \xrightarrow{\{(\mathtt{true},\tau)\}}{}^{*} A^{\dagger}$.
   – Recall $A_2 \xrightarrow{\{(\mathtt{true},\tau)\}}{}^{*} A^{\dagger}$ and $A_2 \xrightarrow{\xi,\alpha} A_2'$. Then, by Lem. 69, $A^{\dagger} \xrightarrow{\xi,\alpha} A^{\ddagger}$ and $A_2' \xrightarrow{\{(\mathtt{true},\tau)\}}{}^{*} A^{\ddagger}$, for some $A^{\ddagger}$.

– Recall $A_1 \Longrightarrow A^\dagger$. Then, by Defn. 34, $A_1 \xrightarrow{\{(\mathtt{true},\tau)\}}{}^* A^\dagger$.
– Recall $A_1 \xrightarrow{\{(\mathtt{true},\tau)\}}{}^* A^\dagger \xrightarrow{\xi,\alpha} A^\ddagger$. Then, by Lem. 69, $A_1 \xrightarrow{\xi,\alpha} A'_1$ and $A'_1 \xrightarrow{\{(\mathtt{true},\tau)\}}{}^* A^\ddagger$, for some A'_1.
– Recall $A'_1 \xrightarrow{\{(\mathtt{true},\tau)\}}{}^* A^\ddagger$ and $A'_2 \xrightarrow{\{(\mathtt{true},\tau)\}}{}^* A^\ddagger$. Then, by [⇒1], $A'_1 \Longrightarrow A^\ddagger$ and $A'_2 \Longrightarrow A^\ddagger$. □

Proof (of Lem. 71). Recall $\mathcal{A}^\dagger \xrightarrow{\Pi}{}^* \mathcal{A}^\S$. Then, by Defn. 35:

– **Base:** [$\to^*$2-Base], such that $\mathcal{A}^\dagger = \mathcal{A}^\S$.
Recall $\mathcal{A} \xrightarrow{\Pi}{}^* \mathcal{A}^\dagger$. Then, $\mathcal{A} \xrightarrow{\Pi}{}^* \mathcal{A}^\S$.
– **Step:** [$\to^*$2-Step], such that $\mathcal{A}^\dagger \xrightarrow{\Pi}{}^* \mathcal{A}^\ddagger \xrightarrow{\pi} \mathcal{A}^\S$ and $\pi \in \Pi$, for some $\mathcal{A}^\ddagger, \pi$.
  • Recall $\mathcal{A} \xrightarrow{\Pi}{}^* \mathcal{A}^\dagger \xrightarrow{\Pi}{}^* \mathcal{A}^\ddagger$. Then, by induction, $\mathcal{A} \xrightarrow{\Pi}{}^* \mathcal{A}^\ddagger$.
  • Recall $\mathcal{A} \xrightarrow{\Pi}{}^* \mathcal{A}^\ddagger \xrightarrow{\pi} \mathcal{A}^\S$ and $\pi \in \Pi$. Then, by [$\to^*$2-Step], $\mathcal{A} \xrightarrow{\Pi}{}^* \mathcal{A}^\S$. □

Proof (of Lem. 72).

1. Recall $L_{\hat{r}} \xrightarrow{\{(\mathtt{true},\tau)\}}{}^* L^\dagger_{\hat{r}}$. Then, by Defn. 34:
   – **Base:** [$\to^*$1-Base], such that $L_{\hat{r}} = L^\dagger_{\hat{r}}$.
     • Recall $\hat{r} \in R$. Then, $R = (R \setminus \{\hat{r}\}) \cup \{\hat{r}\}$
     • Recall $L_{\hat{r}} = L^\dagger_{\hat{r}}$. Then, $\{L_r\}_{r\in\{\hat{r}\}} = \{L^\dagger_r\}_{r\in\{\hat{r}\}}$.
     • By [$\to^*$2-Base], $\{L_r\}_{r\in R} \xrightarrow{\{(\mathtt{true},\tau)\}}{}^* \{L_r\}_{r\in R}$. Then, $\{L_r\}_{r\in R} \xrightarrow{\{(\mathtt{true},\tau)\}}{}^* \{L_r\}_{r\in(R\setminus\{\hat{r}\})\cup\{\hat{r}\}}$. Then, $\{L_r\}_{r\in R} \xrightarrow{\{(\mathtt{true},\tau)\}}{}^* \{L_r\}_{r\in R\setminus\{\hat{r}\}} \cup \{L_r\}_{r\in\{\hat{r}\}}$. Then, $\{L_r\}_{r\in R} \xrightarrow{\{(\mathtt{true},\tau)\}}{}^* \{L_r\}_{r\in R\setminus\{\hat{r}\}} \cup \{L^\dagger_r\}_{r\in\{\hat{r}\}}$.
   – **Step:** [$\to^*$1-Step], such that $L_{\hat{r}} \xrightarrow{\{(\mathtt{true},\tau)\}}{}^* L^*_{\hat{r}} \xrightarrow{\pi} L^\dagger_{\hat{r}}$ and $\pi \in \{(\mathtt{true},\tau)\}$, for some $L^*_{\hat{r}}, \pi$.
     • Recall $L_{\hat{r}} \xrightarrow{\{(\mathtt{true},\tau)\}}{}^* L^*_{\hat{r}}$ and $\hat{r} \in R$. Then, by induction, $\{L_r\}_{r\in R} \xrightarrow{\{(\mathtt{true},\tau)\}}{}^* \{L_r\}_{r\in R\setminus\{\hat{r}\}} \cup \{L^*_r\}_{r\in\{\hat{r}\}}$.
     • Recall $\pi \in \{(\mathtt{true},\tau)\}$. Then, $\pi = (\mathtt{true},\tau)$.
     • Recall $L^*_{\hat{r}} \xrightarrow{\pi} L^\dagger_{\hat{r}}$. Then, $L^*_{\hat{r}} \xrightarrow{\mathtt{true},\tau} L^\dagger_{\hat{r}}$.
     • Recall $L^*_{\hat{r}} \xrightarrow{\mathtt{true},\tau} L^\dagger_{\hat{r}}$ and $\hat{r} \in (R \setminus \{\hat{r}\}) \cup \{\hat{r}\}$. Then, by [$\to$2-Locs1], $\{L_r\}_{r\in R\setminus\{\hat{r}\}} \cup \{L^*_r\}_{r\in\{\hat{r}\}} \xrightarrow{\mathtt{true},\tau} \{L_r\}_{r\in R\setminus\{\hat{r}\}} \cup \{L^*_r\}_{r\in\{\hat{r}\}\setminus\{\hat{r}\}} \cup \{L^\dagger_r\}_{r\in\{\hat{r}\}}$. Then, $\{L_r\}_{r\in R\setminus\{\hat{r}\}} \cup \{L^*_r\}_{r\in\{\hat{r}\}} \xrightarrow{\mathtt{true},\tau} \{L_r\}_{r\in R\setminus\{\hat{r}\}} \cup \{L^*_r\}_{r\in\setminus} \cup \{L^\dagger_r\}_{r\in\{\hat{r}\}}$. Then, $\{L_r\}_{r\in R\setminus\{\hat{r}\}} \cup \{L^*_r\}_{r\in\{\hat{r}\}} \xrightarrow{\mathtt{true},\tau} \{L_r\}_{r\in R\setminus\{\hat{r}\}} \cup \{L^\dagger_r\}_{r\in\{\hat{r}\}}$.
     • Recall $\{L_r\}_{r\in R} \xrightarrow{\{(\mathtt{true},\tau)\}}{}^* \{L_r\}_{r\in R\setminus\{\hat{r}\}} \cup \{L^*_r\}_{r\in\{\hat{r}\}} \xrightarrow{\mathtt{true},\tau} \{L_r\}_{r\in R\setminus\{\hat{r}\}} \cup \{L^\dagger_r\}_{r\in\{\hat{r}\}}$ and $(\mathtt{true},\tau) \in \{(\mathtt{true},\tau)\}$. Then, by [$\to^*$2-Step], $\{L_r\}_{r\in R} \xrightarrow{\{(\mathtt{true},\tau)\}}{}^* \{L_r\}_{r\in R\setminus\{\hat{r}\}} \cup \{L^\dagger_r\}_{r\in\{\hat{r}\}}$. □
2. Recall:
   – **Base:** $\hat{R} = \emptyset$.
   By [$\to^*$2-Base], $\{L_r\}_{r\in R} \xrightarrow{\{(\mathtt{true},\tau)\}}{}^* \{L_r\}_{r\in R}$. Then, $\{L_r\}_{r\in R} \xrightarrow{\{(\mathtt{true},\tau)\}}{}^* \{L_r\}_{r\in R\setminus\emptyset}$. Then, $\{L_r\}_{r\in R} \xrightarrow{\{(\mathtt{true},\tau)\}}{}^* \{L_r\}_{r\in R\setminus\emptyset} \cup \{L^\dagger_r\}_{r\in\emptyset}$. Then, $\{L_r\}_{r\in R} \xrightarrow{\{(\mathtt{true},\tau)\}}{}^* \{L_r\}_{r\in R\setminus\hat{R}} \cup \{L^\dagger_r\}_{r\in\hat{R}}$.
   – **Step:** $\hat{R} = \hat{R}' \cup \{\hat{r}\}$ and $\hat{R}' = \hat{R} \setminus \{\hat{r}\}$, for some $\hat{R}', \hat{r}$.
     • Recall $L_r \xrightarrow{\{(\mathtt{true},\tau)\}}{}^* L^\dagger_r$ for every $r \in \hat{R}$. Then, $L_r \xrightarrow{\{(\mathtt{true},\tau)\}}{}^* L^\dagger_r$ for every $r \in \hat{R}' \cup \{\hat{r}\}$. Then, $L_r \xrightarrow{\{(\mathtt{true},\tau)\}}{}^* L^\dagger_r$ for every $r \in \hat{R}'$.
     • Recall $\hat{R} \subseteq R$. Then, $\hat{R}' \cup \{\hat{r}\} \subseteq R$. Then, $\hat{R}' \subseteq R$.
     • Recall $L_r \xrightarrow{\{(\mathtt{true},\tau)\}}{}^* L^\dagger_r$ for every $r \in \hat{R}' \subseteq R$. Then, by induction, $\{L_r\}_{r\in R} \xrightarrow{\{(\mathtt{true},\tau)\}}{}^* \{L_r\}_{r\in R\setminus\hat{R}'} \cup \{L^\dagger_r\}_{r\in\hat{R}'}$.
     • Recall $\hat{r} \in \hat{R}' \cup \{\hat{r}\}$. Then, $\hat{r} \in \hat{R}$.
     • Recall $\hat{r} \in \hat{R}$. Then, $L_{\hat{r}} \xrightarrow{\{(\mathtt{true},\tau)\}}{}^* L^\dagger_{\hat{r}}$.
     • Recall $\hat{R}' \subseteq R$. Then, $R = (R \setminus \hat{R}') \cup \hat{R}'$.
     • Recall $\hat{r} \in \hat{R} \subseteq R$. Then, $\hat{r} \in R$. Then, $\hat{r} \in (R \setminus \hat{R}') \cup \hat{R}'$.

- Recall $L_{\hat{r}} \xrightarrow{\{(\mathtt{true},\tau)\}}{}^* L^\dagger_{\hat{r}}$ and $\hat{r} \in (R \setminus \hat{R}') \cup \hat{R}'$. Then, by 1, $\{L_r\}_{r\in R\setminus\hat{R}'} \cup \{L^\dagger_r\}_{r\in\hat{R}'} \xrightarrow{\{(\mathtt{true},\tau)\}}{}^* \{L_r\}_{r\in(R\setminus\hat{R}')\setminus\{\hat{r}\}} \cup \{L^\dagger_r\}_{r\in\hat{R}'\setminus\{\hat{r}\}} \cup \{L^\dagger_r\}_{r\in\{\hat{r}\}}$. Then, $\{L_r\}_{r\in R\setminus\hat{R}'} \cup \{L^\dagger_r\}_{r\in\hat{R}'} \xrightarrow{\{(\mathtt{true},\tau)\}}{}^* \{L_r\}_{r\in(R\setminus\hat{R}')\setminus\{\hat{r}\}} \cup \{L^\dagger_r\}_{r\in(\hat{R}'\setminus\{\hat{r}\})\cup\{\hat{r}\}}$. Then, $\{L_r\}_{r\in R\setminus\hat{R}'} \cup \{L^\dagger_r\}_{r\in\hat{R}'} \xrightarrow{\{(\mathtt{true},\tau)\}}{}^* \{L_r\}_{r\in R\setminus(\hat{R}'\cup\{\hat{r}\})} \cup \{L^\dagger_r\}_{r\in\hat{R}'\cup\{\hat{r}\}}$. Then, $\{L_r\}_{r\in R\setminus\hat{R}'} \cup \{L^\dagger_r\}_{r\in\hat{R}'} \xrightarrow{\{(\mathtt{true},\tau)\}}{}^* \{L_r\}_{r\in R\setminus\hat{R}} \cup \{L^\dagger_r\}_{r\in\hat{R}}$.
- Recall $\{L_r\}_{r\in R} \xrightarrow{\{(\mathtt{true},\tau)\}}{}^* \{L_r\}_{r\in R\setminus\hat{R}'} \cup \{L^\dagger_r\}_{r\in\hat{R}'} \xrightarrow{\{(\mathtt{true},\tau)\}}{}^* \{L_r\}_{r\in R\setminus\hat{R}} \cup \{L^\dagger_r\}_{r\in\hat{R}}$. Then, by Lem. 71, $\{L_r\}_{r\in R} \xrightarrow{\{(\mathtt{true},\tau)\}}{}^* \{L_r\}_{r\in R\setminus\hat{R}} \cup \{L^\dagger_r\}_{r\in\hat{R}}$. □

3. Recall $L_r \xrightarrow{\{(\mathtt{true},\tau)\}}{}^* L^\dagger_r$ for every $r \in R \subseteq R$. Then, by 2, $\{L_r\}_{r\in R} \xrightarrow{\{(\mathtt{true},\tau)\}}{}^* \{L_r\}_{r\in R\setminus R} \cup \{L^\dagger_r\}_{r\in R}$. Then, $\{L_r\}_{r\in R} \xrightarrow{\{(\mathtt{true},\tau)\}}{}^* \{L_r\}_{r\in\emptyset} \cup \{L^\dagger_r\}_{r\in R}$. Then, $\{L_r\}_{r\in R} \xrightarrow{\{(\mathtt{true},\tau)\}}{}^* \{L^\dagger_r\}_{r\in R}$. □

Proof (of Lem. 73). Recall $\mathcal{A} \xrightarrow{\{(\mathtt{true},\tau)\}}{}^* \mathcal{A}^\dagger$. Then, by Defn. 35:

- **Base:** [$\rightarrow^*$2-Base], such that $\mathcal{A} = \mathcal{A}^\dagger$.
  By [$\rightarrow^*$3-Base], $(\mathcal{A},\mathcal{S}) \xrightarrow{\{\tau\}}{}^* (\mathcal{A},\mathcal{S})$. Then, $(\mathcal{A},\mathcal{S}) \xrightarrow{\{\tau\}}{}^* (\mathcal{A}^\dagger,\mathcal{S})$.
- **Step:** [$\rightarrow^*$2-Step], such that $\mathcal{A} \xrightarrow{\pi\in\{(\mathtt{true},\tau)\}}{}^* \mathcal{A}^* \xrightarrow{\pi} \mathcal{A}^\dagger$ and $\pi \in \{(\mathtt{true},\tau)\}$, for some $\mathcal{A}^*$.
  - Recall $\mathcal{A} \xrightarrow{\pi\in\{(\mathtt{true},\tau)\}}{}^* \mathcal{A}^*$. Then, by induction, $(\mathcal{A},\mathcal{S}) \xrightarrow{\{\tau\}}{}^* (\mathcal{A}^*,\mathcal{S})$.
  - Recall $\pi \in \{(\mathtt{true},\tau)\}$. Then, $\pi = (\mathtt{true},\tau)$.
  - Recall $\mathcal{A}^* \xrightarrow{\pi} \mathcal{A}^\dagger$. Then, $\mathcal{A}^* \xrightarrow{\mathtt{true},\tau} \mathcal{A}^\dagger$.
  - By Defn. 4, $\mathsf{eval}_{\mathcal{S}}(\mathtt{true}) = \mathtt{true}$. Then, $\mathcal{S} \in \{\hat{\mathcal{S}} \mid \mathsf{eval}_{\hat{\mathcal{S}}}(\mathtt{true}) = \mathtt{true}\}$. Then, by Defn. 13, $\mathcal{S} \in [\![\mathtt{true}]\!]$.
  - By Defn. 19, $\tau = \mathsf{con}_{\mathcal{S}}(\tau)$.
  - Recall $\mathcal{A}^* \xrightarrow{\mathtt{true},\tau} \mathcal{A}^\dagger$ and $\mathcal{S} \in [\![\mathtt{true}]\!]$ and $\tau = \mathsf{con}_{\mathcal{S}}(\tau)$. Then, by [$\rightarrow$3], $(\mathcal{A}^*,\mathcal{S}) \xrightarrow{\tau} (\mathcal{A}^\dagger,\mathsf{effect}(\tau,\mathcal{S}))$. Then, by Defn. 20, $(\mathcal{A}^*,\mathcal{S}) \xrightarrow{\tau} (\mathcal{A}^\dagger,\mathcal{S})$.
  - Recall $(\mathcal{A},\mathcal{S}) \xrightarrow{\{\tau\}}{}^* (\mathcal{A}^*,\mathcal{S}) \xrightarrow{\tau} (\mathcal{A}^\dagger,\mathcal{S})$ and $\tau \in \{\tau\}$. Then, by [$\rightarrow^*$3-Step], $(\mathcal{A},\mathcal{S}) \xrightarrow{\{\tau\}}{}^* (\mathcal{A}^\dagger,\mathcal{S})$. □

Proof (of Lem. 74). Recall $\mathcal{A} \Longrightarrow \mathcal{A}^\dagger$. Then, by Defn. 36, $\mathcal{A} \xrightarrow{\{(\mathtt{true},\tau)\}}{}^* \mathcal{A}^\dagger$. Then, by Lem. 73, $(\mathcal{A},\mathcal{S}) \xrightarrow{\{\tau\}}{}^* (\mathcal{A}^\dagger,\mathcal{S})$. Then, by [$\Rightarrow$3], $(\mathcal{A},\mathcal{S}) \Longrightarrow (\mathcal{A}^\dagger,\mathcal{S})$. □

H Theorems: Deadlock Freedom and Functional Correctness

Proof (of Lem. 75).

1. By Defn. 14:
   - **Base:** $\gamma = \tau$.
     Recall $\neg\chi = \neg\chi$. Then, by Defn. 37, $\phi(\tau, \neg\chi) = \neg\phi(\tau, \chi)$. Then, $\phi(\gamma, \neg\chi) = \neg\phi(\gamma, \chi)$.
   - **Base:** $\gamma = q.y{:=}e$, for some q, y, e.
     By Defn. 7, $\neg\chi[e/q.y] = \neg(\chi[e/q.y])$. Then, by Defn. 37, $\phi(q.y{:=}e, \neg\chi) = \neg\phi(q.y{:=}e, \chi)$. Then, $\phi(\gamma, \neg\chi) = \neg\phi(\gamma, \chi)$.
   - **Base:** $\gamma = p.e \rightarrowtriangle q.y$, for some p, q, y, e. Similar to case "$\gamma = q.y{:=}e$, for some q, y, e".
   - **Base:** $\gamma = i_R^R$, for some R. Similar to case $\gamma = \tau$. □
2. Similar to 1. □
3. Recall $\phi(\gamma, \chi_1 \Rightarrow \chi_2) = \phi(\gamma, \chi_1 \Rightarrow \chi_2)$. Then, by Defn. 6, $\phi(\gamma, \chi_1 \Rightarrow \chi_2) = \phi(\gamma, \neg(\chi_1 \wedge \neg\chi_2))$. Then, by 1, $\phi(\gamma, \chi_1 \Rightarrow \chi_2) = \neg\phi(\gamma, \chi_1 \wedge \neg\chi_2)$. Then, by 2, $\phi(\gamma, \chi_1 \Rightarrow \chi_2) = \neg(\phi(\gamma, \chi_1) \wedge \phi(\gamma, \neg\chi_2))$. Then, by 1, $\phi(\gamma, \chi_1 \Rightarrow \chi_2) = \neg(\phi(\gamma, \chi_1) \wedge \neg\phi(\gamma, \chi_2))$. Then, by Defn. 6, $\phi(\gamma, \chi_1) \Rightarrow \phi(\gamma, \chi_2)$. □

Proof (of Lem. 76). By Defn. 14:

- **Base:** $\gamma = \tau$.
  By Defn. 37, $\phi(\tau, \chi) = \chi$. Then, $\phi(\gamma, \chi) = \chi$.
- **Base:** $\gamma = q.y{:=}e$, for some q, y, e.
  Recall $\mathsf{write}(\gamma) \cap \mathsf{read}(\chi) = \emptyset$. Then, $\mathsf{write}(q.y{:=}e) \cap \mathsf{read}(\chi) = \emptyset$. Then, by Defn. 15, $\{q.y\} \cap \mathsf{read}(\chi) = \emptyset$. Then, by Defn. 9, $\{q.y\} \cap \mathsf{rx}(\chi) = \emptyset$. Then, $q.y \notin \mathsf{rx}(\chi)$. Then, by Lem. 1, $\chi[e/q.y] = \chi$. Then, by Defn. 37, $\phi(q.y{:=}e, \chi) = \chi$. Then, $\phi(\gamma, \chi) = \chi$.
- **Base:** $\gamma = p.e \rightarrowtriangle q.y$, for some p, q, y, e. Similar to case "$\gamma = q.y{:=}e$, for some q, y, e".
- **Base:** $\gamma = i_R^R$, for some R. Similar to case $\gamma = \tau$. □

Proof (of Lem. 77). By Defn. 14:

- **Base:** $\gamma_1 = \tau$.
  Recall $\phi(\gamma_2, \chi) = \phi(\gamma_2, \chi)$. Then, by Defn. 37, $\phi(\tau, \phi(\gamma_2, \chi)) = \phi(\gamma_2, \phi(\tau, \chi))$. Then, $\phi(\gamma_1, \phi(\gamma_2, \chi)) = \phi(\gamma_2, \phi(\gamma_1, \chi))$.
- **Base:** $\gamma_1 = q_1.y_1{:=}e_1$, for some q_1, y_1, e_1.
  By Defn. 14:
  - **Base:** $\gamma_2 = \tau$.
    Recall $\phi(\gamma_1, \chi) = \phi(\gamma_1, \chi)$. Then, by Defn. 37, $\phi(\gamma_1, \phi(\tau, \chi)) = \phi(\tau, \phi(\gamma_1, \chi))$. Then, $\phi(\gamma_1, \phi(\gamma_2, \chi)) = \phi(\gamma_2, \phi(\gamma_1, \chi))$.
  - **Base:** $\gamma_2 = q_2.y_2{:=}e_2$, for some q_2, y_2, e_2.
    - Recall $\mathsf{write}(\gamma_1) \cap \mathsf{write}(\gamma_2) = \emptyset$. Then, $\mathsf{write}(q_1.y_1{:=}e_1) \cap \mathsf{write}(q_2.y_2{:=}e_2) = \emptyset$. Then, by Defn. 15, $\{q_1.y_1\} \cap \{q_2.y_2\} = \emptyset$. Then, $q_1.y_1 \neq q_2.y_2$.
    - Recall $\mathsf{read}(\gamma_1) \cap \mathsf{write}(\gamma_2) = \emptyset$ and $\mathsf{write}(\gamma_1) \cap \mathsf{read}(\gamma_2) = \emptyset$. Then, $\mathsf{read}(q_1.y_1{:=}e_1) \cap \mathsf{write}(q_2.y_2{:=}e_2) = \emptyset$ and $\mathsf{write}(q_1.y_1{:=}e_1) \cap \mathsf{read}(q_2.y_2{:=}e_2) = \emptyset$. Then, by Defn. 15, $\mathsf{rx}(e_1) \cap \{q_2.y_2\} = \emptyset$ and $\{q_1.y_1\} \cap \mathsf{rx}(e_2) = \emptyset$. Then, $q_2.y_2 \notin \mathsf{rx}(e_1)$ and $q_1.y_1 \notin \mathsf{rx}(e_2)$.
    - Recall $q_1.y_1 \neq q_2.y_2$ and $q_2.y_2 \notin \mathsf{rx}(e_1)$ and $q_1.y_1 \notin \mathsf{rx}(e_2)$. Then, by Lem. 2, $\chi[e_1/q_1.y_1][e_2/q_2.y_2] = \chi[e_2/q_2.y_2][e_1/q_1.y_1]$. Then, by Defn. 37, $\phi(q_2.y_2{:=}e_2, \chi[e_1/q_1.y_1]) = \phi(q_1.y_1{:=}e_1, \chi[e_2/q_2.y_2])$. Then, by Defn. 37, $\phi(q_2.y_2{:=}e_2, \phi(q_1.y_1{:=}e_1, \chi)) = \phi(q_1.y_1{:=}e_1, \phi(q_2.y_2{:=}e_2, \chi))$. Then, $\phi(\gamma_2, \phi(\gamma_1, \chi)) = \phi(\gamma_1, \phi(\gamma_2, \chi))$.
  - **Base:** $\gamma_2 = p_2.e_2 \rightarrowtriangle q_2.y_2$, for some p_2, q_2, e_2, y_2. Similar to case "$\gamma_2 = q_2.y_2{:=}e_2$, for some q_2, y_2, e_2".
  - **Base:** $\gamma_2 = i_{R_2}^{R_2}$, for some R_2. Similar to case $\gamma_2 = \tau$.

- **Base:** $\gamma_1 = p_1.e_1 \twoheadrightarrow q_1.y_1$, for some p_1, q_1, e_1, y_1. Similar to case "$\gamma_1 = q_1.y_1 := e_1$, for some q_1, y_1, e_1".
- **Base:** $\gamma_1 = i_{R_1}^{R_1}$, for some R_1. Similar to case $\gamma_1 = \tau$. □

Proof (of Lem. 78). By Defn. 14:

- **Base:** $\gamma = \tau$.
  Recall $[\![\chi_1]\!] \subseteq [\![\chi_2]\!]$. Then, by Defn. 37, $[\![\phi(\tau, \chi_1)]\!] \subseteq [\![\phi(\tau, \chi_2)]\!]$. Then, $[\![\phi(\gamma, \chi_1)]\!] \subseteq [\![\phi(\gamma, \chi_2)]\!]$.
- **Base:** $\gamma = q.y := e$, for some q, y, e.
  Recall $[\![\chi_1]\!] \subseteq [\![\chi_2]\!]$. Then, by Lem. 20, $[\![\chi_1[e/q.y]]\!] \subseteq [\![\chi_2[e/q.y]]\!]$. Then, by Defn. 37, $[\![\phi(q.y := e, \chi_1)]\!] \subseteq [\![\phi(q.y := e, \chi_2)]\!]$. Then, $[\![\phi(\gamma, \chi_1)]\!] \subseteq [\![\phi(\gamma, \chi_2)]\!]$.
- **Base:** $\gamma = p.e \twoheadrightarrow q.y$, for some p, q, y, e. Similar to case "$\gamma = q.y := e$, for some q, y, e".
- **Base:** $\gamma = i_R^R$, for some R. Similar to case $\gamma = \tau$. □

Proof (of Lem. 79). Recall $\mathcal{S} \in [\![\phi(\gamma, \chi)]\!]$. Then, by Defn. 37:

- **Base:** $\gamma = \tau$ and $\phi(\gamma, \tau) = \tau$.
  Recall $\mathcal{S} \in [\![\phi(\gamma, \chi)]\!]$. Then, by Defn. 20, $\mathsf{effect}(\tau, \mathcal{S}) \in [\![\phi(\gamma, \chi)]\!]$. Then, by Defn. 19, $\mathsf{effect}(\mathsf{con}_{\mathcal{S}}(\tau), \mathcal{S}) \in [\![\chi]\!]$. Then, $\mathsf{effect}(\mathsf{con}_{\mathcal{S}}(\gamma), \mathcal{S}) \in [\![\chi]\!]$.
- **Base:** $\gamma = q.y := e$ and $\phi(\gamma, \chi) = \chi[e/q.y]$, for some q, y, e.
  Recall $\mathcal{S} \in [\![\phi(\gamma, \chi)]\!]$. Then, $\mathcal{S} \in [\![\phi(q.y := e, \chi)]\!]$. Then, by Defn. 37, $\mathcal{S} \in [\![\chi[e/q.y]]\!]$. Then, by Lem. 14, $\mathcal{S}[\mathsf{eval}_{\mathcal{S}}(e)/q.y] \in [\![\chi]\!]$. Then, by Defn. 20, $\mathsf{effect}(q.y := \mathsf{eval}_{\mathcal{S}}(e), \mathcal{S}) \in [\![\chi]\!]$. Then, by Defn. 19, $\mathsf{effect}(\mathsf{con}_{\mathcal{S}}(q.y := e), \mathcal{S}) \in [\![\chi]\!]$. Then, $\mathsf{effect}(\mathsf{con}_{\mathcal{S}}(\gamma), \mathcal{S}) \in [\![\chi]\!]$.
- **Base:** $\gamma = p.e \twoheadrightarrow q.y$ and $\phi(\gamma, \chi) = \chi[e/q.y]$, for some p, q, y, e. Similar to case "$\gamma = q.y := e$ and $\phi(\gamma, \chi) = \chi[e/q.y]$, for some q, y, e".
- **Base:** $\gamma = i_R^R$ and $\phi(\gamma, \chi) = \chi$, for some R. Similar to case "$\gamma = \tau$ and $\phi(\gamma, \tau) = \tau$". □

Proof (of Lem. 80). By Defn. 13:

- **Case:** $[\![\forall(\psi \Rightarrow ((\xi^+ \Rightarrow \phi(G, \psi)) \wedge (\xi^- \Rightarrow \chi) \wedge \xi^{\equiv}))]\!] = [\![\mathtt{true}]\!]$.
  - Recall $[\![\phi(\mathbf{while}\, \xi|_0\, \{\psi\}\, G|_\emptyset, \chi)]\!] \subseteq [\![\phi(\mathbf{while}\, \xi|_0\, \{\psi\}\, G|_\emptyset, \chi)]\!]$. Then, by Defn. 38, $[\![\phi(\mathbf{while}\, \xi|_0\, \{\psi\}\, G|_\emptyset, \chi)]\!] \subseteq [\![\psi \wedge \forall(\psi \Rightarrow ((\xi^+ \Rightarrow \phi(G, \psi)) \wedge (\xi^- \Rightarrow \chi) \wedge \xi^{\equiv}))]\!]$. Then, by Defn. 13, $[\![\phi(\mathbf{while}\, \xi|_0\, \{\psi\}\, G|_\emptyset, \chi)]\!] \subseteq [\![\psi]\!] \cap [\![\forall(\psi \Rightarrow ((\xi^+ \Rightarrow \phi(G, \psi)) \wedge (\xi^- \Rightarrow \chi) \wedge \xi^{\equiv}))]\!]$. Then, by Lem. 24, $[\![\phi(\mathbf{while}\, \xi|_0\, \{\psi\}\, G|_\emptyset, \chi)]\!] \subseteq [\![\psi]\!] \cap [\![\psi \Rightarrow ((\xi^+ \Rightarrow \phi(G, \psi)) \wedge (\xi^- \Rightarrow \chi) \wedge \xi^{\equiv})]\!]$. Then, by Lem. 19, $[\![\phi(\mathbf{while}\, \xi|_0\, \{\psi\}\, G|_\emptyset, \chi)]\!] \subseteq [\![\psi]\!] \cap [\![(\xi^+ \Rightarrow \phi(G, \psi)) \wedge (\xi^- \Rightarrow \chi) \wedge \xi^{\equiv}]\!]$. Then, $[\![\phi(\mathbf{while}\, \xi|_0\, \{\psi\}\, G|_\emptyset, \chi)]\!] \subseteq [\![(\xi^+ \Rightarrow \phi(G, \psi)) \wedge (\xi^- \Rightarrow \chi) \wedge \xi^{\equiv}]\!]$.
  - By Lem. 15, $[\![\psi]\!] \subseteq [\![\mathtt{true}]\!]$. Then, $[\![\psi]\!] \subseteq [\![\psi]\!] \cap [\![\mathtt{true}]\!]$. Then, $[\![\psi]\!] \subseteq [\![\psi]\!] \cap [\![\forall(\psi \Rightarrow ((\xi^+ \Rightarrow \phi(G, \psi)) \wedge (\xi^- \Rightarrow \chi) \wedge \xi^{\equiv}))]\!]$. Then, by Defn. 13, $[\![\psi]\!] \subseteq [\![\psi \wedge \forall(\psi \Rightarrow ((\xi^+ \Rightarrow \phi(G, \psi)) \wedge (\xi^- \Rightarrow \chi) \wedge \xi^{\equiv}))]\!]$. Then, by Lem. 81, $[\![\phi(G, \psi)]\!] \subseteq [\![\phi(G, \psi \wedge \forall(\psi \Rightarrow ((\xi^+ \Rightarrow \phi(G, \psi)) \wedge (\xi^- \Rightarrow \chi) \wedge \xi^{\equiv})))]\!]$. Then, by Defn. 38, $[\![\phi(G, \psi)]\!] \subseteq [\![\phi(G, \phi(\mathbf{while}\, \xi|_0\, \{\psi\}\, G|_\emptyset, \chi))]\!]$. Then, by Defn. 38, $[\![\phi(G, \psi)]\!] \subseteq [\![\phi(G\, ;\, \mathbf{while}\, \xi|_0\, \{\psi\}\, G|_\emptyset, \chi)]\!]$. Then, by Lem. 22, $[\![\xi^+ \Rightarrow \phi(G, \psi)]\!] \subseteq [\![\xi^+ \Rightarrow \phi(G\, ;\, \mathbf{while}\, \xi|_0\, \{\psi\}\, G|_\emptyset, \chi)]\!]$.
  - Recall $[\![\chi]\!] \subseteq [\![\chi]\!]$. Then, by Defn. 38, $[\![\chi]\!] \subseteq [\![\phi(\mathbf{skip}, \chi)]\!]$. Then, by Lem. 22, $[\![\xi^- \Rightarrow \chi]\!] \subseteq [\![\xi^- \Rightarrow \phi(\mathbf{skip}, \chi)]\!]$.
  - Recall $[\![\xi^+ \Rightarrow \phi(G, \psi)]\!] \subseteq [\![\xi^+ \Rightarrow \phi(G\, ;\, \mathbf{while}\, \xi|_0\, \{\psi\}\, G|_\emptyset, \chi)]\!]$ and $[\![\xi^- \Rightarrow \chi]\!] \subseteq [\![\xi^- \Rightarrow \phi(\mathbf{skip}, \chi)]\!]$. Then, $[\![\xi^+ \Rightarrow \phi(G, \psi)]\!] \cap [\![\xi^- \Rightarrow \chi]\!] \subseteq [\![\xi^+ \Rightarrow \phi(G\, ;\, \mathbf{while}\, \xi|_0\, \{\psi\}\, G|_\emptyset, \chi)]\!] \cap [\![\xi^- \Rightarrow \phi(\mathbf{skip}, \chi)]\!]$. Then, $[\![\xi^+ \Rightarrow \phi(G, \psi)]\!] \cap [\![\xi^- \Rightarrow \chi]\!] \cap [\![\xi^{\equiv}]\!] \subseteq [\![\xi^+ \Rightarrow \phi(G\, ;\, \mathbf{while}\, \xi|_0\, \{\psi\}\, G|_\emptyset, \chi)]\!] \cap [\![\xi^- \Rightarrow \phi(\mathbf{skip}, \chi)]\!] \cap [\![\xi^{\equiv}]\!]$. Then, by Defn. 13, $[\![(\xi^+ \Rightarrow \phi(G, \psi)) \wedge (\xi^- \Rightarrow \chi) \wedge \xi^{\equiv}]\!] \subseteq [\![(\xi^+ \Rightarrow \phi(G\, ;\, \mathbf{while}\, \xi|_0\, \{\psi\}\, G|_\emptyset, \chi)) \wedge (\xi^- \Rightarrow \phi(\mathbf{skip}, \chi)) \wedge \xi^{\equiv}]\!]$. Then, by Defn. 38, $[\![(\xi^+ \Rightarrow \phi(G, \psi)) \wedge (\xi^- \Rightarrow \chi) \wedge \xi^{\equiv}]\!] \subseteq [\![\phi(\mathbf{if}\, \xi|_0\, (G\, ;\, \mathbf{while}\, \xi|_0\, \{\psi\}\, G|_\emptyset)|_\emptyset\, \mathbf{skip}|_\emptyset, \chi)]\!]$.

- Recall $[\![\phi(\textbf{while}\,\xi|_0\,\{\psi\}\,G|_\emptyset,\chi)]\!] \subseteq [\![(\xi^+ \Rightarrow \phi(G,\psi)) \wedge (\xi^- \Rightarrow \chi) \wedge \xi^\equiv]\!] \subseteq [\![\phi(\textbf{if}\,\xi|_0\,(G\,;\textbf{while}\,\xi|_0\,\{\psi\}\,G|_\emptyset)|_\emptyset\,\textbf{skip}|_\emptyset,\chi)]\!]$. Then, $[\![\phi(\textbf{while}\,\xi|_0\,\{\psi\}\,G|_\emptyset,\chi)]\!] \subseteq [\![\phi(\textbf{if}\,\xi|_0\,(G\,;\textbf{while}\,\xi|_0\,\{\psi\}\,G|_\emptyset)|_\emptyset\,\textbf{skip}|_\emptyset,\chi)]\!]$.

– **Case:** $[\![\forall(\psi \Rightarrow ((\xi^+ \Rightarrow \phi(G,\psi)) \wedge (\xi^- \Rightarrow \chi) \wedge \xi^\equiv))]\!] = [\![\texttt{false}]\!]$.
Recall $\emptyset \subseteq [\![\phi(\textbf{if}\,\xi|_0\,(G\,;\textbf{while}\,\xi|_0\,\{\psi\}\,G|_\emptyset)|_\emptyset\,\textbf{skip}|_\emptyset,\chi)]\!]$. Then, by Lem. 15, $[\![\texttt{false}]\!] \subseteq [\![\phi(\textbf{if}\,\xi|_0\,(G\,;\textbf{while}\,\xi|_0\,\{\psi\}\,G|_\emptyset)|_\emptyset\,\textbf{skip}|_\emptyset,\chi)]\!]$. Then, $[\![\forall(\psi \Rightarrow ((\xi^+ \Rightarrow \phi(G,\psi)) \wedge (\xi^- \Rightarrow \chi) \wedge \xi^\equiv))]\!] \subseteq [\![\phi(\textbf{if}\,\xi|_0\,(G\,;\textbf{while}\,\xi|_0\,\{\psi\}\,G|_\emptyset)|_\emptyset\,\textbf{skip}|_\emptyset,\chi)]\!]$. Then, $[\![\psi]\!] \cap [\![\forall(\psi \Rightarrow ((\xi^+ \Rightarrow \phi(G,\psi)) \wedge (\xi^- \Rightarrow \chi) \wedge \xi^\equiv))]\!] \subseteq [\![\phi(\textbf{if}\,\xi|_0\,(G\,;\textbf{while}\,\xi|_0\,\{\psi\}\,G|_\emptyset)|_\emptyset\,\textbf{skip}|_\emptyset,\chi)]\!]$. Then, by Defn. 13, $[\![\psi \wedge \forall(\psi \Rightarrow ((\xi^+ \Rightarrow \phi(G,\psi)) \wedge (\xi^- \Rightarrow \chi) \wedge \xi^\equiv))]\!] \subseteq [\![\phi(\textbf{if}\,\xi|_0\,(G\,;\textbf{while}\,\xi|_0\,\{\psi\}\,G|_\emptyset)|_\emptyset\,\textbf{skip}|_\emptyset,\chi)]\!]$. Then, by Defn. 38, $[\![\phi(\textbf{while}\,\xi|_0\,\{\psi\}\,G|_\emptyset,\chi)]\!] \subseteq [\![\phi(\textbf{if}\,\xi|_0\,(G\,;\textbf{while}\,\xi|_0\,\{\psi\}\,G|_\emptyset)|_\emptyset\,\textbf{skip}|_\emptyset,\chi)]\!]$. □

Proof (of Lem. 81).

1. By Defn. 21:
   – **Base:** $G = \xi$, for some ξ.
   Recall $[\![\chi_1]\!] \subseteq [\![\chi_2]\!]$. Then, $[\![\xi]\!] \cap [\![\chi_1]\!] \subseteq [\![\xi]\!] \cap [\![\chi_2]\!]$. Then, by Defn. 13, $[\![\xi \wedge \chi_1]\!] \subseteq [\![\xi \wedge \chi_2]\!]$. Then, by Defn. 38, $[\![\phi(\xi,\chi_1)]\!] \subseteq [\![\phi(\xi,\chi_2)]\!]$. Then, $[\![\phi(G,\chi_1)]\!] \subseteq [\![\phi(G,\chi_2)]\!]$.
   – **Base:** $G = \gamma$, for some γ.
   Recall $[\![\chi_1]\!] \subseteq [\![\chi_2]\!]$. Then, by Lem. 78, $[\![\phi(\gamma,\chi_1)]\!] \subseteq [\![\phi(\gamma,\chi_2)]\!]$. Then, $[\![\phi(G,\chi_1)]\!] \subseteq [\![\phi(G,\chi_2)]\!]$.
   – **Base:** $G = \textbf{skip}$.
   Recall $[\![\chi_1]\!] \subseteq [\![\chi_2]\!]$. Then, by Defn. 38, $[\![\phi(\textbf{skip},\chi_1)]\!] \subseteq [\![\phi(\textbf{skip},\chi_2)]\!]$. Then, $[\![\phi(G,\chi_1)]\!] \subseteq [\![\phi(G,\chi_2)]\!]$.
   – **Step:** $G = G_1\,;G_2$, for some G_1, G_2.
   Recall $[\![\chi_1]\!] \subseteq [\![\chi_2]\!]$. Then, by induction, $[\![\phi(G_2,\chi_1)]\!] \subseteq [\![\phi(G_2,\chi_2)]\!]$. Then, by induction, $[\![\phi(G_1,\phi(G_2,\chi_1))]\!] \subseteq [\![\phi(G_1,\phi(G_2,\chi_2))]\!]$. Then, by Defn. 38, $[\![\phi(G_1\,;G_2,\chi_1)]\!] \subseteq [\![\phi(G_1\,;G_2,\chi_2)]\!]$. Then, $[\![\phi(G,\chi_1)]\!] \subseteq [\![\phi(G,\chi_2)]\!]$.
   – **Step:** $G = G_1 \parallel G_2$, for some G_1, G_2.
   Recall:
     - **Case:** $G_1, G_2 \in \underline{\mathbb{G}}$ and $G_1 \# G_2$.
       * Recall $G_1, G_2 \in \underline{\mathbb{G}}$ and $G_1 \# G_2$. Then, by Defn. 38, $\phi(G_1 \parallel G_2,\chi_1) = \phi(G_1,\phi(G_2,\chi_1))$ and $\phi(G_1 \parallel G_2,\chi_2) = \phi(G_1,\phi(G_2,\chi_2))$.
       * Recall $[\![\chi_1]\!] \subseteq [\![\chi_2]\!]$. Then, by induction, $[\![\phi(G_2,\chi_1)]\!] \subseteq [\![\phi(G_2,\chi_2)]\!]$. Then, by induction, $[\![\phi(G_1,\phi(G_2,\chi_1))]\!] \subseteq [\![\phi(G_1,\phi(G_2,\chi_2))]\!]$. Then, $[\![\phi(G_1 \parallel G_2,\chi_1)]\!] \subseteq [\![\phi(G_1 \parallel G_2,\chi_2)]\!]$. Then, $[\![\phi(G,\chi_1)]\!] \subseteq [\![\phi(G,\chi_2)]\!]$.
     - **Case:** Not $G_1, G_2 \in \underline{\mathbb{G}}$, or not $G_1 \# G_2$.
       * Recall not $G_1, G_2 \in \underline{\mathbb{G}}$, or not $G_1 \# G_2$. Then, by Defn. 38, $\phi(G_1 \parallel G_2,\chi_1) = \texttt{false}$ and $\phi(G_1 \parallel G_2,\chi_2) = \texttt{false}$.
       * Recall $[\![\texttt{false}]\!] \subseteq [\![\texttt{false}]\!]$. Then, $[\![\phi(G_1 \parallel G_2,\chi_1)]\!] \subseteq [\![\phi(G_1 \parallel G_2,\chi_2)]\!]$. Then, $[\![\phi(G,\chi_1)]\!] \subseteq [\![\phi(G,\chi_2)]\!]$.
   – **Step:** $G = R.\textbf{if}\,\xi\,G_1\,G_2$, for some R, G_1, G_2, ξ.
   Recall $[\![\chi_1]\!] \subseteq [\![\chi_2]\!]$. Then, by induction, $[\![\phi(G_1,\chi_1)]\!] \subseteq [\![\phi(G_1,\chi_2)]\!]$ and $[\![\phi(G_2,\chi_1)]\!] \subseteq [\![\phi(G_2,\chi_2)]\!]$. Then, by Lem. 22, $[\![\xi^+ \Rightarrow \phi(G_1,\chi_1)]\!] \subseteq [\![\xi^+ \Rightarrow \phi(G_1,\chi_2)]\!]$ and $[\![\xi^- \Rightarrow \phi(G_2,\chi_1)]\!] \subseteq [\![\xi^- \Rightarrow \phi(G_2,\chi_2)]\!]$. Then, $[\![\xi^+ \Rightarrow \phi(G_1,\chi_1)]\!] \cap [\![\xi^- \Rightarrow \phi(G_2,\chi_1)]\!] \subseteq [\![\xi^+ \Rightarrow \phi(G_1,\chi_2)]\!] \cap [\![\xi^- \Rightarrow \phi(G_2,\chi_2)]\!]$. Then, $[\![\xi^+ \Rightarrow \phi(G_1,\chi_1)]\!] \cap [\![\xi^- \Rightarrow \phi(G_2,\chi_1)]\!] \cap [\![\xi^\equiv]\!] \subseteq [\![\xi^+ \Rightarrow \phi(G_1,\chi_2)]\!] \cap [\![\xi^- \Rightarrow \phi(G_2,\chi_2)]\!] \cap [\![\xi^\equiv]\!]$. Then, by Defn. 13, $[\![(\xi^+ \Rightarrow \phi(G_1,\chi_1)) \wedge (\xi^- \Rightarrow \phi(G_2,\chi_1)) \wedge \xi^\equiv]\!] \subseteq [\![(\xi^+ \Rightarrow \phi(G_1,\chi_2)) \wedge (\xi^- \Rightarrow \phi(G_2,\chi_2)) \wedge \xi^\equiv]\!]$. Then, by Defn. 38, $[\![\phi(R.\textbf{if}\,\xi\,G_1\,G_2,\chi_1)]\!] \subseteq [\![\phi(R.\textbf{if}\,\xi\,G_1\,G_2,\chi_2)]\!]$ Then, $[\![\phi(G,\chi_1)]\!] \subseteq [\![\phi(G,\chi_2)]\!]$.
   – **Step:** $G = R.\textbf{while}\,\xi\,\{\psi\}\,\hat{G}$, for some $R, \hat{G}, \psi, \xi$.
   Recall $[\![\chi_1]\!] \subseteq [\![\chi_2]\!]$. Then, by Lem. 22, $[\![\xi^- \Rightarrow \chi_1]\!] \subseteq [\![\xi^- \Rightarrow \chi_2]\!]$. Then, $[\![\xi^+ \Rightarrow \phi(\hat{G},\psi)]\!] \cap [\![\xi^- \Rightarrow \chi_1]\!] \cap [\![\xi^\equiv]\!] \subseteq [\![\xi^+ \Rightarrow \phi(\hat{G},\psi)]\!] \cap [\![\xi^- \Rightarrow \chi_2]\!] \cap [\![\xi^\equiv]\!]$. Then, by Defn. 13, $[\![(\xi^+ \Rightarrow \phi(\hat{G},\psi)) \wedge (\xi^- \Rightarrow \chi_1) \wedge \xi^\equiv]\!] \subseteq [\![(\xi^+ \Rightarrow \phi(\hat{G},\psi)) \wedge (\xi^- \Rightarrow \chi_2) \wedge \xi^\equiv]\!]$.

Then, by Lem. 22,
$[\![\psi \Rightarrow ((\xi^+ \Rightarrow \phi(\hat{G},\psi)) \wedge (\xi^- \Rightarrow \chi_1) \wedge \xi^{\equiv})]\!] \subseteq [\![\psi \Rightarrow ((\xi^+ \Rightarrow \phi(\hat{G},\psi)) \wedge (\xi^- \Rightarrow \chi_2) \wedge \xi^{\equiv})]\!]$.
Then, by Lem. 21, $[\![\forall(\psi \Rightarrow ((\xi^+ \Rightarrow \phi(\hat{G},\psi)) \wedge (\xi^- \Rightarrow \chi_1) \wedge \xi^{\equiv}))]\!] \subseteq [\![\forall(\psi \Rightarrow ((\xi^+ \Rightarrow \phi(\hat{G},\psi)) \wedge (\xi^- \Rightarrow \chi_2) \wedge \xi^{\equiv}))]\!]$. Then, $[\![\psi]\!] \cap [\![\forall(\psi \Rightarrow ((\xi^+ \Rightarrow \phi(\hat{G},\psi)) \wedge (\xi^- \Rightarrow \chi_1) \wedge \xi^{\equiv}))]\!] \subseteq [\![\psi]\!] \cap [\![\forall(\psi \Rightarrow ((\xi^+ \Rightarrow \phi(\hat{G},\psi)) \wedge (\xi^- \Rightarrow \chi_2) \wedge \xi^{\equiv}))]\!]$. Then, by Defn. 13,
$[\![\psi \wedge \forall(\psi \Rightarrow ((\xi^+ \Rightarrow \phi(\hat{G},\psi)) \wedge (\xi^- \Rightarrow \chi_1) \wedge \xi^{\equiv}))]\!] \subseteq [\![\psi \wedge \forall(\psi \Rightarrow ((\xi^+ \Rightarrow \phi(\hat{G},\psi)) \wedge (\xi^- \Rightarrow \chi_2) \wedge \xi^{\equiv}))]\!]$. Then, by Defn. 38,
$[\![\phi(R.\mathbf{while}\,\xi\,\{\psi\}\,\hat{G},\chi_1)]\!] \subseteq [\![\phi(R.\mathbf{while}\,\xi\,\{\psi\}\,\hat{G},\chi_2)]\!]$ Then, $[\![\phi(G,\chi_1)]\!] \subseteq [\![\phi(G,\chi_2)]\!]$.

– **Step:** $G = \mathbf{if}\,\xi|_0\, G_1|_{R_1}\, G_2|_{R_2}$, for some R_1, R_2, G_1, G_2, ξ.
Recall:
  - **Case:** $R_1 = \emptyset = R_2$.
    * Recall $R_1 = \emptyset = R_2$. Then, by Defn. 38,
    $\phi(\mathbf{if}\,\xi|_0\, G_1|_{R_1}\, G_2|_{R_2}, \chi_1) = (\xi^+ \Rightarrow \phi(G_1,\chi_1)) \wedge (\xi^- \Rightarrow \phi(G_2,\chi_1)) \wedge \xi^{\equiv}$ and
    $\phi(\mathbf{if}\,\xi|_0\, G_1|_{R_1}\, G_2|_{R_2}, \chi_2) = (\xi^+ \Rightarrow \phi(G_1,\chi_2)) \wedge (\xi^- \Rightarrow \phi(G_2,\chi_2)) \wedge \xi^{\equiv}$.
    * Recall $[\![\chi_1]\!] \subseteq [\![\chi_2]\!]$. Then, by induction, $[\![\phi(G_1,\chi_1)]\!] \subseteq [\![\phi(G_1,\chi_2)]\!]$ and
    $[\![\phi(G_2,\chi_1)]\!] \subseteq [\![\phi(G_2,\chi_2)]\!]$. Then, by Lem. 22,
    $[\![\xi^+ \Rightarrow \phi(G_1,\chi_1)]\!] \subseteq [\![\xi^+ \Rightarrow \phi(G_1,\chi_2)]\!]$ and
    $[\![\xi^- \Rightarrow \phi(G_2,\chi_1)]\!] \subseteq [\![\xi^- \Rightarrow \phi(G_2,\chi_2)]\!]$. Then,
    $[\![\xi^+ \Rightarrow \phi(G_1,\chi_1)]\!] \cap [\![\xi^- \Rightarrow \phi(G_2,\chi_1)]\!] \subseteq [\![\xi^+ \Rightarrow \phi(G_1,\chi_2)]\!] \cap [\![\xi^- \Rightarrow \phi(G_2,\chi_2)]\!]$.
    Then, $[\![\xi^+ \Rightarrow \phi(G_1,\chi_1)]\!] \cap [\![\xi^- \Rightarrow \phi(G_2,\chi_1)]\!] \cap [\![\xi^{\equiv}]\!] \subseteq [\![\xi^+ \Rightarrow \phi(G_1,\chi_2)]\!] \cap [\![\xi^- \Rightarrow \phi(G_2,\chi_2)]\!] \cap [\![\xi^{\equiv}]\!]$. Then, by Defn. 13, $[\![(\xi^+ \Rightarrow \phi(G_1,\chi_1)) \wedge (\xi^- \Rightarrow \phi(G_2,\chi_1)) \wedge \xi^{\equiv}]\!] \subseteq [\![(\xi^+ \Rightarrow \phi(G_1,\chi_2)) \wedge (\xi^- \Rightarrow \phi(G_2,\chi_2)) \wedge \xi^{\equiv}]\!]$. Then,
    $[\![\phi(\mathbf{if}\,\xi|_0\, G_1|_{R_1}\, G_2|_{R_2}, \chi_1)]\!] \subseteq [\![\phi(\mathbf{if}\,\xi|_0\, G_1|_{R_1}\, G_2|_{R_2}, \chi_2)]\!]$ Then,
    $[\![\phi(G,\chi_1)]\!] \subseteq [\![\phi(G,\chi_2)]\!]$.
  - **Case:** $R_1 = \emptyset \neq R_2$.
    * Recall $R_1 = \emptyset \neq R_2$. Then, by Defn. 38,
    $\phi(\mathbf{if}\,\xi|_0\, G_1|_{R_1}\, G_2|_{R_2}, \chi_1) = \phi(G_2,\chi_1) \wedge \bigwedge\{\xi^- \upharpoonright r\}_{r \in \mathsf{subj}(\xi) \setminus R_2}$ and
    $\phi(\mathbf{if}\,\xi|_0\, G_1|_{R_1}\, G_2|_{R_2}, \chi_2) = \phi(G_2,\chi_2) \wedge \bigwedge\{\xi^- \upharpoonright r\}_{r \in \mathsf{subj}(\xi) \setminus R_2}$.
    * Recall $[\![\chi_1]\!] \subseteq [\![\chi_2]\!]$. Then, by induction, $[\![\phi(G_2,\chi_1)]\!] \subseteq [\![\phi(G_2,\chi_2)]\!]$. Then,
    $[\![\phi(G_2,\chi_1)]\!] \cap [\![\bigwedge\{\xi \upharpoonright r\}_{r \in \mathsf{subj}(\xi) \setminus R_2}]\!] \subseteq [\![\phi(G_2,\chi_2)]\!] \cap [\![\bigwedge\{\xi^- \upharpoonright r\}_{r \in \mathsf{subj}(\xi) \setminus R_2}]\!]$. Then, by Defn. 13,
    $[\![\phi(G_2,\chi_1) \wedge \bigwedge\{\xi^- \upharpoonright r\}_{r \in \mathsf{subj}(\xi) \setminus R_2}]\!] \subseteq [\![\phi(G_2,\chi_2) \wedge \bigwedge\{\xi^- \upharpoonright r\}_{r \in \mathsf{subj}(\xi) \setminus R_2}]\!]$. Then,
    $[\![\phi(\mathbf{if}\,\xi|_0\, G_1|_{R_1}\, G_2|_{R_2}, \chi_1)]\!] \subseteq [\![\phi(\mathbf{if}\,\xi|_0\, G_1|_{R_1}\, G_2|_{R_2}, \chi_2)]\!]$ Then,
    $[\![\phi(G,\chi_1)]\!] \subseteq [\![\phi(G,\chi_2)]\!]$.
  - **Case:** $R_1 \neq \emptyset = R_2$. Similar to case $R_1 = \emptyset \neq R_2$.
  - **Case:** $R_1 \neq \emptyset \neq R_2$.
    * Recall $R_1 = \emptyset \neq R_2$. Then, by Defn. 38, $\phi(\mathbf{if}\,\xi|_0\, G_1|_{R_1}\, G_2|_{R_2}, \chi_1) = \texttt{false}$ and
    $\phi(\mathbf{if}\,\xi|_0\, G_1|_{R_1}\, G_2|_{R_2}, \chi_2) = \texttt{false}$.
    * Recall $[\![\texttt{false}]\!] \subseteq [\![\texttt{false}]\!]$. Then,
    $[\![\phi(\mathbf{if}\,\xi|_0\, G_1|_{R_1}\, G_2|_{R_2}, \chi_1)]\!] \subseteq [\![\phi(\mathbf{if}\,\xi|_0\, G_1|_{R_1}\, G_2|_{R_2}, \chi_2)]\!]$. Then,
    $[\![\phi(G,\chi_1)]\!] \subseteq [\![\phi(G,\chi_2)]\!]$.

– **Step:** $G = \mathbf{while}\,\xi|_0\,\{\psi\}\,\hat{G}|_{\emptyset}$, for some $\hat{G}, \psi, \xi$. Similar to case "$G = R.\mathbf{while}\,\xi\,\{\psi\}\,\hat{G}$, for some $R, \hat{G}, \psi, \xi$". □

2. Recall $[\![\chi_1]\!] = [\![\chi_2]\!]$. Then, $[\![\chi_1]\!] \subseteq [\![\chi_2]\!]$ and $[\![\chi_1]\!] \supseteq [\![\chi_2]\!]$. Then, by 1, $[\![\phi(G,\chi_1)]\!] \subseteq [\![\phi(G,\chi_2)]\!]$ and $[\![\phi(G,\chi_1)]\!] \supseteq [\![\phi(G,\chi_2)]\!]$. Then, $[\![\phi(G,\chi_1)]\!] = [\![\phi(G,\chi_2)]\!]$. □

Proof (of Lem. 82). By Defn. 21:

– **Base:** $G = \xi$, for some ξ.
Recall $[\![\xi]\!] \cap [\![\chi_1]\!] \cap [\![\chi_2]\!] = [\![\xi]\!] \cap [\![\chi_1]\!] \cap [\![\xi]\!] \cap [\![\chi_2]\!]$. Then, by Defn. 13,
$[\![\xi \wedge \chi_1 \wedge \chi_2]\!] = [\![\xi \wedge \chi_1 \wedge \xi \wedge \chi_2]\!]$. Then, by Defn. 38, $[\![\phi(\xi,\chi_1 \wedge \chi_2)]\!] = [\![\phi(\xi,\chi_1) \wedge \phi(\xi,\chi_2)]\!]$.
Then, $[\![\phi(G,\chi_1 \wedge \chi_2)]\!] = [\![\phi(G,\chi_1) \wedge \phi(G,\chi_2)]\!]$.

- **Base:** $G = \gamma$, for some γ.
  Recall $[\![\phi(\gamma, \chi_1 \wedge \chi_2)]\!] = [\![\phi(\gamma, \chi_1 \wedge \chi_2)]\!]$. Then, by Lem. 75,
  $[\![\phi(\gamma, \chi_1 \wedge \chi_2)]\!] = [\![\phi(\gamma, \chi_1) \wedge \phi(\gamma, \chi_2)]\!]$. Then, by Defn. 38,
  $[\![\phi(\gamma, \chi_1 \wedge \chi_2)]\!] = [\![\phi(\gamma, \chi_1) \wedge \phi(\gamma, \chi_2)]\!]$. Then, $[\![\phi(G, \chi_1 \wedge \chi_2)]\!] = [\![\phi(G, \chi_1) \wedge \phi(G, \chi_2)]\!]$.
- **Base:** $G = \mathbf{skip}$.
  Recall $[\![\chi_1 \wedge \chi_2]\!] = [\![\chi_1 \wedge \chi_2]\!]$. Then, by Defn. 38,
  $[\![\phi(\mathbf{skip}, \chi_1 \wedge \chi_2)]\!] = [\![\phi(\mathbf{skip}, \chi_1) \wedge \phi(\mathbf{skip}, \chi_2)]\!]$. Then,
  $[\![\phi(G, \chi_1 \wedge \chi_2)]\!] = [\![\phi(G, \chi_1) \wedge \phi(G, \chi_2)]\!]$.
- **Step:** $G = G_1 \,;\, G_2$, for some G_1, G_2.
  By induction, $[\![\phi(G_2, \chi_1 \wedge \chi_2)]\!] = [\![\phi(G_2, \chi_1) \wedge \phi(G_2, \chi_2)]\!]$. Then, by Lem. 81,
  $[\![\phi(G_1, \phi(G_2, \chi_1 \wedge \chi_2))]\!] = [\![\phi(G_1, \phi(G_2, \chi_1) \wedge \phi(G_2, \chi_2))]\!]$. Then, by induction,
  $[\![\phi(G_1, \phi(G_2, \chi_1 \wedge \chi_2))]\!] = [\![\phi(G_1, \phi(G_2, \chi_1)) \wedge \phi(G_1, \phi(G_2, \chi_2))]\!]$. Then, by Defn. 38,
  $[\![\phi(G_1 \,;\, G_2, \chi_1 \wedge \chi_2)]\!] = [\![\phi(G_1 \,;\, G_2, \chi_1) \wedge \phi(G_1 \,;\, G_2, \chi_2)]\!]$. Then,
  $[\![\phi(G, \chi_1 \wedge \chi_2)]\!] = [\![\phi(G, \chi_1) \wedge \phi(G, \chi_2)]\!]$.
- **Step:** $G = G_1 \parallel G_2$, for some G_1, G_2.
  Recall:
  - **Case:** $G_1, G_2 \in \underline{\mathbb{G}}$ and $G_1 \# G_2$.
    - Recall $G_1, G_2 \in \underline{\mathbb{G}}$ and $G_1 \# G_2$. Then, by Defn. 38,
      $\phi(G_1 \parallel G_2, \chi_1 \wedge \chi_2) = \phi(G_1, \phi(G_2, \chi_1 \wedge \chi_2))$ and $\phi(G_1 \parallel G_2, \chi_1) = \phi(G_1, \phi(G_2, \chi_1))$
      and $\phi(G_1 \parallel G_2, \chi_2) = \phi(G_1, \phi(G_2, \chi_2))$.
    - By induction, $[\![\phi(G_2, \chi_1 \wedge \chi_2)]\!] = [\![\phi(G_2, \chi_1) \wedge \phi(G_2, \chi_2)]\!]$. Then, by Lem. 81,
      $[\![\phi(G_1, \phi(G_2, \chi_1 \wedge \chi_2))]\!] = [\![\phi(G_1, \phi(G_2, \chi_1) \wedge \phi(G_2, \chi_2))]\!]$. Then, by induction,
      $[\![\phi(G_1, \phi(G_2, \chi_1 \wedge \chi_2))]\!] = [\![\phi(G_1, \phi(G_2, \chi_1)) \wedge \phi(G_1, \phi(G_2, \chi_2))]\!]$. Then,
      $[\![\phi(G_1 \parallel G_2, \chi_1 \wedge \chi_2)]\!] = [\![\phi(G_1 \parallel G_2, \chi_1) \wedge \phi(G_1 \parallel G_2, \chi_2)]\!]$. Then,
      $[\![\phi(G, \chi_1 \wedge \chi_2)]\!] = [\![\phi(G, \chi_1) \wedge \phi(G, \chi_2)]\!]$.
  - **Case:** Not $G_1, G_2 \in \underline{\mathbb{G}}$, or not $G_1 \# G_2$.
    - Recall not $G_1, G_2 \in \underline{\mathbb{G}}$, or not $G_1 \# G_2$. Then, by Defn. 38, $\phi(G_1 \parallel G_2, \chi) = \mathtt{false}$
      and $\phi(G_1 \parallel G_2, \chi_1) = \mathtt{false}$ and $\phi(G_1 \parallel G_2, \chi_2) = \mathtt{false}$.
    - Recall $\emptyset = \emptyset \cap \emptyset$. Then, by Lem. 15, $[\![\mathtt{false}]\!] = [\![\mathtt{false}]\!] \cap [\![\mathtt{false}]\!]$. Then, by Defn. 13,
      $[\![\mathtt{false}]\!] = [\![\mathtt{false} \wedge \mathtt{false}]\!]$. Then,
      $[\![\phi(G_1 \parallel G_2, \chi_1 \wedge \chi_2)]\!] = [\![\phi(G_1 \parallel G_2, \chi_1) \wedge \phi(G_1 \parallel G_2, \chi_2)]\!]$. Then,
      $[\![\phi(G, \chi_1 \wedge \chi_2)]\!] = [\![\phi(G, \chi_1) \wedge \phi(G, \chi_2)]\!]$.
- **Step:** $G = R.\mathbf{if}\ \xi\ G_1\ G_2$, for some R, G_1, G_2, ξ.
  By induction, $[\![\phi(G_1, \chi_1 \wedge \chi_2)]\!] = [\![\phi(G_1, \chi_1) \wedge \phi(G_1, \chi_2)]\!]$ and
  $[\![\phi(G_2, \chi_1 \wedge \chi_2)]\!] = [\![\phi(G_2, \chi_1) \wedge \phi(G_2, \chi_2)]\!]$. Then, by Lem. 22,
  $[\![\xi^+ \Rightarrow \phi(G_1, \chi_1 \wedge \chi_2)]\!] = [\![\xi^+ \Rightarrow (\phi(G_1, \chi_1) \wedge \phi(G_1, \chi_2))]\!]$ and
  $[\![\xi^- \Rightarrow \phi(G_2, \chi_1 \wedge \chi_2)]\!] = [\![\xi^- \Rightarrow (\phi(G_2, \chi_1) \wedge \phi(G_2, \chi_2))]\!]$. Then, by Lem. 18,
  $[\![\xi^+ \Rightarrow \phi(G_1, \chi_1 \wedge \chi_2)]\!] = [\![(\xi^+ \Rightarrow \phi(G_1, \chi_1)) \wedge (\xi^+ \Rightarrow \phi(G_1, \chi_2))]\!]$ and
  $[\![\xi^- \Rightarrow \phi(G_2, \chi_1 \wedge \chi_2)]\!] = [\![(\xi^- \Rightarrow \phi(G_2, \chi_1)) \wedge (\xi^- \Rightarrow \phi(G_2, \chi_2))]\!]$. Then,
  $[\![\xi^+ \Rightarrow \phi(G_1, \chi_1 \wedge \chi_2)]\!] \cap [\![\xi^- \Rightarrow \phi(G_2, \chi_1 \wedge \chi_2)]\!] = [\![(\xi^+ \Rightarrow \phi(G_1, \chi_1)) \wedge (\xi^+ \Rightarrow \phi(G_1, \chi_2))]\!] \cap [\![(\xi^- \Rightarrow \phi(G_2, \chi_1)) \wedge (\xi^- \Rightarrow \phi(G_2, \chi_2))]\!]$. Then,
  $[\![\xi^+ \Rightarrow \phi(G_1, \chi_1 \wedge \chi_2)]\!] \cap [\![\xi^- \Rightarrow \phi(G_2, \chi_1 \wedge \chi_2)]\!] \cap [\![\xi^\equiv]\!] = [\![(\xi^+ \Rightarrow \phi(G_1, \chi_1)) \wedge (\xi^+ \Rightarrow \phi(G_1, \chi_2))]\!] \cap [\![(\xi^- \Rightarrow \phi(G_2, \chi_1)) \wedge (\xi^- \Rightarrow \phi(G_2, \chi_2))]\!] \cap [\![\xi^\equiv]\!] \cap [\![\xi^\equiv]\!]$. Then, by Defn. 13,
  $[\![(\xi^+ \Rightarrow \phi(G_1, \chi_1 \wedge \chi_2)) \wedge (\xi^- \Rightarrow \phi(G_2, \chi_1 \wedge \chi_2)) \wedge \xi^\equiv]\!] = [\![(\xi^+ \Rightarrow \phi(G_1, \chi_1)) \wedge (\xi^+ \Rightarrow \phi(G_1, \chi_2)) \wedge (\xi^- \Rightarrow \phi(G_2, \chi_1)) \wedge (\xi^- \Rightarrow \phi(G_2, \chi_2)) \wedge \xi^\equiv \wedge \xi^\equiv]\!]$. Then, by Defn. 38,
  $[\![\phi(R.\mathbf{if}\ \xi\ G_1\ G_2, \chi_1 \wedge \chi_2)]\!] = [\![\phi(R.\mathbf{if}\ \xi\ G_1\ G_2, \chi_1) \wedge \phi(R.\mathbf{if}\ \xi\ G_1\ G_2, \chi_2)]\!]$. Then,
  $[\![\phi(G, \chi_1 \wedge \chi_2)]\!] = [\![\phi(G, \chi_1) \wedge \phi(G, \chi_2)]\!]$.
- **Step:** $G = R.\mathbf{while}\ \xi\ \{\psi\}\ \hat{G}$, for some $R, \hat{G}, \psi, \xi$.
  By Lem. 18, $[\![\xi^- \Rightarrow (\chi_1 \wedge \chi_2)]\!] = [\![(\xi^- \Rightarrow \chi_1) \wedge (\xi^- \Rightarrow \chi_2)]\!]$. Then,
  $[\![\xi^- \Rightarrow (\chi_1 \wedge \chi_2)]\!] \cap [\![(\xi^+ \Rightarrow \phi(\hat{G}, \psi)) \wedge \xi^\equiv]\!] = [\![(\xi^- \Rightarrow \chi_1) \wedge (\xi^- \Rightarrow \chi_2)]\!] \cap [\![(\xi^+ \Rightarrow \phi(\hat{G}, \psi)) \wedge \xi^\equiv]\!] \cap [\![(\xi^+ \Rightarrow \phi(\hat{G}, \psi)) \wedge \xi^\equiv]\!]$. Then, by Defn. 13, $[\![(\xi^- \Rightarrow (\chi_1 \wedge \chi_2)) \wedge (\xi^+ \Rightarrow \phi(\hat{G}, \psi)) \wedge \xi^\equiv]\!] = [\![(\xi^- \Rightarrow \chi_1) \wedge (\xi^- \Rightarrow \chi_2) \wedge (\xi^+ \Rightarrow \phi(\hat{G}, \psi)) \wedge \xi^\equiv \wedge (\xi^+ \Rightarrow \phi(\hat{G}, \psi)) \wedge \xi^\equiv]\!]$.
  Then, by Lem. 22, $[\![\psi \Rightarrow ((\xi^- \Rightarrow (\chi_1 \wedge \chi_2)) \wedge (\xi^+ \Rightarrow \phi(\hat{G}, \psi)) \wedge \xi^\equiv)]\!] = [\![\psi \Rightarrow ((\xi^- \Rightarrow \chi_1) \wedge (\xi^- \Rightarrow \chi_2) \wedge (\xi^+ \Rightarrow \phi(\hat{G}, \psi)) \wedge \xi^\equiv \wedge (\xi^+ \Rightarrow \phi(\hat{G}, \psi)) \wedge \xi^\equiv)]\!]$. Then, by Lem. 18,

$[\![\psi \Rightarrow ((\xi^- \Rightarrow (\chi_1 \wedge \chi_2)) \wedge (\xi^+ \Rightarrow \phi(\hat{G},\psi)) \wedge \xi^\equiv)]\!] = [\![(\psi \Rightarrow ((\xi^- \Rightarrow \chi_1) \wedge (\xi^+ \Rightarrow \phi(\hat{G},\psi)) \wedge \xi^\equiv)) \wedge (\psi \Rightarrow ((\xi^- \Rightarrow \chi_2) \wedge (\xi^+ \Rightarrow \phi(\hat{G},\psi)) \wedge \xi^\equiv))]\!]$. Then, by Lem. 21, $[\![\forall(\psi \Rightarrow ((\xi^- \Rightarrow (\chi_1 \wedge \chi_2)) \wedge (\xi^+ \Rightarrow \phi(\hat{G},\psi)) \wedge \xi^\equiv))]\!] = [\![\forall((\psi \Rightarrow ((\xi^- \Rightarrow \chi_1) \wedge (\xi^+ \Rightarrow \phi(\hat{G},\psi)) \wedge \xi^\equiv)) \wedge (\psi \Rightarrow ((\xi^- \Rightarrow \chi_2) \wedge (\xi^+ \Rightarrow \phi(\hat{G},\psi)) \wedge \xi^\equiv)))]\!]$. Then, by Lem. 16, $[\![\forall(\psi \Rightarrow ((\xi^- \Rightarrow (\chi_1 \wedge \chi_2)) \wedge (\xi^+ \Rightarrow \phi(\hat{G},\psi)) \wedge \xi^\equiv))]\!] = [\![\forall(\psi \Rightarrow ((\xi^- \Rightarrow \chi_1) \wedge (\xi^+ \Rightarrow \phi(\hat{G},\psi)) \wedge \xi^\equiv)) \wedge \forall(\psi \Rightarrow ((\xi^- \Rightarrow \chi_2) \wedge (\xi^+ \Rightarrow \phi(\hat{G},\psi)) \wedge \xi^\equiv))]\!]$. Then, $[\![\psi]\!] \cap [\![\forall(\psi \Rightarrow ((\xi^- \Rightarrow (\chi_1 \wedge \chi_2)) \wedge (\xi^+ \Rightarrow \phi(\hat{G},\psi)) \wedge \xi^\equiv))]\!] = [\![\psi]\!] \cap [\![\psi]\!] \cap [\![\forall(\psi \Rightarrow ((\xi^- \Rightarrow \chi_1) \wedge (\xi^+ \Rightarrow \phi(\hat{G},\psi)) \wedge \xi^\equiv)) \wedge \forall(\psi \Rightarrow ((\xi^- \Rightarrow \chi_2) \wedge (\xi^+ \Rightarrow \phi(\hat{G},\psi)) \wedge \xi^\equiv))]\!]$. Then, by Defn. 13, $[\![\psi \wedge \forall(\psi \Rightarrow ((\xi^- \Rightarrow (\chi_1 \wedge \chi_2)) \wedge (\xi^+ \Rightarrow \phi(\hat{G},\psi)) \wedge \xi^\equiv))]\!] = [\![\psi \wedge \psi \wedge \forall(\psi \Rightarrow ((\xi^- \Rightarrow \chi_1) \wedge (\xi^+ \Rightarrow \phi(\hat{G},\psi)) \wedge \xi^\equiv)) \wedge \forall(\psi \Rightarrow ((\xi^- \Rightarrow \chi_2) \wedge (\xi^+ \Rightarrow \phi(\hat{G},\psi)) \wedge \xi^\equiv))]\!]$. Then, by Defn. 38,
$[\![\phi(R.\mathbf{while}\,\xi\,\{\psi\}\,\hat{G},\chi_1 \wedge \chi_2)]\!] = [\![\phi(R.\mathbf{while}\,\xi\,\{\psi\}\,\hat{G},\chi_1) \wedge \phi(R.\mathbf{while}\,\xi\,\{\psi\}\,\hat{G},\chi_2)]\!]$. Then,
$[\![\phi(G,\chi_1 \wedge \chi_2)]\!] = [\![\phi(G,\chi_1) \wedge \phi(G,\chi_2)]\!]$.

– **Step:** $G = \mathbf{if}\ \xi|_0\ G_1|_{R_1}\ G_2|_{R_2}$, for some R_1, R_2, G_1, G_2, ξ.
  Recall:
  - **Case:** $R_1 = \emptyset = R_2$.
    * Recall $R_1 = \emptyset = R_2$. Then, by Defn. 38,
      $\phi(\mathbf{if}\ \xi|_0\ G_1|_{R_1}\ G_2|_{R_2},\chi_1 \wedge \chi_2) = (\xi^+ \Rightarrow \phi(G_1,\chi_1 \wedge \chi_2)) \wedge (\xi^- \Rightarrow \phi(G_2,\chi_1 \wedge \chi_2)) \wedge \xi^\equiv$ and $\phi(\mathbf{if}\ \xi|_0\ G_1|_{R_1}\ G_2|_{R_2},\chi_1) = (\xi^+ \Rightarrow \phi(G_1,\chi_1)) \wedge (\xi^- \Rightarrow \phi(G_2,\chi_1)) \wedge \xi^\equiv$ and $\phi(\mathbf{if}\ \xi|_0\ G_1|_{R_1}\ G_2|_{R_2},\chi_2) = (\xi^+ \Rightarrow \phi(G_1,\chi_2)) \wedge (\xi^- \Rightarrow \phi(G_2,\chi_2)) \wedge \xi^\equiv$.
    * By induction, $[\![\phi(G_1,\chi_1 \wedge \chi_2)]\!] = [\![\phi(G_1,\chi_1) \wedge \phi(G_1,\chi_2)]\!]$ and $[\![\phi(G_2,\chi_1 \wedge \chi_2)]\!] = [\![\phi(G_2,\chi_1) \wedge \phi(G_2,\chi_2)]\!]$. Then, by Lem. 22, $[\![\xi^+ \Rightarrow \phi(G_1,\chi_1 \wedge \chi_2)]\!] = [\![\xi^+ \Rightarrow (\phi(G_1,\chi_1) \wedge \phi(G_1,\chi_2))]\!]$ and $[\![\xi^- \Rightarrow \phi(G_2,\chi_1 \wedge \chi_2)]\!] = [\![\xi^- \Rightarrow (\phi(G_2,\chi_1) \wedge \phi(G_2,\chi_2))]\!]$. Then, by Lem. 18, $[\![\xi^+ \Rightarrow \phi(G_1,\chi_1 \wedge \chi_2)]\!] = [\![(\xi^+ \Rightarrow \phi(G_1,\chi_1)) \wedge (\xi^+ \Rightarrow \phi(G_1,\chi_2))]\!]$ and $[\![\xi^- \Rightarrow \phi(G_2,\chi_1 \wedge \chi_2)]\!] = [\![(\xi^- \Rightarrow \phi(G_2,\chi_1)) \wedge (\xi^- \Rightarrow \phi(G_2,\chi_2))]\!]$. Then, $[\![\xi^+ \Rightarrow \phi(G_1,\chi_1 \wedge \chi_2)]\!] \cap [\![\xi^- \Rightarrow \phi(G_2,\chi_1 \wedge \chi_2)]\!] = [\![(\xi^+ \Rightarrow \phi(G_1,\chi_1)) \wedge (\xi^+ \Rightarrow \phi(G_1,\chi_2))]\!] \cap [\![(\xi^- \Rightarrow \phi(G_2,\chi_1)) \wedge (\xi^- \Rightarrow \phi(G_2,\chi_2))]\!]$. Then, $[\![\xi^+ \Rightarrow \phi(G_1,\chi_1 \wedge \chi_2)]\!] \cap [\![\xi^- \Rightarrow \phi(G_2,\chi_1 \wedge \chi_2)]\!] \cap [\![\xi^\equiv]\!] = [\![(\xi^+ \Rightarrow \phi(G_1,\chi_1)) \wedge (\xi^+ \Rightarrow \phi(G_1,\chi_2))]\!] \cap [\![(\xi^- \Rightarrow \phi(G_2,\chi_1)) \wedge (\xi^- \Rightarrow \phi(G_2,\chi_2))]\!] \cap [\![\xi^\equiv]\!] \cap [\![\xi^\equiv]\!]$. Then, by Defn. 13, $[\![(\xi^+ \Rightarrow \phi(G_1,\chi_1 \wedge \chi_2)) \wedge (\xi^- \Rightarrow \phi(G_2,\chi_1 \wedge \chi_2)) \wedge \xi^\equiv]\!] = [\![(\xi^+ \Rightarrow \phi(G_1,\chi_1)) \wedge (\xi^+ \Rightarrow \phi(G_1,\chi_2)) \wedge (\xi^- \Rightarrow \phi(G_2,\chi_1)) \wedge (\xi^- \Rightarrow \phi(G_2,\chi_2)) \wedge \xi^\equiv \wedge \xi^\equiv]\!]$.
      Then, $[\![\phi(\mathbf{if}\ \xi|_0\ G_1|_{R_1}\ G_2|_{R_2},\chi_1 \wedge \chi_2)]\!] = [\![\phi(\mathbf{if}\ \xi|_0\ G_1|_{R_1}\ G_2|_{R_2},\chi_1) \wedge \phi(\mathbf{if}\ \xi|_0\ G_1|_{R_1}\ G_2|_{R_2},\chi_2)]\!]$. Then,
      $[\![\phi(G,\chi_1 \wedge \chi_2)]\!] = [\![\phi(G,\chi_1) \wedge \phi(G,\chi_2)]\!]$.
  - **Case:** $R_1 = \emptyset \neq R_2$.
    * Recall $R_1 = \emptyset \neq R_2$. Then, by Defn. 38,
      $\phi(\mathbf{if}\ \xi|_0\ G_1|_{R_1}\ G_2|_{R_2},\chi_1 \wedge \chi_2) = \phi(G_2,\chi_1 \wedge \chi_2) \wedge \bigwedge\{\xi^- \upharpoonright r\}_{r \in \mathsf{subj}(\xi) \setminus R_2}$ and $\phi(\mathbf{if}\ \xi|_0\ G_1|_{R_1}\ G_2|_{R_2},\chi_1) = \phi(G_2,\chi_1) \wedge \bigwedge\{\xi^- \upharpoonright r\}_{r \in \mathsf{subj}(\xi) \setminus R_2}$ and $\phi(\mathbf{if}\ \xi|_0\ G_1|_{R_1}\ G_2|_{R_2},\chi_2) = \phi(G_2,\chi_2) \wedge \bigwedge\{\xi^- \upharpoonright r\}_{r \in \mathsf{subj}(\xi) \setminus R_2}$.
    * By induction, $[\![\phi(G_2,\chi_1 \wedge \chi_2)]\!] = [\![\phi(G_2,\chi_1) \wedge \phi(G_2,\chi_2)]\!]$. Then, $[\![\phi(G_2,\chi_1 \wedge \chi_2)]\!] \cap [\![\bigwedge\{\xi^- \upharpoonright r\}_{r \in \mathsf{subj}(\xi) \setminus R_2}]\!] = [\![\phi(G_2,\chi_1) \wedge \phi(G_2,\chi_2)]\!] \cap [\![\bigwedge\{\xi^- \upharpoonright r\}_{r \in \mathsf{subj}(\xi) \setminus R_2}]\!] \cap [\![\bigwedge\{\xi^- \upharpoonright r\}_{r \in \mathsf{subj}(\xi) \setminus R_2}]\!]$. Then, by Defn. 13, $[\![\phi(G_2,\chi_1 \wedge \chi_2) \wedge \bigwedge\{\xi^- \upharpoonright r\}_{r \in \mathsf{subj}(\xi) \setminus R_2}]\!] = [\![\phi(G_2,\chi_1) \wedge \phi(G_2,\chi_2) \wedge \bigwedge\{\xi^- \upharpoonright r\}_{r \in \mathsf{subj}(\xi) \setminus R_2} \wedge \bigwedge\{\xi^- \upharpoonright r\}_{r \in \mathsf{subj}(\xi) \setminus R_2}]\!]$. Then, $[\![\phi(\mathbf{if}\ \xi|_0\ G_1|_{R_1}\ G_2|_{R_2},\chi_1 \wedge \chi_2)]\!] = [\![\phi(\mathbf{if}\ \xi|_0\ G_1|_{R_1}\ G_2|_{R_2},\chi_1) \wedge \phi(\mathbf{if}\ \xi|_0\ G_1|_{R_1}\ G_2|_{R_2},\chi_2)]\!]$. Then,
      $[\![\phi(G,\chi_1 \wedge \chi_2)]\!] = [\![\phi(G,\chi_1) \wedge \phi(G,\chi_2)]\!]$.
  - **Case:** $R_1 \neq \emptyset = R_2$. Similar to case $R_1 = \emptyset \neq R_2$.
  - **Case:** $R_1 \neq \emptyset \neq R_2$.
    * Recall $R_1 = \emptyset \neq R_2$. Then, by Defn. 38, $\phi(\mathbf{if}\ \xi|_0\ G_1|_{R_1}\ G_2|_{R_2},\chi_1 \wedge \chi_2) = \mathtt{false}$ and $\phi(\mathbf{if}\ \xi|_0\ G_1|_{R_1}\ G_2|_{R_2},\chi_1) = \mathtt{false}$ and $\phi(\mathbf{if}\ \xi|_0\ G_1|_{R_1}\ G_2|_{R_2},\chi_2) = \mathtt{false}$.
    * Recall $[\![\mathtt{false}]\!] = [\![\mathtt{false}]\!] \cap [\![\mathtt{false}]\!]$. Then, $[\![\phi(\mathbf{if}\ \xi|_0\ G_1|_{R_1}\ G_2|_{R_2},\chi_1 \wedge \chi_2)]\!] = [\![\phi(\mathbf{if}\ \xi|_0\ G_1|_{R_1}\ G_2|_{R_2},\chi_1) \wedge \phi(\mathbf{if}\ \xi|_0\ G_1|_{R_1}\ G_2|_{R_2},\chi_2)]\!]$. Then,
      $[\![\phi(G,\chi_1 \wedge \chi_2)]\!] = [\![\phi(G,\chi_1) \wedge \phi(G,\chi_2)]\!]$.

– **Step:** $G = \textbf{while}\,\xi|_0\ \{\psi\}\ \hat{G}|_\emptyset$, for some $\hat{G}, \psi, \xi$. Similar to case "$G = R.\textbf{while}\,\xi\ \{\psi\}\ \hat{G}$, for some $R, \hat{G}, \psi, \xi$". □

Proof (of Lem. 83). By Defn. 21:

– **Base:** $\underline{G} = q.y{:=}e$, for some q, y, e.
Recall $\mathsf{write}(q.y{:=}e) \cap \emptyset = \emptyset$. Then, by Lem. 3, $\mathsf{write}(q.y{:=}e) \cap \mathsf{read}(\mathtt{false}) = \emptyset$. Then, by Lem. 76, $\phi(q.y{:=}e, \mathtt{false}) = \mathtt{false}$. Then, $[\![\phi(q.y{:=}e, \mathtt{false})]\!] = [\![\mathtt{false}]\!]$. Then, $[\![\phi(\underline{G}, \mathtt{false})]\!] = [\![\mathtt{false}]\!]$.
– **Base:** $\underline{G} = p.e \rightarrow q.y$, for some p, q, y, e. Similar to case "$\underline{G} = q.y{:=}e$, for some q, y, e".
– **Base:** $\underline{G} = \textbf{skip}$.
By Defn. 38, $\phi(\textbf{skip}, \mathtt{false}) = \mathtt{false}$. Then, $[\![\phi(\textbf{skip}, \mathtt{false})]\!] = [\![\mathtt{false}]\!]$. Then, $[\![\phi(\underline{G}, \mathtt{false})]\!] = [\![\mathtt{false}]\!]$.
– **Step:** $\underline{G} = \underline{G_1}\ ;\underline{G_2}$, for some $\underline{G_1}, \underline{G_2}$.
By induction, $[\![\phi(\underline{G_2}, \mathtt{false})]\!] = [\![\mathtt{false}]\!]$. Then, by Lem. 81, $[\![\phi(\underline{G_1}, \phi(\underline{G_2}, \mathtt{false}))]\!] = [\![\phi(\underline{G_1}, \mathtt{false})]\!]$. Then, by induction, $[\![\phi(\underline{G_1}, \phi(\underline{G_2}, \mathtt{false}))]\!] = [\![\mathtt{false}]\!]$. Then, by Defn. 38, $[\![\phi(\underline{G_1}\ ;\underline{G_2}, \mathtt{false})]\!] = [\![\mathtt{false}]\!]$. Then, $[\![\phi(\underline{G}, \mathtt{false})]\!] = [\![\mathtt{false}]\!]$.
– **Step:** $\underline{G} = \underline{G_1} \parallel \underline{G_2}$, for some $\underline{G_1}, \underline{G_2}$.
Recall:
  - **Case:** $\underline{G_1}, \underline{G_2} \in \mathbb{G}$ and $\underline{G_1} \# \underline{G_2}$.
    * Recall $\underline{G_1}, \underline{G_2} \in \mathbb{G}$ and $\underline{G_1} \# \underline{G_2}$. Then, by Defn. 38, $\phi(\underline{G_1} \parallel \underline{G_2}, \mathtt{false}) = \phi(\underline{G_1}, \phi(\underline{G_2}, \mathtt{false}))$.
    * By induction, $[\![\phi(\underline{G_2}, \mathtt{false})]\!] = [\![\mathtt{false}]\!]$. Then, by Lem. 81, $[\![\phi(\underline{G_1}, \phi(\underline{G_2}, \mathtt{false}))]\!] = [\![\phi(\underline{G_1}, \mathtt{false})]\!]$. Then, by induction, $[\![\phi(\underline{G_1}, \phi(\underline{G_2}, \mathtt{false}))]\!] = [\![\mathtt{false}]\!]$. Then, $[\![\phi(\underline{G_1} \parallel \underline{G_2}, \mathtt{false})]\!] = [\![\mathtt{false}]\!]$. Then, $[\![\phi(\underline{G}, \mathtt{false})]\!] = [\![\mathtt{false}]\!]$.
  - **Case:** Not $\underline{G_1}, \underline{G_2} \in \mathbb{G}$, or not $\underline{G_1} \# \underline{G_2}$. Then, by Defn. 38, $\phi(\underline{G_1} \parallel \underline{G_2}, \chi) = \mathtt{false}$. Then, $[\![\phi(\underline{G_1} \parallel \underline{G_2}, \chi)]\!] = [\![\mathtt{false}]\!]$. Then, $[\![\phi(\underline{G}, \mathtt{false})]\!] = [\![\mathtt{false}]\!]$.
– **Step:** $\underline{G} = R.\textbf{if}\,\xi\ \underline{G_1}\ \underline{G_2}$, for some $R, \underline{G_1}, \underline{G_2}, \xi$.
  - By Lem. 27, $\mathcal{S} \in [\![\xi^\equiv]\!]$ implies $\mathcal{S} \in [\![\xi^+]\!]$ or $\mathcal{S} \in [\![\xi^-]\!]$, for every $\mathcal{S}$. Then, $\mathcal{S} \in [\![\xi^\equiv]\!]$ implies $\mathcal{S} \in [\![\xi^+]\!] \cup [\![\xi^-]\!]$, for every $\mathcal{S}$. Then, $[\![\xi^\equiv]\!] \subseteq [\![\xi^+]\!] \cup [\![\xi^-]\!]$. Then, $[\![\xi^\equiv]\!] = [\![\xi^\equiv]\!] \cap ([\![\xi^+]\!] \cup [\![\xi^-]\!])$.
  - By Defn. 38, $\phi(R.\textbf{if}\,\xi\ \underline{G_1}\ \underline{G_2}, \mathtt{false}) = (\xi^+ \Rightarrow \phi(\underline{G_1}, \mathtt{false})) \wedge (\xi^- \Rightarrow \phi(\underline{G_2}, \mathtt{false})) \wedge \xi^\equiv$. Then, $[\![\phi(R.\textbf{if}\,\xi\ \underline{G_1}\ \underline{G_2}, \mathtt{false})]\!] = [\![(\xi^+ \Rightarrow \phi(\underline{G_1}, \mathtt{false})) \wedge (\xi^- \Rightarrow \phi(\underline{G_2}, \mathtt{false})) \wedge \xi^\equiv]\!]$. Then, $[\![\phi(\underline{G}, \mathtt{false})]\!] = [\![(\xi^+ \Rightarrow \phi(\underline{G_1}, \mathtt{false})) \wedge (\xi^- \Rightarrow \phi(\underline{G_2}, \mathtt{false})) \wedge \xi^\equiv]\!]$. Then, by Defn. 13, $[\![\phi(\underline{G}, \mathtt{false})]\!] = [\![\xi^+ \Rightarrow \phi(\underline{G_1}, \mathtt{false})]\!] \cap [\![\xi^- \Rightarrow \phi(\underline{G_2}, \mathtt{false})]\!] \cap [\![\xi^\equiv]\!]$. Then, $[\![\phi(\underline{G}, \mathtt{false})]\!] = [\![\xi^+ \Rightarrow \phi(\underline{G_1}, \mathtt{false})]\!] \cap [\![\xi^- \Rightarrow \phi(\underline{G_2}, \mathtt{false})]\!] \cap [\![\xi^\equiv]\!] \cap ([\![\xi^+]\!] \cup [\![\xi^-]\!])$. Then, $[\![\phi(\underline{G}, \mathtt{false})]\!] = ([\![\xi^+ \Rightarrow \phi(\underline{G_1}, \mathtt{false})]\!] \cap [\![\xi^- \Rightarrow \phi(\underline{G_2}, \mathtt{false})]\!] \cap [\![\xi^\equiv]\!] \cap [\![\xi^+]\!]) \cup ([\![\xi^+ \Rightarrow \phi(\underline{G_1}, \mathtt{false})]\!] \cap [\![\xi^- \Rightarrow \phi(\underline{G_2}, \mathtt{false})]\!] \cap [\![\xi^\equiv]\!] \cap [\![\xi^-]\!])$. Then, by Lem. 19, $[\![\phi(\underline{G}, \mathtt{false})]\!] = ([\![\phi(\underline{G_1}, \mathtt{false})]\!] \cap [\![\xi^- \Rightarrow \phi(\underline{G_2}, \mathtt{false})]\!] \cap [\![\xi^\equiv]\!] \cap [\![\xi^+]\!]) \cup ([\![\xi^+ \Rightarrow \phi(\underline{G_1}, \mathtt{false})]\!] \cap [\![\phi(\underline{G_2}, \mathtt{false})]\!] \cap [\![\xi^\equiv]\!] \cap [\![\xi^-]\!])$. Then, by induction, $[\![\phi(\underline{G}, \mathtt{false})]\!] = ([\![\mathtt{false}]\!] \cap [\![\xi^- \Rightarrow \phi(\underline{G_2}, \mathtt{false})]\!] \cap [\![\xi^\equiv]\!] \cap [\![\xi^+]\!]) \cup ([\![\xi^+ \Rightarrow \phi(\underline{G_1}, \mathtt{false})]\!] \cap [\![\mathtt{false}]\!] \cap [\![\xi^\equiv]\!] \cap [\![\xi^-]\!])$. Then, by Lem. 15, $[\![\phi(\underline{G}, \mathtt{false})]\!] = (\emptyset \cap [\![\xi^- \Rightarrow \phi(\underline{G_2}, \mathtt{false})]\!] \cap [\![\xi^\equiv]\!] \cap [\![\xi^+]\!]) \cup ([\![\xi^+ \Rightarrow \phi(\underline{G_1}, \mathtt{false})]\!] \cap \emptyset \cap [\![\xi^\equiv]\!] \cap [\![\xi^-]\!])$. Then, $[\![\phi(\underline{G}, \mathtt{false})]\!] = \emptyset \cup \emptyset$. Then, $[\![\phi(\underline{G}, \mathtt{false})]\!] = \emptyset$. Then, by Lem. 15, $[\![\phi(\underline{G}, \mathtt{false})]\!] = [\![\mathtt{false}]\!]$.
– **Step:** $\underline{G} = \textbf{if}\,\xi|_0\ \underline{G_1}|_{R_1}\ \underline{G_2}|_{R_2}$, for some $R_1, R_2, \underline{G_1}, \underline{G_2}, \xi$.
Recall:
  - **Case:** $R_1 = \emptyset = R_2$.
    * By Lem. 27, $\mathcal{S} \in [\![\xi^\equiv]\!]$ implies $\mathcal{S} \in [\![\xi^+]\!]$ or $\mathcal{S} \in [\![\xi^-]\!]$, for every $\mathcal{S}$. Then, $\mathcal{S} \in [\![\xi^\equiv]\!]$ implies $\mathcal{S} \in [\![\xi^+]\!] \cup [\![\xi^-]\!]$, for every $\mathcal{S}$. Then, $[\![\xi^\equiv]\!] \subseteq [\![\xi^+]\!] \cup [\![\xi^-]\!]$. Then, $[\![\xi^\equiv]\!] = [\![\xi^\equiv]\!] \cap ([\![\xi^+]\!] \cup [\![\xi^-]\!])$.

    * Recall $R_1 = \emptyset = R_2$. Then, by Defn. 38,
      $\phi(\textbf{if } \xi|_0\ \underline{G_1}|_{R_1}\ \underline{G_2}|_{R_2}, \texttt{false}) = (\xi^+ \Rightarrow \phi(\underline{G_1}, \texttt{false})) \wedge (\xi^- \Rightarrow \phi(\underline{G_2}, \texttt{false})) \wedge \xi^{\equiv}$.
      Then,
      $\llbracket \phi(\textbf{if } \xi|_0\ \underline{G_1}|_{R_1}\ \underline{G_2}|_{R_2}, \texttt{false}) \rrbracket = \llbracket (\xi^+ \Rightarrow \phi(\underline{G_1}, \texttt{false})) \wedge (\xi^- \Rightarrow \phi(\underline{G_2}, \texttt{false})) \wedge \xi^{\equiv} \rrbracket$.
      Then, $\llbracket \phi(\underline{G}, \texttt{false}) \rrbracket = \llbracket (\xi^+ \Rightarrow \phi(\underline{G_1}, \texttt{false})) \wedge (\xi^- \Rightarrow \phi(\underline{G_2}, \texttt{false})) \wedge \xi^{\equiv} \rrbracket$. Then, by Defn. 13, $\llbracket \phi(\underline{G}, \texttt{false}) \rrbracket = \llbracket \xi^+ \Rightarrow \phi(\underline{G_1}, \texttt{false}) \rrbracket \cap \llbracket \xi^- \Rightarrow \phi(\underline{G_2}, \texttt{false}) \rrbracket \cap \llbracket \xi^{\equiv} \rrbracket$.
      Then,
      $\llbracket \phi(\underline{G}, \texttt{false}) \rrbracket = \llbracket \xi^+ \Rightarrow \phi(\underline{G_1}, \texttt{false}) \rrbracket \cap \llbracket \xi^- \Rightarrow \phi(\underline{G_2}, \texttt{false}) \rrbracket \cap \llbracket \xi^{\equiv} \rrbracket \cap (\llbracket \xi^+ \rrbracket \cup \llbracket \xi^- \rrbracket)$.
      Then,
      $\llbracket \phi(\underline{G}, \texttt{false}) \rrbracket = (\llbracket \xi^+ \Rightarrow \phi(\underline{G_1}, \texttt{false}) \rrbracket \cap \llbracket \xi^- \Rightarrow \phi(\underline{G_2}, \texttt{false}) \rrbracket \cap \llbracket \xi^{\equiv} \rrbracket \cap \llbracket \xi^+ \rrbracket) \cup (\llbracket \xi^+ \Rightarrow \phi(\underline{G_1}, \texttt{false}) \rrbracket \cap \llbracket \xi^- \Rightarrow \phi(\underline{G_2}, \texttt{false}) \rrbracket \cap \llbracket \xi^{\equiv} \rrbracket \cap \llbracket \xi^- \rrbracket)$. Then, by Lem. 19,
      $\llbracket \phi(\underline{G}, \texttt{false}) \rrbracket = (\llbracket \phi(\underline{G_1}, \texttt{false}) \rrbracket \cap \llbracket \xi^- \Rightarrow \phi(\underline{G_2}, \texttt{false}) \rrbracket \cap \llbracket \xi^{\equiv} \rrbracket \cap \llbracket \xi^+ \rrbracket) \cup (\llbracket \xi^+ \Rightarrow \phi(\underline{G_1}, \texttt{false}) \rrbracket \cap \llbracket \phi(\underline{G_2}, \texttt{false}) \rrbracket \cap \llbracket \xi^{\equiv} \rrbracket \cap \llbracket \xi^- \rrbracket)$. Then, by induction,
      $\llbracket \phi(\underline{G}, \texttt{false}) \rrbracket = (\llbracket \texttt{false} \rrbracket \cap \llbracket \xi^- \Rightarrow \phi(\underline{G_2}, \texttt{false}) \rrbracket \cap \llbracket \xi^{\equiv} \rrbracket \cap \llbracket \xi^+ \rrbracket) \cup (\llbracket \xi^+ \Rightarrow \phi(\underline{G_1}, \texttt{false}) \rrbracket \cap \llbracket \texttt{false} \rrbracket \cap \llbracket \xi^{\equiv} \rrbracket \cap \llbracket \xi^- \rrbracket)$. Then, by Lem. 15, $\llbracket \phi(\underline{G}, \texttt{false}) \rrbracket =$
      $(\emptyset \cap \llbracket \xi^- \Rightarrow \phi(\underline{G_2}, \texttt{false}) \rrbracket \cap \llbracket \xi^{\equiv} \rrbracket \cap \llbracket \xi^+ \rrbracket) \cup (\llbracket \xi^+ \Rightarrow \phi(\underline{G_1}, \texttt{false}) \rrbracket \cap \emptyset \cap \llbracket \xi^{\equiv} \rrbracket \cap \llbracket \xi^- \rrbracket)$.
      Then, $\llbracket \phi(\underline{G}, \texttt{false}) \rrbracket = \emptyset \cup \emptyset$. Then, $\llbracket \phi(\underline{G}, \texttt{false}) \rrbracket = \emptyset$. Then, by Lem. 15,
      $\llbracket \phi(\underline{G}, \texttt{false}) \rrbracket = \llbracket \texttt{false} \rrbracket$. Then, $\llbracket \phi(\underline{G}, \texttt{false}) \rrbracket = \llbracket \texttt{false} \rrbracket$.
  - **Case:** $R_1 = \emptyset \neq R_2$. Then, by Defn. 38,
    $\phi(\textbf{if } \xi|_0\ \underline{G_1}|_{R_1}\ \underline{G_2}|_{R_2}, \texttt{false}) = \phi(\underline{G_2}, \texttt{false}) \wedge \bigwedge\{\xi^- \restriction r\}_{r \in \mathsf{subj}(\xi) \setminus R_2}$. Then,
    $\llbracket \phi(\textbf{if } \xi|_0\ \underline{G_1}|_{R_1}\ \underline{G_2}|_{R_2}, \texttt{false}) \rrbracket = \llbracket \phi(\underline{G_2}, \texttt{false}) \wedge \bigwedge\{\xi^- \restriction r\}_{r \in \mathsf{subj}(\xi) \setminus R_2} \rrbracket$. Then,
    $\llbracket \phi(\underline{G}, \texttt{false}) \rrbracket = \llbracket \phi(\underline{G_2}, \texttt{false}) \wedge \bigwedge\{\xi^- \restriction r\}_{r \in \mathsf{subj}(\xi) \setminus R_2} \rrbracket$. Then, by Defn. 13,
    $\llbracket \phi(\underline{G}, \texttt{false}) \rrbracket = \llbracket \phi(\underline{G_2}, \texttt{false}) \rrbracket \cap \llbracket \bigwedge\{\xi^- \restriction r\}_{r \in \mathsf{subj}(\xi) \setminus R_2} \rrbracket$. Then, by induction,
    $\llbracket \phi(\underline{G}, \texttt{false}) \rrbracket = \llbracket \texttt{false} \rrbracket \cap \llbracket \bigwedge\{\xi^- \restriction r\}_{r \in \mathsf{subj}(\xi) \setminus R_2} \rrbracket$. Then, by Lem. 15,
    $\llbracket \phi(\underline{G}, \texttt{false}) \rrbracket = \emptyset \cap \llbracket \bigwedge\{\xi^- \restriction r\}_{r \in \mathsf{subj}(\xi) \setminus R_2} \rrbracket$. Then, $\llbracket \phi(\underline{G}, \texttt{false}) \rrbracket = \emptyset$. Then, by Lem. 15,
    $\llbracket \phi(\underline{G}, \texttt{false}) \rrbracket = \llbracket \texttt{false} \rrbracket$.
  - **Case:** $R_1 \neq \emptyset = R_2$. Similar to case $R_1 = \emptyset \neq R_2$.
  - **Case:** $R_1 \neq \emptyset \neq R_2$. Then, by Defn. 38, $\phi(\textbf{if } \xi|_0\ \underline{G_1}|_{R_1}\ \underline{G_2}|_{R_2}, \chi) = \texttt{false}$. Then,
    $\llbracket \phi(\textbf{if } \xi|_0\ \underline{G_1}|_{R_1}\ \underline{G_2}|_{R_2}, \chi) \rrbracket = \llbracket \texttt{false} \rrbracket$. Then, $\llbracket \phi(\underline{G}, \texttt{false}) \rrbracket = \llbracket \texttt{false} \rrbracket$.
- **Step:** $\underline{G} = \textbf{if } \xi|_0\ G_1|_\emptyset\ \underline{G_2}|_{R_2 \cup \{r\}}$, for some $R_2, G_1, \underline{G_2}, r, \xi$.
  Recall $\emptyset = \emptyset \neq R_2 \cup \{r\}$. Then, by Defn. 38,
  $\phi(\textbf{if } \xi|_0\ G_1|_\emptyset\ \underline{G_2}|_{R_2 \cup \{r\}}, \texttt{false}) = \phi(\underline{G_2}, \texttt{false}) \wedge \bigwedge\{\xi^- \restriction r\}_{r \in \mathsf{subj}(\xi) \setminus (R_2 \cup \{r\})}$. Then,
  $\llbracket \phi(\textbf{if } \xi|_0\ G_1|_\emptyset\ \underline{G_2}|_{R_2 \cup \{r\}}, \texttt{false}) \rrbracket = \llbracket \phi(\underline{G_2}, \texttt{false}) \wedge \bigwedge\{\xi^- \restriction r\}_{r \in \mathsf{subj}(\xi) \setminus (R_2 \cup \{r\})} \rrbracket$. Then,
  $\llbracket \phi(\underline{G}, \texttt{false}) \rrbracket = \llbracket \phi(\underline{G_2}, \texttt{false}) \wedge \bigwedge\{\xi^- \restriction r\}_{r \in \mathsf{subj}(\xi) \setminus (R_2 \cup \{r\})} \rrbracket$. Then, by Defn. 13,
  $\llbracket \phi(\underline{G}, \texttt{false}) \rrbracket = \llbracket \phi(\underline{G_2}, \texttt{false}) \rrbracket \cap \llbracket \bigwedge\{\xi^- \restriction r\}_{r \in \mathsf{subj}(\xi) \setminus (R_2 \cup \{r\})} \rrbracket$. Then, by induction,
  $\llbracket \phi(\underline{G}, \texttt{false}) \rrbracket = \llbracket \texttt{false} \rrbracket \cap \llbracket \bigwedge\{\xi^- \restriction r\}_{r \in \mathsf{subj}(\xi) \setminus (R_2 \cup \{r\})} \rrbracket$. Then, by Lem. 15,
  $\llbracket \phi(\underline{G}, \texttt{false}) \rrbracket = \emptyset \cap \llbracket \bigwedge\{\xi^- \restriction r\}_{r \in \mathsf{subj}(\xi) \setminus (R_2 \cup \{r\})} \rrbracket$. Then, $\llbracket \phi(\underline{G}, \texttt{false}) \rrbracket = \emptyset$. Then, by Lem. 15, $\llbracket \phi(\underline{G}, \texttt{false}) \rrbracket = \llbracket \texttt{false} \rrbracket$.
- **Step:** $\underline{G} = \textbf{if } \xi|_0\ \underline{G_1}|_{R_1 \cup \{r\}}\ G_2|_\emptyset$, for some $R_1, \underline{G_1}, G_2, r, \xi$. Similar to case "$\underline{G} = \textbf{if } \xi|_0\ G_1|_\emptyset\ \underline{G_2}|_{R_2 \cup \{r\}}$, for some $R_2, G_1, \underline{G_2}, r, \xi$". □

Proof (of Lem. 84). By Defn. 21:

- **Base:** $\underline{G} = q.y := e$, for some q, y, e.
  Recall $\llbracket \phi(\underline{G}, \chi) \rrbracket \neq \emptyset$. Then, $\llbracket \phi(q.y := e, \chi) \rrbracket \neq \emptyset$. Then, by Defn. 38, $\llbracket \chi[e/q.y] \rrbracket \neq \emptyset$. Then, Then, $\mathcal{S} \in \llbracket \chi[e/q.y] \rrbracket$, for some $\mathcal{S}$. Then, by Lem. 14, $\mathcal{S}[\mathsf{eval}_{\mathcal{S}}(e)/q.y] \in \llbracket \chi \rrbracket$. Then, $\llbracket \chi \rrbracket \neq \emptyset$.
- **Base:** $\underline{G} = p.e \twoheadrightarrow q.y$, for some p, q, y, e. Similar to case "$\underline{G} = q.y := e$, for some q, y, e".
- **Base:** $\underline{G} = \textbf{skip}$.
  Recall $\llbracket \phi(\underline{G}, \chi) \rrbracket \neq \emptyset$. Then, $\llbracket \phi(\textbf{skip}, \chi) \rrbracket \neq \emptyset$. Then, by Defn. 38, $\llbracket \chi \rrbracket \neq \emptyset$.
- **Step:** $\underline{G} = \underline{G_1}\,;\underline{G_2}$, for some $\underline{G_1}, \underline{G_2}$.
  Recall $\llbracket \phi(\underline{G}, \chi) \rrbracket \neq \emptyset$. Then, $\llbracket \phi(\underline{G_1}\,;\underline{G_2}, \chi) \rrbracket \neq \emptyset$. Then, by Defn. 38, $\llbracket \phi(\underline{G_1}, \phi(\underline{G_2}, \chi)) \rrbracket \neq \emptyset$. Then, by induction, $\llbracket \phi(\underline{G_2}, \chi) \rrbracket \neq \emptyset$. Then, by induction, $\llbracket \chi \rrbracket \neq \emptyset$.
- **Step:** $\underline{G} = \underline{G_1} \parallel \underline{G_2}$, for some $\underline{G_1}, \underline{G_2}$.
  Recall $\llbracket \phi(\underline{G}, \chi) \rrbracket \neq \emptyset$. Then, $\llbracket \phi(\underline{G_1} \parallel \underline{G_2}, \chi) \rrbracket \neq \emptyset$. Then, by Defn. 38:

- **Case:** $\llbracket \phi(\underline{G}_1, \phi(\underline{G}_2, \chi)) \rrbracket \neq \emptyset$. Then, by induction, $\llbracket \phi(\underline{G}_2, \chi) \rrbracket \neq \emptyset$. Then, by induction, $\llbracket \chi \rrbracket \neq \emptyset$.
- **Case:** $\llbracket \texttt{false} \rrbracket \neq \emptyset$. Then, by Lem. 15, $\emptyset \neq \emptyset$. Then, **false**.

– **Step:** $\underline{G} = R.\textbf{if}\ \xi\ \underline{G}_1\ \underline{G}_2$, for some $R, \underline{G}_1, \underline{G}_2, \xi$.

- Recall $\llbracket \phi(\underline{G}, \chi) \rrbracket \neq \emptyset$. Then, $\llbracket \phi(R.\textbf{if}\ \xi\ \underline{G}_1\ \underline{G}_2, \chi) \rrbracket \neq \emptyset$. Then, by Defn. 38, $\llbracket (\xi^+ \Rightarrow \phi(\underline{G}_1, \chi)) \wedge (\xi^- \Rightarrow \phi(\underline{G}_2, \chi)) \wedge \xi^{\equiv} \rrbracket \neq \emptyset$. Then, by Defn. 13, $\llbracket \xi^+ \Rightarrow \phi(\underline{G}_1, \chi) \rrbracket \cap \llbracket \xi^- \Rightarrow \phi(\underline{G}_2, \chi) \rrbracket \cap \llbracket \xi^{\equiv} \rrbracket \neq \emptyset$. Then, $\mathcal{S} \in \llbracket \xi^+ \Rightarrow \phi(\underline{G}_1, \chi) \rrbracket \cap \llbracket \xi^- \Rightarrow \phi(\underline{G}_2, \chi) \rrbracket \cap \llbracket \xi^{\equiv} \rrbracket$, for some $\mathcal{S}$.
- Recall $\mathcal{S} \in \llbracket \xi^+ \Rightarrow \phi(\underline{G}_1, \chi) \rrbracket \cap \llbracket \xi^- \Rightarrow \phi(\underline{G}_2, \chi) \rrbracket \cap \llbracket \xi^{\equiv} \rrbracket$. Then, $\mathcal{S} \in \llbracket \xi^{\equiv} \rrbracket$. Then, by Lem. 27:
  - **Case:** $\mathcal{S} \in \llbracket \xi^+ \rrbracket$.
    - Recall $\mathcal{S} \in \llbracket \xi^+ \Rightarrow \phi(\underline{G}_1, \chi) \rrbracket \cap \llbracket \xi^- \Rightarrow \phi(\underline{G}_2, \chi) \rrbracket \cap \llbracket \xi^{\equiv} \rrbracket$. Then, $\mathcal{S} \in \llbracket \xi^+ \Rightarrow \phi(\underline{G}_1, \chi) \rrbracket$.
    - Recall $\mathcal{S} \in \llbracket \xi^+ \rrbracket$ and $\mathcal{S} \in \llbracket \xi^+ \Rightarrow \phi(\underline{G}_1, \chi) \rrbracket$. Then, $\mathcal{S} \in \llbracket \xi^+ \rrbracket \cap \llbracket \xi^+ \Rightarrow \phi(\underline{G}_1, \chi) \rrbracket$.
    - By Lem. 19, $\llbracket \xi^+ \rrbracket \cap \llbracket \xi^+ \Rightarrow \phi(\underline{G}_1, \chi) \rrbracket \subseteq \llbracket \xi^+ \rrbracket \cap \llbracket \phi(\underline{G}_1, \chi) \rrbracket$. Then, $\llbracket \xi^+ \rrbracket \cap \llbracket \xi^+ \Rightarrow \phi(\underline{G}_1, \chi) \rrbracket \subseteq \llbracket \phi(\underline{G}_1, \chi) \rrbracket$.
    - Recall $\mathcal{S} \in \llbracket \xi^+ \rrbracket \cap \llbracket \xi^+ \Rightarrow \phi(\underline{G}_1, \chi) \rrbracket \subseteq \llbracket \phi(\underline{G}_1, \chi) \rrbracket$. Then, $\mathcal{S} \in \llbracket \phi(\underline{G}_1, \chi) \rrbracket$. Then, $\llbracket \phi(\underline{G}_1, \chi) \rrbracket \neq \emptyset$. Then, by induction, $\llbracket \chi \rrbracket \neq \emptyset$.
  - **Case:** $\mathcal{S} \in \llbracket \xi^- \rrbracket$. Similar to case $\mathcal{S} \in \llbracket \xi^+ \rrbracket$.

– **Step:** $\underline{G} = \textbf{if}\ \xi|_0\ \underline{G}_1|_{R_1}\ \underline{G}_2|_{R_2}$, for some $R_1, R_2, \underline{G}_1, \underline{G}_2, \xi$.
Recall $\llbracket \phi(\underline{G}, \chi) \rrbracket \neq \emptyset$. Then, $\llbracket \phi(\textbf{if}\ \xi|_0\ \underline{G}_1|_{R_1}\ \underline{G}_2|_{R_2}, \chi) \rrbracket \neq \emptyset$. Then, by Defn. 38:

- **Case:** $\llbracket (\xi^+ \Rightarrow \phi(\underline{G}_1, \chi)) \wedge (\xi^- \Rightarrow \phi(\underline{G}_2, \chi)) \wedge \xi^{\equiv} \rrbracket \neq \emptyset$.
  - Recall $\llbracket (\xi^+ \Rightarrow \phi(\underline{G}_1, \chi)) \wedge (\xi^- \Rightarrow \phi(\underline{G}_2, \chi)) \wedge \xi^{\equiv} \rrbracket \neq \emptyset$. Then, by Defn. 13, $\llbracket \xi^+ \Rightarrow \phi(\underline{G}_1, \chi) \rrbracket \cap \llbracket \xi^- \Rightarrow \phi(\underline{G}_2, \chi) \rrbracket \cap \llbracket \xi^{\equiv} \rrbracket \neq \emptyset$. Then, $\mathcal{S} \in \llbracket \xi^+ \Rightarrow \phi(\underline{G}_1, \chi) \rrbracket \cap \llbracket \xi^- \Rightarrow \phi(\underline{G}_2, \chi) \rrbracket \cap \llbracket \xi^{\equiv} \rrbracket$, for some $\mathcal{S}$.
  - Recall $\mathcal{S} \in \llbracket \xi^+ \Rightarrow \phi(\underline{G}_1, \chi) \rrbracket \cap \llbracket \xi^- \Rightarrow \phi(\underline{G}_2, \chi) \rrbracket \cap \llbracket \xi^{\equiv} \rrbracket$. Then, $\mathcal{S} \in \llbracket \xi^{\equiv} \rrbracket$. Then, by Lem. 27:
    - **Case:** $\mathcal{S} \in \llbracket \xi^+ \rrbracket$.
      - Recall $\mathcal{S} \in \llbracket \xi^+ \Rightarrow \phi(\underline{G}_1, \chi) \rrbracket \cap \llbracket \xi^- \Rightarrow \phi(\underline{G}_2, \chi) \rrbracket \cap \llbracket \xi^{\equiv} \rrbracket$. Then, $\mathcal{S} \in \llbracket \xi^+ \Rightarrow \phi(\underline{G}_1, \chi) \rrbracket$.
      - Recall $\mathcal{S} \in \llbracket \xi^+ \rrbracket$ and $\mathcal{S} \in \llbracket \xi^+ \Rightarrow \phi(\underline{G}_1, \chi) \rrbracket$. Then, $\mathcal{S} \in \llbracket \xi^+ \rrbracket \cap \llbracket \xi^+ \Rightarrow \phi(\underline{G}_1, \chi) \rrbracket$.
      - By Lem. 19, $\llbracket \xi^+ \rrbracket \cap \llbracket \xi^+ \Rightarrow \phi(\underline{G}_1, \chi) \rrbracket \subseteq \llbracket \xi^+ \rrbracket \cap \llbracket \phi(\underline{G}_1, \chi) \rrbracket$. Then, $\llbracket \xi^+ \rrbracket \cap \llbracket \xi^+ \Rightarrow \phi(\underline{G}_1, \chi) \rrbracket \subseteq \llbracket \phi(\underline{G}_1, \chi) \rrbracket$.
      - Recall $\mathcal{S} \in \llbracket \xi^+ \rrbracket \cap \llbracket \xi^+ \Rightarrow \phi(\underline{G}_1, \chi) \rrbracket \subseteq \llbracket \phi(\underline{G}_1, \chi) \rrbracket$. Then, $\mathcal{S} \in \llbracket \phi(\underline{G}_1, \chi) \rrbracket$. Then, $\llbracket \phi(\underline{G}_1, \chi) \rrbracket \neq \emptyset$. Then, by induction, $\llbracket \chi \rrbracket \neq \emptyset$.
    - **Case:** $\mathcal{S} \in \llbracket \xi^- \rrbracket$. Similar to case $\mathcal{S} \in \llbracket \xi^+ \rrbracket$.
- **Case:** $\llbracket \phi(\underline{G}_2, \chi) \wedge \bigwedge \{\xi^- \upharpoonright r\}_{r \in \mathsf{subj}(\xi) \setminus R_2} \rrbracket \neq \emptyset$. Then, by Defn. 13, $\llbracket \phi(\underline{G}_2, \chi) \rrbracket \cap \llbracket \{\xi^- \upharpoonright r\}_{r \in \mathsf{subj}(\xi) \setminus R_2} \rrbracket \neq \emptyset$. Then, $\llbracket \phi(\underline{G}_2, \chi) \rrbracket \neq \emptyset$. Then, by induction, $\llbracket \chi \rrbracket \neq \emptyset$.
- **Case:** $\phi(\underline{G}_1, \chi) \wedge \bigwedge \{\xi^+ \upharpoonright r\}_{r \in \mathsf{subj}(\xi) \setminus R_1}$. Similar to case "$\llbracket \phi(\underline{G}_2, \chi) \wedge \bigwedge \{\xi^- \upharpoonright r\}_{r \in \mathsf{subj}(\xi) \setminus R_2} \rrbracket \neq \emptyset$".
- **Case:** $\llbracket \texttt{false} \rrbracket \neq \emptyset$. Then, by Lem. 15, $\emptyset \neq \emptyset$. Then, **false**.

– **Step:** $\underline{G} = \textbf{if}\ \xi|_0\ G_1|_{\emptyset}\ \underline{G}_2|_{R_2 \cup \{r\}}$, for some $R_2, G_1, \underline{G}_2, r, \xi$.

- Recall $\emptyset = \emptyset \neq R_2 \cup \{r\}$. Then, by Defn. 38, $\phi(\textbf{if}\ \xi|_0\ G_1|_{\emptyset}\ \underline{G}_2|_{R_2 \cup \{r\}}, \chi) = \phi(\underline{G}_2, \chi) \wedge \bigwedge \{\xi^- \upharpoonright r\}_{r \in \mathsf{subj}(\xi) \setminus (R_2 \cup \{r\})}$.
- Recall $\llbracket \phi(\underline{G}, \chi) \rrbracket \neq \emptyset$. Then, $\llbracket \phi(\textbf{if}\ \xi|_0\ G_1|_{\emptyset}\ \underline{G}_2|_{R_2 \cup \{r\}}, \chi) \rrbracket \neq \emptyset$. Then, $\llbracket \phi(\underline{G}_2, \chi) \wedge \bigwedge \{\xi^- \upharpoonright r\}_{r \in \mathsf{subj}(\xi) \setminus (R_2 \cup \{r\})} \rrbracket \neq \emptyset$. Then, by Defn. 13, $\llbracket \phi(\underline{G}_2, \chi) \rrbracket \cap \llbracket \bigwedge \{\xi^- \upharpoonright r\}_{r \in \mathsf{subj}(\xi) \setminus (R_2 \cup \{r\})} \rrbracket \neq \emptyset$. Then, $\llbracket \phi(\underline{G}_2, \chi) \rrbracket \neq \emptyset$. Then, by induction, $\llbracket \chi \rrbracket \neq \emptyset$.

– **Step:** $\underline{G} = \textbf{if}\ \xi|_0\ \underline{G}_1|_{R_1 \cup \{r\}}\ G_2|_{\emptyset}$, for some $R_1, \underline{G}_1, G_2, r, \xi$. Similar to case "$\underline{G} = \textbf{if}\ \xi|_0\ G_1|_{\emptyset}\ \underline{G}_2|_{R_2 \cup \{r\}}$, for some $R_2, G_1, \underline{G}_2, r, \xi$". □

Proof (of Lem. 85). By Defn. 21:

- **Base:** $\underline{G} = q.y := e$, for some q, y, e.
  - Recall $\underline{G} \# \xi$. Then, $q.y := e \# \xi$. Then, by Defn. 23, $\mathsf{write}(q.y := e) \cap \mathsf{read}(\xi) = \emptyset$. Then, by Lem. 76, $\phi(q.y := e, \xi) = \xi$.
  - Recall $\llbracket \phi(\underline{G}, \chi) \rrbracket \cap \llbracket \xi \rrbracket \subseteq \llbracket \xi \rrbracket$. Then, $\llbracket \phi(\underline{G}, \chi) \rrbracket \cap \llbracket \xi \rrbracket \subseteq \llbracket \phi(q.y := e, \xi) \rrbracket$. Then, by Defn. 13, $\llbracket \phi(\underline{G}, \chi) \wedge \xi \rrbracket \subseteq \llbracket \phi(q.y := e, \xi) \rrbracket$. Then, $\llbracket \phi(\underline{G}, \chi) \wedge \xi \rrbracket \subseteq \llbracket \phi(\underline{G}, \xi) \rrbracket$.
- **Base:** $\underline{G} = p.e \twoheadrightarrow q.y$, for some p, q, y, e. Similar to case "$\underline{G} = q.y := e$, for some q, y, e".
- **Base:** $\underline{G} = \mathbf{skip}$.
  Recall $\llbracket \phi(\underline{G}, \chi) \rrbracket \cap \llbracket \xi \rrbracket \subseteq \llbracket \xi \rrbracket$. Then, by Defn. 13, $\llbracket \phi(\underline{G}, \chi) \wedge \xi \rrbracket \subseteq \llbracket \xi \rrbracket$. Then, by Defn. 38, $\llbracket \phi(\underline{G}, \chi) \wedge \xi \rrbracket \subseteq \llbracket \phi(\mathbf{skip}, \xi) \rrbracket$. Then, $\llbracket \phi(\underline{G}, \chi) \wedge \xi \rrbracket \subseteq \llbracket \phi(\underline{G}, \xi) \rrbracket$.
- **Step:** $\underline{G} = \underline{G}_1 \,;\, \underline{G}_2$, for some $\underline{G}_1, \underline{G}_2$.
  - Recall $\underline{G} \# \xi$. Then, $\underline{G}_1 \,;\, \underline{G}_2 \# \xi$. Then, by Lem. 38, $\underline{G}_1 \# \xi$ and $\underline{G}_2 \# \xi$.
  - Recall $\underline{G}_1 \# \xi$. Then, by induction, $\llbracket \phi(\underline{G}_1, \phi(\underline{G}_2, \chi)) \wedge \xi \rrbracket \subseteq \llbracket \phi(\underline{G}_1, \xi) \rrbracket$. Then, by Defn. 38, $\llbracket \phi(\underline{G}_1 \,;\, \underline{G}_2, \chi) \wedge \xi \rrbracket \subseteq \llbracket \phi(\underline{G}_1, \xi) \rrbracket$. Then, $\llbracket \phi(\underline{G}, \chi) \wedge \xi \rrbracket \subseteq \llbracket \phi(\underline{G}_1, \xi) \rrbracket$. Then, by Defn. 13, $\llbracket \phi(\underline{G}, \chi) \rrbracket \cap \llbracket \xi \rrbracket \subseteq \llbracket \phi(\underline{G}_1, \xi) \rrbracket$. Then, $\llbracket \phi(\underline{G}, \chi) \rrbracket \cap \llbracket \xi \rrbracket \subseteq \llbracket \phi(\underline{G}, \chi) \rrbracket \cap \llbracket \phi(\underline{G}_1, \xi) \rrbracket$. Then, by Defn. 13, $\llbracket \phi(\underline{G}, \chi) \wedge \xi \rrbracket \subseteq \llbracket \phi(\underline{G}, \chi) \wedge \phi(\underline{G}_1, \xi) \rrbracket$.
  - Recall $\underline{G}_2 \# \xi$. Then, by induction, $\llbracket \phi(\underline{G}_2, \chi) \wedge \xi \rrbracket \subseteq \llbracket \phi(\underline{G}_2, \xi) \rrbracket$. Then, by Lem. 81, $\llbracket \phi(\underline{G}_1, \phi(\underline{G}_2, \chi) \wedge \xi) \rrbracket \subseteq \llbracket \phi(\underline{G}_1, \phi(\underline{G}_2, \xi)) \rrbracket$. Then, by Lem. 82, $\llbracket \phi(\underline{G}_1, \phi(\underline{G}_2, \chi)) \wedge \phi(\underline{G}_1, \xi) \rrbracket \subseteq \llbracket \phi(\underline{G}_1, \phi(\underline{G}_2, \xi)) \rrbracket$. Then, by Defn. 38, $\llbracket \phi(\underline{G}_1 \,;\, \underline{G}_2, \chi) \wedge \phi(\underline{G}_1, \xi) \rrbracket \subseteq \llbracket \phi(\underline{G}_1 \,;\, \underline{G}_2, \xi) \rrbracket$. Then, $\llbracket \phi(\underline{G}, \chi) \wedge \phi(\underline{G}_1, \xi) \rrbracket \subseteq \llbracket \phi(\underline{G}, \xi) \rrbracket$.
  - Recall $\llbracket \phi(G, \chi) \wedge \xi \rrbracket \subseteq \llbracket \phi(\underline{G}, \chi) \wedge \phi(\underline{G}_1, \xi) \rrbracket \subseteq \llbracket \phi(\underline{G}, \xi) \rrbracket$. Then, $\llbracket \phi(\underline{G}, \chi) \wedge \xi \rrbracket \subseteq \llbracket \phi(\underline{G}, \xi) \rrbracket$.
- **Step:** $\underline{G} = \underline{G}_1 \parallel \underline{G}_2$, for some $\underline{G}_1, \underline{G}_2$.
  Recall:
  - **Case:** $\underline{G}_1, \underline{G}_2 \in \mathbb{G}$ and $\underline{G}_1 \# \underline{G}_2$.
    * Recall $\underline{G} \# \xi$. Then, $\underline{G}_1 \parallel \underline{G}_2 \# \xi$. Then, by Lem. 38, $\underline{G}_1 \# \xi$ and $\underline{G}_2 \# \xi$.
    * Recall $\underline{G}_1, \underline{G}_2 \in \mathbb{G}$ and $\underline{G}_1 \# \underline{G}_2$. Then, by Defn. 38, $\phi(\underline{G}_1 \parallel \underline{G}_2, \chi) = \phi(\underline{G}_1, \phi(\underline{G}_2, \chi))$ and $\phi(\underline{G}_1 \parallel \underline{G}_2, \xi) = \phi(\underline{G}_1, \phi(\underline{G}_2, \xi))$.
    * Recall $\underline{G}_1 \# \xi$. Then, by induction, $\llbracket \phi(\underline{G}_1, \phi(\underline{G}_2, \chi)) \wedge \xi \rrbracket \subseteq \llbracket \phi(\underline{G}_1, \xi) \rrbracket$. Then, $\llbracket \phi(\underline{G}_1 \parallel \underline{G}_2, \chi) \wedge \xi \rrbracket \subseteq \llbracket \phi(\underline{G}_1, \xi) \rrbracket$. Then, $\llbracket \phi(\underline{G}, \chi) \wedge \xi \rrbracket \subseteq \llbracket \phi(\underline{G}_1, \xi) \rrbracket$. Then, by Defn. 13, $\llbracket \phi(\underline{G}, \chi) \rrbracket \cap \llbracket \xi \rrbracket \subseteq \llbracket \phi(\underline{G}_1, \xi) \rrbracket$. Then, $\llbracket \phi(\underline{G}, \chi) \rrbracket \cap \llbracket \xi \rrbracket \subseteq \llbracket \phi(\underline{G}, \chi) \rrbracket \cap \llbracket \phi(\underline{G}_1, \xi) \rrbracket$. Then, by Defn. 13, $\llbracket \phi(\underline{G}, \chi) \wedge \xi \rrbracket \subseteq \llbracket \phi(\underline{G}, \chi) \wedge \phi(\underline{G}_1, \xi) \rrbracket$.
    * Recall $\underline{G}_2 \# \xi$. Then, by induction, $\llbracket \phi(\underline{G}_2, \chi) \wedge \xi \rrbracket \subseteq \llbracket \phi(\underline{G}_2, \xi) \rrbracket$. Then, by Lem. 81, $\llbracket \phi(\underline{G}_1, \phi(\underline{G}_2, \chi) \wedge \xi) \rrbracket \subseteq \llbracket \phi(\underline{G}_1, \phi(\underline{G}_2, \xi)) \rrbracket$. Then, by Lem. 82, $\llbracket \phi(\underline{G}_1, \phi(\underline{G}_2, \chi)) \wedge \phi(\underline{G}_1, \xi) \rrbracket \subseteq \llbracket \phi(\underline{G}_1, \phi(\underline{G}_2, \xi)) \rrbracket$. Then, $\llbracket \phi(\underline{G}_1 \parallel \underline{G}_2, \chi) \wedge \phi(\underline{G}_1, \xi) \rrbracket \subseteq \llbracket \phi(\underline{G}_1 \parallel \underline{G}_2, \xi) \rrbracket$. Then, $\llbracket \phi(\underline{G}, \chi) \wedge \phi(\underline{G}_1, \xi) \rrbracket \subseteq \llbracket \phi(\underline{G}, \xi) \rrbracket$.
    * Recall $\llbracket \phi(\underline{G}, \chi) \wedge \xi \rrbracket \subseteq \llbracket \phi(\underline{G}, \chi) \wedge \phi(\underline{G}_1, \xi) \rrbracket \subseteq \llbracket \phi(\underline{G}, \xi) \rrbracket$. Then, $\llbracket \phi(\underline{G}, \chi) \wedge \xi \rrbracket \subseteq \llbracket \phi(\underline{G}, \xi) \rrbracket$.
  - **Case:** Not $\underline{G}_1, \underline{G}_2 \in \mathbb{G}$, or not $\underline{G}_1 \# \underline{G}_2$.
    * Recall not $\underline{G}_1, \underline{G}_2 \in \mathbb{G}$, or not $\underline{G}_1 \# \underline{G}_2$. Then, by Defn. 38, $\phi(\underline{G}_1 \parallel \underline{G}_2, \chi) = \mathtt{false}$.
    * Recall $\emptyset \cap \llbracket \xi \rrbracket \subseteq \llbracket \phi(\underline{G}, \xi) \rrbracket$. Then, by Lem. 15, $\llbracket \mathtt{false} \rrbracket \cap \llbracket \xi \rrbracket \subseteq \llbracket \phi(\underline{G}, \xi) \rrbracket$. Then, by Defn. 13, $\llbracket \mathtt{false} \wedge \xi \rrbracket \subseteq \llbracket \phi(\underline{G}, \xi) \rrbracket$. Then, $\llbracket \phi(\underline{G}_1 \parallel \underline{G}_2, \chi) \wedge \xi \rrbracket \subseteq \llbracket \phi(\underline{G}, \xi) \rrbracket$. hen, $\llbracket \phi(\underline{G}, \chi) \wedge \xi \rrbracket \subseteq \llbracket \phi(\underline{G}, \xi) \rrbracket$.
- **Step:** $\underline{G} = R.\mathbf{if}\, \hat{\xi}\, \underline{G}_1\, \underline{G}_2$, for some $R, \underline{G}_1, \underline{G}_2, \hat{\xi}$.
  Recall $\underline{G} \# \xi$. Then, $R.\mathbf{if}\, \hat{\xi}\, \underline{G}_1\, \underline{G}_2 \# \xi$. Then, by Lem. 38, $\underline{G}_1 \# \xi$ and $\underline{G}_2 \# \xi$. Then, by induction, $\llbracket \phi(\underline{G}_1, \chi) \wedge \xi \rrbracket \subseteq \llbracket \phi(\underline{G}_1, \xi) \rrbracket$ and $\llbracket \phi(\underline{G}_2, \chi) \wedge \xi \rrbracket \subseteq \llbracket \phi(\underline{G}_2, \xi) \rrbracket$. Then, by Lem. 22, $\llbracket \hat{\xi}^+ \Rightarrow (\phi(\underline{G}_1, \chi) \wedge \xi) \rrbracket \subseteq \llbracket \hat{\xi}^+ \Rightarrow \phi(\underline{G}_1, \xi) \rrbracket$ and $\llbracket \hat{\xi}^- \Rightarrow (\phi(\underline{G}_2, \chi) \wedge \xi) \rrbracket \subseteq \llbracket \hat{\xi}^- \Rightarrow \phi(\underline{G}_2, \xi) \rrbracket$. Then, by Lem. 18, $\llbracket \hat{\xi}^+ \Rightarrow \phi(\underline{G}_1, \chi) \rrbracket \cap \llbracket \hat{\xi}^+ \Rightarrow \xi \rrbracket \subseteq \llbracket \hat{\xi}^+ \Rightarrow \phi(\underline{G}_1, \xi) \rrbracket$ and $\llbracket \hat{\xi}^- \Rightarrow \phi(\underline{G}_2, \chi) \rrbracket \cap \llbracket \hat{\xi}^- \Rightarrow \xi \rrbracket \subseteq \llbracket \hat{\xi}^- \Rightarrow \phi(\underline{G}_2, \xi) \rrbracket$. Then, by Lem. 25, $\llbracket \hat{\xi}^+ \Rightarrow \phi(\underline{G}_1, \chi) \rrbracket \cap \llbracket \xi \rrbracket \subseteq \llbracket \hat{\xi}^+ \Rightarrow \phi(\underline{G}_1, \xi) \rrbracket$ and $\llbracket \hat{\xi}^- \Rightarrow \phi(\underline{G}_2, \chi) \rrbracket \cap \llbracket \xi \rrbracket \subseteq \llbracket \hat{\xi}^- \Rightarrow \phi(\underline{G}_2, \xi) \rrbracket$. Then, $\llbracket \hat{\xi}^+ \Rightarrow \phi(\underline{G}_1, \chi) \rrbracket \cap \llbracket \hat{\xi}^- \Rightarrow \phi(\underline{G}_2, \chi) \rrbracket \cap \llbracket \xi \rrbracket \subseteq \llbracket \hat{\xi}^+ \Rightarrow \phi(\underline{G}_1, \xi) \rrbracket \cap \llbracket \hat{\xi}^- \Rightarrow \phi(\underline{G}_2, \xi) \rrbracket$. Then, $\llbracket \hat{\xi}^+ \Rightarrow \phi(\underline{G}_1, \chi) \rrbracket \cap \llbracket \hat{\xi}^- \Rightarrow \phi(\underline{G}_2, \chi) \rrbracket \cap \llbracket \hat{\xi}^{\equiv} \rrbracket \cap \llbracket \xi \rrbracket \subseteq \llbracket \hat{\xi}^+ \Rightarrow \phi(\underline{G}_1, \xi) \rrbracket \cap \llbracket \hat{\xi}^- \Rightarrow \phi(\underline{G}_2, \xi) \rrbracket \cap \llbracket \hat{\xi}^{\equiv} \rrbracket$. Then, by Defn. 13,

$\llbracket(\hat{\xi}^+ \Rightarrow \phi(\underline{G}_1,\chi)) \wedge (\hat{\xi}^- \Rightarrow \phi(\underline{G}_2,\chi)) \wedge \hat{\xi}^{\equiv} \wedge \xi\rrbracket \subseteq \llbracket(\hat{\xi}^+ \Rightarrow \phi(\underline{G}_1,\xi)) \wedge (\hat{\xi}^- \Rightarrow \phi(\underline{G}_2,\xi)) \wedge \hat{\xi}^{\equiv}\rrbracket$. Then, by Defn. 38, $\llbracket\phi(R.\mathbf{if}\ \hat{\xi}\ \underline{G}_1\ \underline{G}_2,\chi) \wedge \xi\rrbracket \subseteq \llbracket\phi(R.\mathbf{if}\ \hat{\xi}\ \underline{G}_1\ \underline{G}_2,\xi)\rrbracket$. Then, $\llbracket\phi(\underline{G},\chi) \wedge \xi\rrbracket \subseteq \llbracket\phi(\underline{G},\xi)\rrbracket$.

- **Step:** $\underline{G} = \mathbf{if}\ \hat{\xi}|_0\ \underline{G}_1|_{R_1}\ \underline{G}_2|_{R_2}$, for some $R_1, R_2, \underline{G}_1, \underline{G}_2, \hat{\xi}$.
  Recall:
  - **Case:** $R_1 = \emptyset = R_2$.
    - ∗ Recall $\underline{G} \# \xi$. Then, $\mathbf{if}\ \hat{\xi}|_0\ \underline{G}_1|_{R_1}\ \underline{G}_2|_{R_2} \# \xi$.
    - ∗ Recall $R_1 = \emptyset = R_2$. Then, by Defn. 38, $\phi(\mathbf{if}\ \hat{\xi}|_0\ \underline{G}_1|_{R_1}\ \underline{G}_2|_{R_2},\chi) = (\hat{\xi}^+ \Rightarrow \phi(\underline{G}_1,\chi)) \wedge (\hat{\xi}^- \Rightarrow \phi(\underline{G}_2,\chi)) \wedge \hat{\xi}^{\equiv}$ and $\phi(\mathbf{if}\ \hat{\xi}|_0\ \underline{G}_1|_{R_1}\ \underline{G}_2|_{R_2},\xi) = (\hat{\xi}^+ \Rightarrow \phi(\underline{G}_1,\xi)) \wedge (\hat{\xi}^- \Rightarrow \phi(\underline{G}_2,\xi)) \wedge \hat{\xi}^{\equiv}$.
    - ∗ Recall $\mathbf{if}\ \hat{\xi}|_0\ \underline{G}_1|_{R_1}\ \underline{G}_2|_{R_2} \# \xi$ and $R_1 = \emptyset = R_2$. Then, by Lem. 38, $\underline{G}_1 \# \xi$ and $\underline{G}_2 \# \xi$. Then, by induction, $\llbracket\phi(\underline{G}_1,\chi) \wedge \xi\rrbracket \subseteq \llbracket\phi(\underline{G}_1,\xi)\rrbracket$ and $\llbracket\phi(\underline{G}_2,\chi) \wedge \xi\rrbracket \subseteq \llbracket\phi(\underline{G}_2,\xi)\rrbracket$. Then, by Lem. 22, $\llbracket\hat{\xi}^+ \Rightarrow (\phi(\underline{G}_1,\chi) \wedge \xi)\rrbracket \subseteq \llbracket\hat{\xi}^+ \Rightarrow \phi(\underline{G}_1,\xi)\rrbracket$ and $\llbracket\hat{\xi}^- \Rightarrow (\phi(\underline{G}_2,\chi) \wedge \xi)\rrbracket \subseteq \llbracket\hat{\xi}^- \Rightarrow \phi(\underline{G}_2,\xi)\rrbracket$. Then, by Lem. 18, $\llbracket\hat{\xi}^+ \Rightarrow \phi(\underline{G}_1,\chi)\rrbracket \cap \llbracket\hat{\xi}^+ \Rightarrow \xi\rrbracket \subseteq \llbracket\hat{\xi}^+ \Rightarrow \phi(\underline{G}_1,\xi)\rrbracket$ and $\llbracket\hat{\xi}^- \Rightarrow \phi(\underline{G}_2,\chi)\rrbracket \cap \llbracket\hat{\xi}^- \Rightarrow \xi\rrbracket \subseteq \llbracket\hat{\xi}^- \Rightarrow \phi(\underline{G}_2,\xi)\rrbracket$. Then, by Lem. 25, $\llbracket\hat{\xi}^+ \Rightarrow \phi(\underline{G}_1,\chi)\rrbracket \cap \llbracket\xi\rrbracket \subseteq \llbracket\hat{\xi}^+ \Rightarrow \phi(\underline{G}_1,\xi)\rrbracket$ and $\llbracket\hat{\xi}^- \Rightarrow \phi(\underline{G}_2,\chi)\rrbracket \cap \llbracket\xi\rrbracket \subseteq \llbracket\hat{\xi}^- \Rightarrow \phi(\underline{G}_2,\xi)\rrbracket$. Then, $\llbracket\hat{\xi}^+ \Rightarrow \phi(\underline{G}_1,\chi)\rrbracket \cap \llbracket\hat{\xi}^- \Rightarrow \phi(\underline{G}_2,\chi)\rrbracket \cap \llbracket\xi\rrbracket \subseteq \llbracket\hat{\xi}^+ \Rightarrow \phi(\underline{G}_1,\xi)\rrbracket \cap \llbracket\hat{\xi}^- \Rightarrow \phi(\underline{G}_2,\xi)\rrbracket$. Then, $\llbracket\hat{\xi}^+ \Rightarrow \phi(\underline{G}_1,\chi)\rrbracket \cap \llbracket\hat{\xi}^- \Rightarrow \phi(\underline{G}_2,\chi)\rrbracket \cap \llbracket\hat{\xi}^{\equiv}\rrbracket \cap \llbracket\xi\rrbracket \subseteq \llbracket\hat{\xi}^+ \Rightarrow \phi(\underline{G}_1,\xi)\rrbracket \cap \llbracket\hat{\xi}^- \Rightarrow \phi(\underline{G}_2,\xi)\rrbracket \cap \llbracket\hat{\xi}^{\equiv}\rrbracket$. Then, by Defn. 13, $\llbracket(\hat{\xi}^+ \Rightarrow \phi(\underline{G}_1,\chi)) \wedge (\hat{\xi}^- \Rightarrow \phi(\underline{G}_2,\chi)) \wedge \hat{\xi}^{\equiv} \wedge \xi\rrbracket \subseteq \llbracket(\hat{\xi}^+ \Rightarrow \phi(\underline{G}_1,\xi)) \wedge (\hat{\xi}^- \Rightarrow \phi(\underline{G}_2,\xi)) \wedge \hat{\xi}^{\equiv}\rrbracket$. Then, $\llbracket\phi(\mathbf{if}\ \hat{\xi}|_0\ \underline{G}_1|_{R_1}\ \underline{G}_2|_{R_2} \wedge \xi,\chi) \subseteq \llbracket\phi(\mathbf{if}\ \hat{\xi}|_0\ \underline{G}_1|_{R_1}\ \underline{G}_2|_{R_2},\xi)\rrbracket\rrbracket$. Then, $\llbracket\phi(\underline{G},\chi) \wedge \xi\rrbracket \subseteq \llbracket\phi(\underline{G},\xi)\rrbracket$.
  - **Case:** $R_1 = \emptyset \neq R_2$.
    - ∗ Recall $\underline{G} \# \xi$. Then, $\mathbf{if}\ \hat{\xi}|_0\ \underline{G}_1|_{R_1}\ \underline{G}_2|_{R_2} \# \xi$.
    - ∗ Recall $R_1 = \emptyset \neq R_2$. Then, by Defn. 38, $\phi(\mathbf{if}\ \hat{\xi}|_0\ \underline{G}_1|_{R_1}\ \underline{G}_2|_{R_2},\chi) = \phi(\underline{G}_2,\chi) \wedge \bigwedge\{\hat{\xi}^- \upharpoonright r\}_{r \in \mathsf{subj}(\hat{\xi}) \setminus R_2}$ and $\phi(\mathbf{if}\ \hat{\xi}|_0\ \underline{G}_1|_{R_1}\ \underline{G}_2|_{R_2},\xi) = \phi(\underline{G}_2,\xi) \wedge \bigwedge\{\hat{\xi}^- \upharpoonright r\}_{r \in \mathsf{subj}(\hat{\xi}) \setminus R_2}$.
    - ∗ Recall $\mathbf{if}\ \hat{\xi}|_0\ \underline{G}_1|_{R_1}\ \underline{G}_2|_{R_2} \# \xi$ and $R_1 = \emptyset \neq R_2$. Then, by Lem. 38, $\underline{G}_2 \# \xi$. Then, by induction, $\llbracket\phi(\underline{G}_2,\chi) \wedge \xi\rrbracket \subseteq \llbracket\phi(\underline{G}_2,\xi)\rrbracket$. Then, $\llbracket\phi(\underline{G}_2,\chi) \wedge \xi\rrbracket \cap \llbracket\bigwedge\{\hat{\xi}^- \upharpoonright r\}_{r \in \mathsf{subj}(\hat{\xi}) \setminus R_2}\rrbracket \subseteq \llbracket\phi(\underline{G}_2,\xi)\rrbracket \cap \llbracket\bigwedge\{\hat{\xi}^- \upharpoonright r\}_{r \in \mathsf{subj}(\hat{\xi}) \setminus R_2}\rrbracket$. Then, by Defn. 13, $\llbracket\phi(\underline{G}_2,\chi) \wedge \xi \wedge \bigwedge\{\hat{\xi}^- \upharpoonright r\}_{r \in \mathsf{subj}(\hat{\xi}) \setminus R_2}\rrbracket \subseteq \llbracket\phi(\underline{G}_2,\xi) \wedge \bigwedge\{\hat{\xi}^- \upharpoonright r\}_{r \in \mathsf{subj}(\hat{\xi}) \setminus R_2}\rrbracket$. Then, $\llbracket\phi(\mathbf{if}\ \hat{\xi}|_0\ \underline{G}_1|_{R_1}\ \underline{G}_2|_{R_2},\chi) \wedge \xi\rrbracket \subseteq \llbracket\phi(\mathbf{if}\ \hat{\xi}|_0\ \underline{G}_1|_{R_1}\ \underline{G}_2|_{R_2},\xi)\rrbracket$. Then, $\llbracket\phi(\underline{G},\chi) \wedge \xi\rrbracket \subseteq \llbracket\phi(\underline{G},\xi)\rrbracket$.
  - **Case:** $R_1 \neq \emptyset = R_2$. Similar to case $R_1 = \emptyset \neq R_2$.
  - **Case:** $R_1 \neq \emptyset \neq R_2$.
    - ∗ Recall $R_1 \neq \emptyset \neq R_2$. Then, by Defn. 38, $\phi(\mathbf{if}\ \hat{\xi}|_0\ \underline{G}_1|_{R_1}\ \underline{G}_2|_{R_2},\chi) = \texttt{false}$.
    - ∗ Recall $\emptyset \cap \llbracket\xi\rrbracket \subseteq \llbracket\phi(\underline{G},\xi)\rrbracket$. Then, by Lem. 15, $\llbracket\texttt{false}\rrbracket \cap \llbracket\xi\rrbracket \subseteq \llbracket\phi(\underline{G},\xi)\rrbracket$. Then, by Defn. 13, $\llbracket\texttt{false} \wedge \xi\rrbracket \subseteq \llbracket\phi(\underline{G},\xi)\rrbracket$. Then, $\llbracket\phi(\mathbf{if}\ \hat{\xi}|_0\ \underline{G}_1|_{R_1}\ \underline{G}_2|_{R_2},\chi) \wedge \xi\rrbracket \subseteq \llbracket\phi(\underline{G},\xi)\rrbracket$. Then, $\llbracket\phi(\underline{G},\chi) \wedge \xi\rrbracket \subseteq \llbracket\phi(\underline{G},\xi)\rrbracket$.
- **Step:** $\underline{G} = \mathbf{if}\ \hat{\xi}|_0\ G_1|_\emptyset\ \underline{G}_2|_{R_2 \cup \{r\}}$, for some $R_2, G_1, \underline{G}_2, r, \hat{\xi}$.
  - Recall $\underline{G} \# \xi$. Then, $\mathbf{if}\ \hat{\xi}|_0\ G_1|_\emptyset\ \underline{G}_2|_{R_2 \cup \{r\}} \# \xi$.
  - Recall $\emptyset = \emptyset \neq R_2 \cup \{r\}$. Then, by Defn. 38, $\phi(\mathbf{if}\ \hat{\xi}|_0\ G_1|_\emptyset\ \underline{G}_2|_{R_2 \cup \{r\}},\chi) = \phi(\underline{G}_2,\chi) \wedge \bigwedge\{\hat{\xi}^- \upharpoonright r\}_{r \in \mathsf{subj}(\hat{\xi}) \setminus (R_2 \cup \{r\})}$ and $\phi(\mathbf{if}\ \hat{\xi}|_0\ G_1|_\emptyset\ \underline{G}_2|_{R_2 \cup \{r\}},\xi) = \phi(\underline{G}_2,\xi) \wedge \bigwedge\{\hat{\xi}^- \upharpoonright r\}_{r \in \mathsf{subj}(\hat{\xi}) \setminus (R_2 \cup \{r\})}$.
  - Recall $\mathbf{if}\ \hat{\xi}|_0\ G_1|_\emptyset\ \underline{G}_2|_{R_2 \cup \{r\}} \# \xi$ and $\emptyset = \emptyset \neq R_2 \cup \{r\}$. Then, by Lem. 38, $\underline{G}_2 \# \xi$. Then, by induction, $\llbracket\phi(\underline{G}_2,\chi) \wedge \xi\rrbracket \subseteq \llbracket\phi(\underline{G}_2,\xi)\rrbracket$. Then, $\llbracket\phi(\underline{G}_2,\chi) \wedge \xi\rrbracket \cap \llbracket\bigwedge\{\hat{\xi}^- \upharpoonright r\}_{r \in \mathsf{subj}(\hat{\xi}) \setminus (R_2 \cup \{r\})}\rrbracket \subseteq \llbracket\phi(\underline{G}_2,\xi)\rrbracket \cap \llbracket\bigwedge\{\hat{\xi}^- \upharpoonright r\}_{r \in \mathsf{subj}(\hat{\xi}) \setminus (R_2 \cup \{r\})}\rrbracket$. Then, by Defn. 13, $\llbracket\phi(\underline{G}_2,\chi) \wedge \xi \wedge \bigwedge\{\hat{\xi}^- \upharpoonright r\}_{r \in \mathsf{subj}(\hat{\xi}) \setminus (R_2 \cup \{r\})}\rrbracket \subseteq \llbracket\phi(\underline{G}_2,\xi) \wedge \bigwedge\{\hat{\xi}^- \upharpoonright r\}_{r \in \mathsf{subj}(\hat{\xi}) \setminus (R_2 \cup \{r\})}\rrbracket$. Then, $\llbracket\phi(\mathbf{if}\ \hat{\xi}|_0\ G_1|_\emptyset\ \underline{G}_2|_{R_2 \cup \{r\}},\chi) \wedge \xi\rrbracket \subseteq \llbracket\phi(\mathbf{if}\ \hat{\xi}|_0\ G_1|_\emptyset\ \underline{G}_2|_{R_2 \cup \{r\}},\xi)\rrbracket$. Then, $\llbracket\phi(\underline{G},\chi) \wedge \xi\rrbracket \subseteq \llbracket\phi(\underline{G},\xi)\rrbracket$.

- **Step:** $\underline{G} = \textbf{if}\ \hat{\xi}|_0\ \underline{G_1}|_{R_1 \cup \{r\}}\ G_2|_\emptyset$, for some $R_1, \underline{G_1}, G_2, r, \hat{\xi}$. Similar to case "$\underline{G} = \textbf{if}\ \hat{\xi}|_0\ G_1|_\emptyset\ \underline{G_2}|_{R_2 \cup \{r\}}$, for some $R_2, G_1, \underline{G_2}, r, \hat{\xi}$". □

Proof (of Lem. 86). By Defn. 21:

- **Base:** $\underline{G} = q.y{:=}e$, for some q, y, e.
  Recall $\underline{G} \# \gamma$. Then, $q.y{:=}e \# \gamma$. Then, by Defn. 23, $\mathsf{read}(q.y{:=}e) \cap \mathsf{write}(\gamma) = \emptyset$ and $\mathsf{write}(q.y{:=}e) \cap \mathsf{read}(\gamma) = \emptyset$ and $\mathsf{write}(q.y{:=}e) \cap \mathsf{write}(\gamma) = \emptyset$. Then, by Lem. 77, $\phi(q.y{:=}e, \phi(\gamma, \chi)) = \phi(\gamma, \phi(q.y{:=}e, \chi))$. Then, $\phi(\underline{G}, \phi(\gamma, \chi)) = \phi(\gamma, \phi(\underline{G}, \chi))$. Then, $[\![\phi(\underline{G}, \phi(\gamma, \chi))]\!] = [\![\phi(\gamma, \phi(\underline{G}, \chi))]\!]$.
- **Base:** $\underline{G} = p.e \twoheadrightarrow q.y$, for some p, q, y, e. Similar to case "$\underline{G} = q.y{:=}e$, for some q, y, e".
- **Base:** $\underline{G} = \textbf{skip}$.
  Recall $\phi(\gamma, \chi) = \phi(\gamma, \chi)$. Then, by Defn. 38, $\phi(\textbf{skip}, \phi(\gamma, \chi)) = \phi(\gamma, \phi(\textbf{skip}, \chi))$. Then, $[\![\phi(\underline{G}, \phi(\gamma, \chi))]\!] = [\![\phi(\gamma, \phi(\underline{G}, \chi))]\!]$.
- **Step:** $\underline{G} = \underline{G_1} \,;\, \underline{G_2}$, for some $\underline{G_1}, \underline{G_2}$.
  - Recall $\underline{G} \# \xi$. Then, $\underline{G_1} \,;\, \underline{G_2} \# \xi$. Then, by Lem. 38, $\underline{G_1} \# \xi$ and $\underline{G_2} \# \xi$.
  - Recall $\underline{G_2} \# \xi$. Then, by induction, $[\![\phi(\underline{G_2}, \phi(\gamma, \chi))]\!] = [\![\phi(\gamma, \phi(\underline{G_2}, \chi))]\!]$. Then, by Lem. 81, $[\![\phi(\underline{G_1}, \phi(\underline{G_2}, \phi(\gamma, \chi)))]\!] = [\![\phi(\underline{G_1}, \phi(\gamma, \phi(\underline{G_2}, \chi)))]\!]$.
  - Recall $\underline{G_1} \# \xi$. Then, by induction, $[\![\phi(\underline{G_1}, \phi(\gamma, \phi(\underline{G_2}, \chi)))]\!] = [\![\phi(\gamma, \phi(\underline{G_1}, \phi(\underline{G_2}, \chi)))]\!]$.
  - Recall $[\![\phi(\underline{G_1}, \phi(\underline{G_2}, \phi(\gamma, \chi)))]\!] = [\![\phi(\underline{G_1}, \phi(\gamma, \phi(\underline{G_2}, \chi)))]\!] = [\![\phi(\gamma, \phi(\underline{G_1}, \phi(\underline{G_2}, \chi)))]\!]$. Then, $[\![\phi(\underline{G_1}, \phi(\underline{G_2}, \phi(\gamma, \chi)))]\!] = [\![\phi(\gamma, \phi(\underline{G_1}, \phi(\underline{G_2}, \chi)))]\!]$. Then, by Defn. 38, $[\![\phi(\underline{G_1} \,;\, \underline{G_2}, \phi(\gamma, \chi))]\!] = [\![\phi(\gamma, \phi(\underline{G_1} \,;\, \underline{G_2}, \chi))]\!]$. Then, $[\![\phi(\underline{G}, \phi(\gamma, \chi))]\!] = [\![\phi(\gamma, \phi(\underline{G}, \chi))]\!]$.
- **Step:** $\underline{G} = \underline{G_1} \parallel \underline{G_2}$, for some $\underline{G_1}, \underline{G_2}$.
  Recall:
  - **Case:** $\underline{G_1}, \underline{G_2} \in \mathbb{G}$ and $\underline{G_1} \# \underline{G_2}$.
    * Recall $\underline{G} \# \xi$. Then, $\underline{G_1} \parallel \underline{G_2} \# \xi$. Then, by Lem. 38, $\underline{G_1} \# \xi$ and $\underline{G_2} \# \xi$.
    * Recall $\underline{G_2} \# \xi$. Then, by induction, $[\![\phi(\underline{G_2}, \phi(\gamma, \chi))]\!] = [\![\phi(\gamma, \phi(\underline{G_2}, \chi))]\!]$. Then, by Lem. 81, $[\![\phi(\underline{G_1}, \phi(\underline{G_2}, \phi(\gamma, \chi)))]\!] = [\![\phi(\underline{G_1}, \phi(\gamma, \phi(\underline{G_2}, \chi)))]\!]$.
    * Recall $\underline{G_1} \# \xi$. Then, by induction, $[\![\phi(\underline{G_1}, \phi(\gamma, \phi(\underline{G_2}, \chi)))]\!] = [\![\phi(\gamma, \phi(\underline{G_1}, \phi(\underline{G_2}, \chi)))]\!]$.
    * Recall $\underline{G_1}, \underline{G_2} \in \mathbb{G}$ and $\underline{G_1} \# \underline{G_2}$. Then, by Defn. 38, $\phi(\underline{G_1} \parallel \underline{G_2}, \chi) = \phi(\underline{G_1}, \phi(\underline{G_2}, \chi))$ and $\phi(\underline{G_1} \parallel \underline{G_2}, \phi(\gamma, \chi)) = \phi(\underline{G_1}, \phi(\underline{G_2}, \phi(\gamma, \chi)))$.
    * Recall $[\![\phi(\underline{G_1}, \phi(\underline{G_2}, \phi(\gamma, \chi)))]\!] = [\![\phi(\underline{G_1}, \phi(\gamma, \phi(\underline{G_2}, \chi)))]\!] = [\![\phi(\gamma, \phi(\underline{G_1}, \phi(\underline{G_2}, \chi)))]\!]$. Then, $[\![\phi(\underline{G_1}, \phi(\underline{G_2}, \phi(\gamma, \chi)))]\!] = [\![\phi(\gamma, \phi(\underline{G_1}, \phi(\underline{G_2}, \chi)))]\!]$. Then, $[\![\phi(\underline{G_1} \parallel \underline{G_2}, \phi(\gamma, \chi))]\!] = [\![\phi(\gamma, \phi(\underline{G_1} \parallel \underline{G_2}, \chi))]\!]$. Then, $[\![\phi(\underline{G}, \phi(\gamma, \chi))]\!] = [\![\phi(\gamma, \phi(\underline{G}, \chi))]\!]$.
  - **Case:** Not $\underline{G_1}, \underline{G_2} \in \mathbb{G}$, or not $\underline{G_1} \# \underline{G_2}$.
    * Recall $\mathsf{write}(\gamma) \cap \emptyset = \emptyset$. Then, by Lem. 3, $\mathsf{write}(\gamma) \cap \mathsf{read}(\mathtt{false}) = \emptyset$. Then, by Lem. 76, $\phi(\gamma, \mathtt{false}) = \mathtt{false}$.
    * Recall not $\underline{G_1}, \underline{G_2} \in \mathbb{G}$, or not $\underline{G_1} \# \underline{G_2}$. Then, by Defn. 38, $\phi(\underline{G_1} \parallel \underline{G_2}, \chi) = \mathtt{false}$ and $\phi(\underline{G_1} \parallel \underline{G_2}, \phi(\gamma, \chi)) = \mathtt{false}$.
    * Recall $[\![\mathtt{false}]\!] = [\![\mathtt{false}]\!]$. Then, $[\![\mathtt{false}]\!] = [\![\phi(\gamma, \mathtt{false})]\!]$. Then, $[\![\phi(\underline{G_1} \parallel \underline{G_2}, \phi(\gamma, \chi))]\!] = [\![\phi(\gamma, \phi(\underline{G_1} \parallel \underline{G_2}, \chi))]\!]$. Then, $[\![\phi(\underline{G}, \phi(\gamma, \chi))]\!] = [\![\phi(\gamma, \phi(\underline{G}, \chi))]\!]$.
- **Step:** $\underline{G} = R.\textbf{if}\ \xi\ \underline{G_1}\ \underline{G_2}$, for some $R, \underline{G_1}, \underline{G_2}, \xi$.
  - Recall $\underline{G} \# \gamma$. Then, $R.\textbf{if}\ \xi\ \underline{G_1}\ \underline{G_2} \# \gamma$. Then, by Lem. 38, $\xi \# \gamma$ and $\underline{G_1} \# \gamma$ and $\underline{G_2} \# \gamma$.
  - Recall $\xi \# \gamma$. Then, by Lem. 37, $\gamma \# \xi$. Then, by Defn. 23, $\mathsf{write}(\gamma) \cap \mathsf{read}(\xi) = \emptyset$. Then, by Lem. 12, $\mathsf{write}(\gamma) \cap \mathsf{read}(\xi^+) = \emptyset$ and $\mathsf{write}(\gamma) \cap \mathsf{read}(\xi^-) = \emptyset$ and $\mathsf{write}(\gamma) \cap \mathsf{read}(\xi^\equiv) = \emptyset$. Then, by Lem. 76, $\phi(\gamma, \xi^+) = \xi^+$ and $\phi(\gamma, \xi^-) = \xi^-$ and $\phi(\gamma, \xi^\equiv) = \xi^\equiv$.

- Recall $\underline{G}_1 \# \gamma$ and $\underline{G}_2 \# \gamma$. Then, by induction, $[\![\phi(\underline{G}_1, \phi(\gamma, \chi))]\!] = [\![\phi(\gamma, \phi(\underline{G}_1, \chi))]\!]$ and $[\![\phi(\underline{G}_2, \phi(\gamma, \chi))]\!] = [\![\phi(\gamma, \phi(\underline{G}_2, \chi))]\!]$. Then, by Lem. 22, $[\![\xi^+ \Rightarrow \phi(\underline{G}_1, \phi(\gamma, \chi))]\!] = [\![\xi^+ \Rightarrow \phi(\gamma, \phi(\underline{G}_1, \chi))]\!]$ and $[\![\xi^- \Rightarrow \phi(\underline{G}_2, \phi(\gamma, \chi))]\!] = [\![\xi^- \Rightarrow \phi(\gamma, \phi(\underline{G}_2, \chi))]\!]$. Then, $[\![\xi^+ \Rightarrow \phi(\underline{G}_1, \phi(\gamma, \chi))]\!] \cap [\![\xi^- \Rightarrow \phi(\underline{G}_2, \phi(\gamma, \chi))]\!] = [\![\xi^+ \Rightarrow \phi(\gamma, \phi(\underline{G}_1, \chi))]\!] \cap [\![\xi^- \Rightarrow \phi(\gamma, \phi(\underline{G}_2, \chi))]\!]$. Then, $[\![\xi^+ \Rightarrow \phi(\underline{G}_1, \phi(\gamma, \chi))]\!] \cap [\![\xi^- \Rightarrow \phi(\underline{G}_2, \phi(\gamma, \chi))]\!] \cap [\![\xi^{\equiv}]\!] = [\![\xi^+ \Rightarrow \phi(\gamma, \phi(\underline{G}_1, \chi))]\!] \cap [\![\xi^- \Rightarrow \phi(\gamma, \phi(\underline{G}_2, \chi))]\!] \cap [\![\xi^{\equiv}]\!]$. Then, $[\![\xi^+ \Rightarrow \phi(\underline{G}_1, \phi(\gamma, \chi))]\!] \cap [\![\xi^- \Rightarrow \phi(\underline{G}_2, \phi(\gamma, \chi))]\!] \cap [\![\xi^{\equiv}]\!] = [\![\phi(\gamma, \xi^+) \Rightarrow \phi(\gamma, \phi(\underline{G}_1, \chi))]\!] \cap [\![\phi(\gamma, \xi^-) \Rightarrow \phi(\gamma, \phi(\underline{G}_2, \chi))]\!] \cap [\![\phi(\gamma, \xi^{\equiv})]\!]$. Then, by Defn. 13, $[\![(\xi^+ \Rightarrow \phi(\underline{G}_1, \phi(\gamma, \chi))) \wedge (\xi^- \Rightarrow \phi(\underline{G}_2, \phi(\gamma, \chi))) \wedge \xi^{\equiv}]\!] = [\![(\phi(\gamma, \xi^+) \Rightarrow \phi(\gamma, \phi(\underline{G}_1, \chi))) \wedge (\phi(\gamma, \xi^-) \Rightarrow \phi(\gamma, \phi(\underline{G}_2, \chi))) \wedge \phi(\gamma, \xi^{\equiv})]\!]$. Then, by Lem. 75, $[\![(\xi^+ \Rightarrow \phi(\underline{G}_1, \phi(\gamma, \chi))) \wedge (\xi^- \Rightarrow \phi(\underline{G}_2, \phi(\gamma, \chi))) \wedge \xi^{\equiv}]\!] = [\![\phi(\gamma, \xi^+ \Rightarrow \phi(\underline{G}_1, \chi)) \wedge \phi(\gamma, \xi^- \Rightarrow \phi(\underline{G}_2, \chi)) \wedge \phi(\gamma, \xi^{\equiv})]\!]$. Then, by Lem. 75, $[\![(\xi^+ \Rightarrow \phi(\underline{G}_1, \phi(\gamma, \chi))) \wedge (\xi^- \Rightarrow \phi(\underline{G}_2, \phi(\gamma, \chi))) \wedge \xi^{\equiv}]\!] = [\![\phi(\gamma, (\xi^+ \Rightarrow \phi(\underline{G}_1, \chi)) \wedge (\xi^- \Rightarrow \phi(\underline{G}_2, \chi)) \wedge \xi^{\equiv})]\!]$. Then, by Defn. 38, $[\![\phi(R.\mathbf{if}\ \xi\ \underline{G}_1\ \underline{G}_2, \phi(\gamma, \chi))]\!] = [\![\phi(\gamma, \phi(R.\mathbf{if}\ \xi\ \underline{G}_1\ \underline{G}_2, \chi))]\!]$. Then, $[\![\phi(\underline{G}, \phi(\gamma, \chi))]\!] = [\![\phi(\gamma, \phi(\underline{G}, \chi))]\!]$.

– **Step:** $\underline{G} = \mathbf{if}\ \xi|_0\ \underline{G}_1|_{R_1}\ \underline{G}_2|_{R_2}$, for some $R_1, R_2, \underline{G}_1, \underline{G}_2, \xi$.
Recall:
- **Case:** $R_1 = \emptyset = R_2$.
  - ∗ Recall $\underline{G} \# \xi$. Then, $\mathbf{if}\ \xi|_0\ \underline{G}_1|_{R_1}\ \underline{G}_2|_{R_2} \# \xi$.
  - ∗ Recall $\mathbf{if}\ \xi|_0\ \underline{G}_1|_{R_1}\ \underline{G}_2|_{R_2} \# \xi$ and $R_1 = \emptyset = R_2$. Then, by Lem. 38, $\xi \# \gamma$ and $\underline{G}_1 \# \gamma$ and $\underline{G}_2 \# \gamma$.
  - ∗ Recall $\xi \# \gamma$. Then, by Lem. 37, $\gamma \# \xi$. Then, by Defn. 23, $\mathsf{write}(\gamma) \cap \mathsf{read}(\xi) = \emptyset$. Then, by Lem. 12, $\mathsf{write}(\gamma) \cap \mathsf{read}(\xi^+) = \emptyset$ and $\mathsf{write}(\gamma) \cap \mathsf{read}(\xi^-) = \emptyset$ and $\mathsf{write}(\gamma) \cap \mathsf{read}(\xi^{\equiv}) = \emptyset$. Then, by Lem. 76, $\phi(\gamma, \xi^+) = \xi^+$ and $\phi(\gamma, \xi^-) = \xi^-$ and $\phi(\gamma, \xi^{\equiv}) = \xi^{\equiv}$.
  - ∗ Recall $R_1 = \emptyset = R_2$. Then, by Defn. 38, $\phi(\mathbf{if}\ \xi|_0\ \underline{G}_1|_{R_1}\ \underline{G}_2|_{R_2}, \chi) = (\xi^+ \Rightarrow \phi(\underline{G}_1, \chi)) \wedge (\xi^- \Rightarrow \phi(\underline{G}_2, \chi)) \wedge \xi^{\equiv}$ and $\phi(\mathbf{if}\ \xi|_0\ \underline{G}_1|_{R_1}\ \underline{G}_2|_{R_2}, \phi(\gamma, \chi)) = (\xi^+ \Rightarrow \phi(\underline{G}_1, \phi(\gamma, \chi))) \wedge (\xi^- \Rightarrow \phi(\underline{G}_2, \phi(\gamma, \chi))) \wedge \xi^{\equiv}$.
  - ∗ Recall $\underline{G}_1 \# \gamma$ and $\underline{G}_2 \# \gamma$. Then, by induction, $[\![\phi(\underline{G}_1, \phi(\gamma, \chi))]\!] = [\![\phi(\gamma, \phi(\underline{G}_1, \chi))]\!]$ and $[\![\phi(\underline{G}_2, \phi(\gamma, \chi))]\!] = [\![\phi(\gamma, \phi(\underline{G}_2, \chi))]\!]$. Then, by Lem. 22, $[\![\xi^+ \Rightarrow \phi(\underline{G}_1, \phi(\gamma, \chi))]\!] = [\![\xi^+ \Rightarrow \phi(\gamma, \phi(\underline{G}_1, \chi))]\!]$ and $[\![\xi^- \Rightarrow \phi(\underline{G}_2, \phi(\gamma, \chi))]\!] = [\![\xi^- \Rightarrow \phi(\gamma, \phi(\underline{G}_2, \chi))]\!]$. Then, $[\![\xi^+ \Rightarrow \phi(\underline{G}_1, \phi(\gamma, \chi))]\!] \cap [\![\xi^- \Rightarrow \phi(\underline{G}_2, \phi(\gamma, \chi))]\!] = [\![\xi^+ \Rightarrow \phi(\gamma, \phi(\underline{G}_1, \chi))]\!] \cap [\![\xi^- \Rightarrow \phi(\gamma, \phi(\underline{G}_2, \chi))]\!]$. Then, $[\![\xi^+ \Rightarrow \phi(\underline{G}_1, \phi(\gamma, \chi))]\!] \cap [\![\xi^- \Rightarrow \phi(\underline{G}_2, \phi(\gamma, \chi))]\!] \cap [\![\xi^{\equiv}]\!] = [\![\xi^+ \Rightarrow \phi(\gamma, \phi(\underline{G}_1, \chi))]\!] \cap [\![\xi^- \Rightarrow \phi(\gamma, \phi(\underline{G}_2, \chi))]\!] \cap [\![\xi^{\equiv}]\!]$. Then, $[\![\xi^+ \Rightarrow \phi(\underline{G}_1, \phi(\gamma, \chi))]\!] \cap [\![\xi^- \Rightarrow \phi(\underline{G}_2, \phi(\gamma, \chi))]\!] \cap [\![\xi^{\equiv}]\!] = [\![\phi(\gamma, \xi^+) \Rightarrow \phi(\gamma, \phi(\underline{G}_1, \chi))]\!] \cap [\![\phi(\gamma, \xi^-) \Rightarrow \phi(\gamma, \phi(\underline{G}_2, \chi))]\!] \cap [\![\phi(\gamma, \xi^{\equiv})]\!]$. Then, by Defn. 13, $[\![(\xi^+ \Rightarrow \phi(\underline{G}_1, \phi(\gamma, \chi))) \wedge (\xi^- \Rightarrow \phi(\underline{G}_2, \phi(\gamma, \chi))) \wedge \xi^{\equiv}]\!] = [\![(\phi(\gamma, \xi^+) \Rightarrow \phi(\gamma, \phi(\underline{G}_1, \chi))) \wedge (\phi(\gamma, \xi^-) \Rightarrow \phi(\gamma, \phi(\underline{G}_2, \chi))) \wedge \phi(\gamma, \xi^{\equiv})]\!]$. Then, by Lem. 75, $[\![(\xi^+ \Rightarrow \phi(\underline{G}_1, \phi(\gamma, \chi))) \wedge (\xi^- \Rightarrow \phi(\underline{G}_2, \phi(\gamma, \chi))) \wedge \xi^{\equiv}]\!] = [\![\phi(\gamma, \xi^+ \Rightarrow \phi(\underline{G}_1, \chi)) \wedge \phi(\gamma, \xi^- \Rightarrow \phi(\underline{G}_2, \chi)) \wedge \phi(\gamma, \xi^{\equiv})]\!]$. Then, by Lem. 75, $[\![(\xi^+ \Rightarrow \phi(\underline{G}_1, \phi(\gamma, \chi))) \wedge (\xi^- \Rightarrow \phi(\underline{G}_2, \phi(\gamma, \chi))) \wedge \xi^{\equiv}]\!] = [\![\phi(\gamma, (\xi^+ \Rightarrow \phi(\underline{G}_1, \chi)) \wedge (\xi^- \Rightarrow \phi(\underline{G}_2, \chi)) \wedge \xi^{\equiv})]\!]$. Then, $[\![\phi(\mathbf{if}\ \xi|_0\ \underline{G}_1|_{R_1}\ \underline{G}_2|_{R_2}, \phi(\gamma, \chi))]\!] = [\![\phi(\gamma, \phi(\mathbf{if}\ \xi|_0\ \underline{G}_1|_{R_1}\ \underline{G}_2|_{R_2}, \chi))]\!]$. Then, $[\![\phi(\underline{G}, \phi(\gamma, \chi))]\!] = [\![\phi(\gamma, \phi(\underline{G}, \chi))]\!]$.
- **Case:** $R_1 = \emptyset \neq R_2$.
  - ∗ Recall $\underline{G} \# \xi$. Then, $\mathbf{if}\ \xi|_0\ \underline{G}_1|_{R_1}\ \underline{G}_2|_{R_2} \# \xi$.
  - ∗ Recall $\mathbf{if}\ \xi|_0\ \underline{G}_1|_{R_1}\ \underline{G}_2|_{R_2} \# \xi$ and $R_1 = \emptyset \neq R_2$. Then, by Lem. 38, $\xi \upharpoonright r \# \gamma$ for every $r \in \mathsf{subj}(\xi) \setminus R_2$, and $\underline{G}_2 \# \gamma$.
  - ∗ Recall $\xi \upharpoonright r \# \gamma$ for every $r \in \mathsf{subj}(\xi) \setminus R_2$. Then, by Lem. 39, $\bigwedge\{(\xi \upharpoonright r)^-\}_{r \in \mathsf{subj}(\xi) \setminus R_2} \# \gamma$. Then, by Lem. 11, $\bigwedge\{\xi^- \upharpoonright r\}_{r \in \mathsf{subj}(\xi) \setminus R_2} \# \gamma$. Then, by Lem. 37, $\gamma \# \bigwedge\{\xi^- \upharpoonright r\}_{r \in \mathsf{subj}(\xi) \setminus R_2}$. Then, by Defn. 23,

$\mathsf{write}(\gamma) \cap \mathsf{read}(\bigwedge\{\xi^- \upharpoonright r\}_{r \in \mathsf{subj}(\xi) \setminus R_2}) = \emptyset$. Then, by Lem. 76,
$\phi(\gamma, \bigwedge\{\xi^- \upharpoonright r\}_{r \in \mathsf{subj}(\xi) \setminus R_2}) = \bigwedge\{\xi^- \upharpoonright r\}_{r \in \mathsf{subj}(\xi) \setminus R_2}$.

∗ Recall $R_1 = \emptyset \neq R_2$. Then, by Defn. 38,
$\phi(\mathbf{if}\ \xi|_0\ \underline{G}_1|_{R_1}\ \underline{G}_2|_{R_2}, \chi) = \phi(\underline{G}_2, \chi) \wedge \bigwedge\{\xi^- \upharpoonright r\}_{r \in \mathsf{subj}(\xi) \setminus R_2}$ and
$\phi(\mathbf{if}\ \xi|_0\ \underline{G}_1|_{R_1}\ \underline{G}_2|_{R_2}, \phi(\gamma, \chi)) = \phi(\underline{G}_2, \phi(\gamma, \chi)) \wedge \bigwedge\{\xi^- \upharpoonright r\}_{r \in \mathsf{subj}(\xi) \setminus R_2}$.

∗ Recall $\underline{G}_2 \# \gamma$. Then, by induction, $[\![\phi(\underline{G}_2, \phi(\gamma, \chi))]\!] = [\![\phi(\gamma, \phi(\underline{G}_2, \chi))]\!]$. Then,
$[\![\phi(\underline{G}_2, \phi(\gamma, \chi))]\!] \cap [\![\bigwedge\{\xi^- \upharpoonright r\}_{r \in \mathsf{subj}(\xi) \setminus R_2}]\!] = [\![\phi(\gamma, \phi(\underline{G}_2, \chi))]\!] \cap [\![\bigwedge\{\xi^- \upharpoonright r\}_{r \in \mathsf{subj}(\xi) \setminus R_2}]\!]$.
Then, $[\![\phi(\underline{G}_2, \phi(\gamma, \chi))]\!] \cap [\![\bigwedge\{\xi^- \upharpoonright r\}_{r \in \mathsf{subj}(\xi) \setminus R_2}]\!] =$
$[\![\phi(\gamma, \phi(\underline{G}_2, \chi))]\!] \cap [\![\phi(\gamma, \bigwedge\{\xi^- \upharpoonright r\}_{r \in \mathsf{subj}(\xi) \setminus R_2})]\!]$. Then, by Defn. 13,
$[\![\phi(\underline{G}_2, \phi(\gamma, \chi)) \wedge \bigwedge\{\xi^- \upharpoonright r\}_{r \in \mathsf{subj}(\xi) \setminus R_2}]\!] = [\![\phi(\gamma, \phi(\underline{G}_2, \chi)) \wedge \phi(\gamma, \bigwedge\{\xi^- \upharpoonright r\}_{r \in \mathsf{subj}(\xi) \setminus R_2})]\!]$.
Then, by Lem. 75,
$[\![\phi(\underline{G}_2, \phi(\gamma, \chi)) \wedge \bigwedge\{\xi^- \upharpoonright r\}_{r \in \mathsf{subj}(\xi) \setminus R_2}]\!] = [\![\phi(\gamma, \phi(\underline{G}_2, \chi) \wedge \bigwedge\{\xi^- \upharpoonright r\}_{r \in \mathsf{subj}(\xi) \setminus R_2})]\!]$.
Then, $[\![\phi(\mathbf{if}\ \xi|_0\ \underline{G}_1|_{R_1}\ \underline{G}_2|_{R_2}, \phi(\gamma, \chi))]\!] = [\![\phi(\gamma, \phi(\mathbf{if}\ \xi|_0\ \underline{G}_1|_{R_1}\ \underline{G}_2|_{R_2}, \chi))]\!]$. Then,
$[\![\phi(\underline{G}, \phi(\gamma, \chi))]\!] = [\![\phi(\gamma, \phi(\underline{G}, \chi))]\!]$.

- **Case:** $R_1 \neq \emptyset = R_2$. Similar to case $R_1 = \emptyset \neq R_2$.
- **Case:** $R_1 \neq \emptyset \neq R_2$.
  ∗ Recall $\mathsf{write}(\gamma) \cap \emptyset = \emptyset$. Then, by Lem. 3, $\mathsf{write}(\gamma) \cap \mathsf{read}(\mathtt{false}) = \emptyset$. Then, by Lem. 76, $\phi(\gamma, \mathtt{false}) = \mathtt{false}$.
  ∗ Recall $R_1 \neq \emptyset \neq R_2$. Then, by Defn. 38, $\phi(\mathbf{if}\ \xi|_0\ \underline{G}_1|_{R_1}\ \underline{G}_2|_{R_2}, \chi) = \mathtt{false}$ and $\phi(\mathbf{if}\ \xi|_0\ \underline{G}_1|_{R_1}\ \underline{G}_2|_{R_2}, \phi(\gamma, \chi)) = \mathtt{false}$.
  ∗ Recall $[\![\mathtt{false}]\!] = [\![\mathtt{false}]\!]$. Then, $[\![\mathtt{false}]\!] = [\![\phi(\gamma, \mathtt{false})]\!]$. Then,
  $[\![\phi(\mathbf{if}\ \xi|_0\ \underline{G}_1|_{R_1}\ \underline{G}_2|_{R_2}, \phi(\gamma, \chi))]\!] = [\![\phi(\gamma, \phi(\mathbf{if}\ \xi|_0\ \underline{G}_1|_{R_1}\ \underline{G}_2|_{R_2}, \chi))]\!]$. Then,
  $[\![\phi(\underline{G}, \phi(\gamma, \chi))]\!] = [\![\phi(\gamma, \phi(\underline{G}, \chi))]\!]$.

– **Step:** $\underline{G} = \mathbf{if}\ \xi|_0\ G_1|_\emptyset\ \underline{G}_2|_{R_2 \cup \{r\}}$, for some $R_2, G_1, \underline{G}_2, r, \xi$.
- Recall $\underline{G} \# \xi$. Then, $\mathbf{if}\ \xi|_0\ G_1|_\emptyset\ \underline{G}_2|_{R_2 \cup \{r\}} \# \xi$.
- Recall $\mathbf{if}\ \xi|_0\ G_1|_\emptyset\ \underline{G}_2|_{R_2 \cup \{r\}} \# \xi$ and $\emptyset = \emptyset \neq R_2 \cup \{r\}$. Then, by Lem. 38, $\xi \upharpoonright r \# \gamma$ for every $r \in \mathsf{subj}(\xi) \setminus (R_2 \cup \{r\})$, and $\underline{G}_2 \# \gamma$.
- Recall $\xi \upharpoonright r \# \gamma$ for every $r \in \mathsf{subj}(\xi) \setminus (R_2 \cup \{r\})$. Then, by Lem. 39,
$\bigwedge\{(\xi \upharpoonright r)^-\}_{r \in \mathsf{subj}(\xi) \setminus (R_2 \cup \{r\})} \# \gamma$. Then, by Lem. 11, $\bigwedge\{\xi^- \upharpoonright r\}_{r \in \mathsf{subj}(\xi) \setminus (R_2 \cup \{r\})} \# \gamma$.
Then, by Lem. 37, $\gamma \# \bigwedge\{\xi^- \upharpoonright r\}_{r \in \mathsf{subj}(\xi) \setminus (R_2 \cup \{r\})}$. Then, by Defn. 23,
$\mathsf{write}(\gamma) \cap \mathsf{read}(\bigwedge\{\xi^- \upharpoonright r\}_{r \in \mathsf{subj}(\xi) \setminus (R_2 \cup \{r\})}) = \emptyset$. Then, by Lem. 76,
$\phi(\gamma, \bigwedge\{\xi^- \upharpoonright r\}_{r \in \mathsf{subj}(\xi) \setminus (R_2 \cup \{r\})}) = \bigwedge\{\xi^- \upharpoonright r\}_{r \in \mathsf{subj}(\xi) \setminus (R_2 \cup \{r\})}$.
- Recall $\emptyset = \emptyset \neq R_2 \cup \{r\}$. Then, by Defn. 38,
$\phi(\mathbf{if}\ \xi|_0\ G_1|_\emptyset\ \underline{G}_2|_{R_2 \cup \{r\}}, \chi) = \phi(\underline{G}_2, \chi) \wedge \bigwedge\{\xi^- \upharpoonright r\}_{r \in \mathsf{subj}(\xi) \setminus (R_2 \cup \{r\})}$ and
$\phi(\mathbf{if}\ \xi|_0\ G_1|_\emptyset\ \underline{G}_2|_{R_2 \cup \{r\}}, \phi(\gamma, \chi)) = \phi(\underline{G}_2, \phi(\gamma, \chi)) \wedge \bigwedge\{\xi^- \upharpoonright r\}_{r \in \mathsf{subj}(\xi) \setminus (R_2 \cup \{r\})}$.
- Recall $\underline{G}_2 \# \gamma$. Then, by induction, $[\![\phi(\underline{G}_2, \phi(\gamma, \chi))]\!] = [\![\phi(\gamma, \phi(\underline{G}_2, \chi))]\!]$. Then,
$[\![\phi(\underline{G}_2, \phi(\gamma, \chi))]\!] \cap [\![\bigwedge\{\xi^- \upharpoonright r\}_{r \in \mathsf{subj}(\xi) \setminus (R_2 \cup \{r\})}]\!] =$
$[\![\phi(\gamma, \phi(\underline{G}_2, \chi))]\!] \cap [\![\bigwedge\{\xi^- \upharpoonright r\}_{r \in \mathsf{subj}(\xi) \setminus (R_2 \cup \{r\})}]\!]$. Then,
$[\![\phi(\underline{G}_2, \phi(\gamma, \chi))]\!] \cap [\![\bigwedge\{\xi^- \upharpoonright r\}_{r \in \mathsf{subj}(\xi) \setminus (R_2 \cup \{r\})}]\!] =$
$[\![\phi(\gamma, \phi(\underline{G}_2, \chi))]\!] \cap [\![\phi(\gamma, \bigwedge\{\xi^- \upharpoonright r\}_{r \in \mathsf{subj}(\xi) \setminus (R_2 \cup \{r\})})]\!]$. Then, by Defn. 13,
$[\![\phi(\underline{G}_2, \phi(\gamma, \chi)) \wedge \bigwedge\{\xi^- \upharpoonright r\}_{r \in \mathsf{subj}(\xi) \setminus (R_2 \cup \{r\})}]\!] =$
$[\![\phi(\gamma, \phi(\underline{G}_2, \chi)) \wedge \phi(\gamma, \bigwedge\{\xi^- \upharpoonright r\}_{r \in \mathsf{subj}(\xi) \setminus (R_2 \cup \{r\})})]\!]$. Then, by Lem. 75,
$[\![\phi(\underline{G}_2, \phi(\gamma, \chi)) \wedge \bigwedge\{\xi^- \upharpoonright r\}_{r \in \mathsf{subj}(\xi) \setminus (R_2 \cup \{r\})}]\!] =$
$[\![\phi(\gamma, \phi(\underline{G}_2, \chi) \wedge \bigwedge\{\xi^- \upharpoonright r\}_{r \in \mathsf{subj}(\xi) \setminus (R_2 \cup \{r\})})]\!]$. Then,
$[\![\phi(\mathbf{if}\ \xi|_0\ G_1|_\emptyset\ \underline{G}_2|_{R_2 \cup \{r\}}, \phi(\gamma, \chi))]\!] = [\![\phi(\gamma, \phi(\mathbf{if}\ \xi|_0\ G_1|_\emptyset\ \underline{G}_2|_{R_2 \cup \{r\}}, \chi))]\!]$. Then,
$[\![\phi(\underline{G}, \phi(\gamma, \chi))]\!] = [\![\phi(\gamma, \phi(\underline{G}, \chi))]\!]$.

– **Step:** $\underline{G} = \mathbf{if}\ \xi|_0\ \underline{G}_1|_{R_1 \cup \{r\}}\ G_2|_\emptyset$, for some $R_1, \underline{G}_1, G_2, r, \xi$. Similar to case "$\underline{G} = \mathbf{if}\ \xi|_0\ G_1|_\emptyset\ \underline{G}_2|_{R_2 \cup \{r\}}$, for some $R_2, G_1, \underline{G}_2, r, \xi$". □

Proof (of Lem. 87). Recall $\checkmark_R(G)$. Then, by Defn. 27:

– **Base:** [✓1-Act1], such that $G = q.y{:=}e$, for some q, y, e.
By Defn. 21, $q.y{:=}e \in \underline{\mathbb{G}}$. Then, $G \in \underline{\mathbb{G}}$.

- **Base:** [$\checkmark$1-Act2]. Similar to case [$\checkmark$1-Act1].
- **Base:** [$\checkmark$1-Skip]. Similar to case [$\checkmark$1-Act1].
- **Step:** [$\checkmark$1-Seq], such that $G = G_1 \,;\, G_2$ and $\checkmark_R(G_1)$ and $\checkmark_R(G_2)$, for some G_1, G_2.
  - Recall $[\![\phi(G,\chi)]\!] \neq \emptyset$. Then, $[\![\phi(G_1 \,;\, G_2,\chi)]\!] \neq \emptyset$. Then, by Defn. 38, $[\![\phi(G_1,\phi(G_2,\chi))]\!] \neq \emptyset$.
  - Recall $\checkmark_R(G_1)$ and $[\![\phi(G_1,\phi(G_2,\chi))]\!] \neq \emptyset$. Then:
    - **Case:** $\mathsf{subj}(G_1) = R$.
      Recall $\checkmark_R(G_2)$. Then, by Lem. 47, $\mathsf{subj}(G_2) \subseteq R$. Then, $R \cup \mathsf{subj}(G_2) = R$. Then, $\mathsf{subj}(G_1) \cup \mathsf{subj}(G_2) = R$. Then, by Defn. 24, $\mathsf{subj}(G_1 \,;\, G_2) = R$. Then, $\mathsf{subj}(G) = R$.
    - **Case:** $G_1 \in \underline{\mathbb{G}}$.
      - Recall $G_1 \in \underline{\mathbb{G}}$ and $[\![\phi(G_1,\phi(G_2,\chi))]\!] \neq \emptyset$. Then, by Lem. 84, $[\![\phi(G_2,\chi)]\!] \neq \emptyset$.
      - Recall $\checkmark_R(G_2)$ and $[\![\phi(G_2,\chi)]\!] \neq \emptyset$. Then, by induction:
        - **Case:** $\mathsf{subj}(G_2) = R$.
          Recall $\checkmark_R(G_1)$. Then, by Lem. 47, $\mathsf{subj}(G_1) \subseteq R$. Then, $\mathsf{subj}(G_1) \cup R = R$. Then, $\mathsf{subj}(G_1) \cup \mathsf{subj}(G_2) = R$. Then, by Defn. 24, $\mathsf{subj}(G_1 \,;\, G_2) = R$. Then, $\mathsf{subj}(G) = R$.
        - **Case:** $G_2 \in \underline{\mathbb{G}}$.
          Recall $G_1, G_2 \in \underline{\mathbb{G}}$. Then, by Defn. 21, $G_1 \,;\, G_2 \in \underline{\mathbb{G}}$. Then, $G \in \underline{\mathbb{G}}$.
- **Step:** [$\checkmark$1-Par], such that $G = G_1 \parallel G_2$ and $\checkmark_R(G_1)$ and $\checkmark_R(G_2)$, for some G_1, G_2.
  Recall $[\![\phi(G,\chi)]\!] \neq \emptyset$. Then, $[\![\phi(G_1 \parallel G_2,\chi)]\!] \neq \emptyset$. Then, by Defn. 38:
  - **Case:** $[\![\phi(G_1,\phi(G_2,\chi))]\!] \neq \emptyset$.
    Recall $\checkmark_R(G_1)$ and $[\![\phi(G_1,\phi(G_2,\chi))]\!] \neq \emptyset$. Then:
    - **Case:** $\mathsf{subj}(G_1) = R$.
      Recall $\checkmark_R(G_2)$. Then, by Lem. 47, $\mathsf{subj}(G_2) \subseteq R$. Then, $R \cup \mathsf{subj}(G_2) = R$. Then, $\mathsf{subj}(G_1) \cup \mathsf{subj}(G_2) = R$. Then, by Defn. 24, $\mathsf{subj}(G_1 \parallel G_2) = R$. Then, $\mathsf{subj}(G) = R$.
    - **Case:** $G_1 \in \underline{\mathbb{G}}$.
      - Recall $G_1 \in \underline{\mathbb{G}}$ and $[\![\phi(G_1,\phi(G_2,\chi))]\!] \neq \emptyset$. Then, by Lem. 84, $[\![\phi(G_2,\chi)]\!] \neq \emptyset$.
      - Recall $\checkmark_R(G_2)$ and $[\![\phi(G_2,\chi)]\!] \neq \emptyset$. Then, by induction:
        - **Case:** $\mathsf{subj}(G_2) = R$.
          Recall $\checkmark_R(G_1)$. Then, by Lem. 47, $\mathsf{subj}(G_1) \subseteq R$. Then, $\mathsf{subj}(G_1) \cup R = R$. Then, $\mathsf{subj}(G_1) \cup \mathsf{subj}(G_2) = R$. Then, by Defn. 24, $\mathsf{subj}(G_1 \parallel G_2) = R$. Then, $\mathsf{subj}(G) = R$.
        - **Case:** $G_2 \in \underline{\mathbb{G}}$.
          Recall $G_1, G_2 \in \underline{\mathbb{G}}$. Then, by Defn. 21, $G_1 \parallel G_2 \in \underline{\mathbb{G}}$. Then, $G \in \underline{\mathbb{G}}$.
  - **Case:** $[\![\mathtt{false}]\!] \neq \emptyset$. Then, by Lem. 15, $\emptyset \neq \emptyset$. Then, **false**,
- **Step:** [$\checkmark$1-If], such that $G = R.\mathbf{if}\ \xi\ G_1\ G_2$ and $\checkmark_R(G_1)$ and $\checkmark_R(G_2)$ and $R = \mathsf{subj}(\xi)$, for some G_1, G_2, ξ.
  Recall $\checkmark_R(G_1)$ and $\checkmark_R(G_2)$. Then, by Lem. 47, $\mathsf{subj}(G_1) \subseteq R$ and $\mathsf{subj}(G_2) \subseteq R$. Then, $\mathsf{subj}(G_1) \cup \mathsf{subj}(G_2) \subseteq R$. Then, $R \cup \mathsf{subj}(G_1) \cup \mathsf{subj}(G_2) = R$. Then, $\mathsf{subj}(\xi) \cup \mathsf{subj}(G_1) \cup \mathsf{subj}(G_2) = R$. Then, by Defn. 24, $\mathsf{subj}(R.\mathbf{if}\ \xi\ G_1\ G_2) = R$. Then, $\mathsf{subj}(G) = R$.
- **Step:** [$\checkmark$1-While]. Similar to case [$\checkmark$1-If].
- **Step:** [$\checkmark$1-NIf], such that $G = \mathbf{if}\ \xi|_0\ G_1|_{R_1}\ G_2|_{R_2}$ and $\checkmark_R(G_1)$ and $\checkmark_R(G_2)$ and $R = \mathsf{subj}(\xi)$, for some R_1, R_2, G_1, G_2, ξ.
  - **Case:** $R_1 = \emptyset = R_2$.
    Recall $\checkmark_R(G_1)$ and $\checkmark_R(G_2)$. Then, by Lem. 47, $\mathsf{subj}(G_1) \subseteq R$ and $\mathsf{subj}(G_2) \subseteq R$. Then, $\mathsf{subj}(G_1) \cup \mathsf{subj}(G_2) \subseteq R$. Then, $R \cup \mathsf{subj}(G_1) \cup \mathsf{subj}(G_2) = R$. Then, $\mathsf{subj}(\xi) \cup \mathsf{subj}(G_1) \cup \mathsf{subj}(G_2) = R$. Then, $(\mathsf{subj}(\xi) \setminus \emptyset) \cup (\mathsf{subj}(G_1) \setminus \emptyset) \cup (\mathsf{subj}(G_2) \setminus \emptyset) = R$. Then, $(\mathsf{subj}(\xi) \setminus (\emptyset \cup \emptyset)) \cup (\mathsf{subj}(G_1) \setminus \emptyset) \cup (\mathsf{subj}(G_2) \setminus \emptyset) = R$. Then, $(\mathsf{subj}(\xi) \setminus (R_1 \cup R_2)) \cup (\mathsf{subj}(G_1) \setminus R_2) \cup (\mathsf{subj}(G_2) \setminus R_1) = R$. Then, by Defn. 24, $\mathsf{subj}(\mathbf{if}\ \xi|_0\ G_1|_{R_1}\ G_2|_{R_2}) = R$. Then, $\mathsf{subj}(G) = R$.
  - **Case:** $R_1 = \emptyset \neq R_2$.
    - Recall $R_1 = \emptyset \neq R_2$. Then, by Defn. 38, $\phi(\mathbf{if}\ \xi|_0\ G_1|_{R_1}\ G_2|_{R_2}, \chi) = \phi(G_2,\chi) \wedge \bigwedge\{\xi^- \restriction r\}_{r \in \mathsf{subj}(\xi) \setminus R_2}$.

- * Recall $[\![\phi(G,\chi)]\!] \neq \emptyset$. Then, $[\![\phi(\mathbf{if}\ \xi|_0\ G_1|_{R_1}\ G_2|_{R_2},\chi)]\!] \neq \emptyset$. Then, $[\![\phi(G_2,\chi) \wedge \bigwedge\{\xi^- \upharpoonright r\}_{r\in\mathsf{subj}(\xi)\setminus R_2}]\!] \neq \emptyset$. Then, by Defn. 13, $[\![\phi(G_2,\chi)]\!] \cap [\![\bigwedge\{\xi^- \upharpoonright r\}_{r\in\mathsf{subj}(\xi)\setminus R_2}]\!] \neq \emptyset$. Then, $[\![\phi(G_2,\chi)]\!] \neq \emptyset$.
  * Recall $\checkmark_R(G_2)$ and $[\![\phi(G_2,\chi)]\!] \neq \emptyset$. Then, by induction:
    · **Case:** $\mathsf{subj}(G_2) = R$.
      · Recall Recall $R \subseteq R$. Then, $\mathsf{subj}(\xi) \subseteq R$. Then, $\mathsf{subj}(\xi) \setminus (R_1 \cup R_2) \subseteq R$.
      · Recall $\checkmark_R(G_1)$. Then, by Lem. 47, $\mathsf{subj}(G_1) \subseteq R$. Then, $\mathsf{subj}(G_1) \setminus R_2 \subseteq R$.
      · Recall $\mathsf{subj}(\xi) \setminus (R_1 \cup R_2) \subseteq R$ and $\mathsf{subj}(G_1) \setminus R_2 \subseteq R$. Then, $(\mathsf{subj}(\xi) \setminus (R_1 \cup R_2)) \cup (\mathsf{subj}(G_1) \setminus R_2) \subseteq R$. Then, $(\mathsf{subj}(\xi) \setminus (R_1 \cup R_2)) \cup (\mathsf{subj}(G_1) \setminus R_2) \cup R = R$. Then, $(\mathsf{subj}(\xi) \setminus (R_1 \cup R_2)) \cup (\mathsf{subj}(G_1) \setminus R_2) \cup \mathsf{subj}(G_2) = R$. Then, $(\mathsf{subj}(\xi) \setminus (R_1 \cup R_2)) \cup (\mathsf{subj}(G_1) \setminus R_2) \cup (\mathsf{subj}(G_2) \setminus \emptyset) = R$. Then, $(\mathsf{subj}(\xi) \setminus (R_1 \cup R_2)) \cup (\mathsf{subj}(G_1) \setminus R_2) \cup (\mathsf{subj}(G_2) \setminus R_1) = R$. Then, by Defn. 24, $\mathsf{subj}(\mathbf{if}\ \xi|_0\ G_1|_{R_1}\ G_2|_{R_2}) = R$. Then, $\mathsf{subj}(G) = R$.
    · **Case:** $G_2 \in \underline{\mathbb{G}}$.
      Recall $G_2 \in \underline{\mathbb{G}}$ and $R_2 \neq \emptyset$. Then, by Defn. 21, $\mathbf{if}\ \xi|_0\ G_1|_\emptyset\ G_2|_{R_2} \in \underline{\mathbb{G}}$. Then, $\mathbf{if}\ \xi|_0\ G_1|_{R_1}\ G_2|_{R_2} \in \underline{\mathbb{G}}$. Then, $G \in \underline{\mathbb{G}}$.
- **Case:** $R_1 \neq \emptyset = R_2$. Similar to case $R_1 = \emptyset \neq R_2$.
- **Case:** $R_1 \neq \emptyset \neq R_2$.
  * Recall $R_1 \neq \emptyset \neq R_2$. Then, by Defn. 38, $\phi(\mathbf{if}\ \xi|_0\ G_1|_{R_1}\ G_2|_{R_2},\chi) = \mathtt{false}$.
  * Recall $[\![\phi(G,\chi)]\!] \neq \emptyset$. Then, $[\![\phi(\mathbf{if}\ \xi|_0\ G_1|_{R_1}\ G_2|_{R_2},\chi)]\!] \neq \emptyset$. Then, $[\![\mathtt{false}]\!] \neq \emptyset$. Then, by Lem. 15, $\emptyset \neq \emptyset$. Then, **false**.

– **Step:** [✓1-NWHILE]. Similar to case [✓1-IF]. □

Proof (of Lem. 88).

1. Recall $G\downarrow$. Then, by Defn. 28:
   – **Base:** [↓1-SKIP], such that $G = \mathbf{skip}$.
   Recall $[\![\chi]\!] \subseteq [\![\chi]\!]$. Then, by Defn. 38, $[\![\phi(\mathbf{skip},\chi)]\!] = [\![\chi]\!]$. Then, $[\![\phi(G,\chi)]\!] \subseteq [\![\chi]\!]$.
   – **Step:** [↓1-SEQ], such that $G = G_1\,;G_2$ and $G_1\downarrow$ and $G_2\downarrow$, for some G_1, G_2.
   - Recall $\checkmark_R(G)$. Then, $\checkmark_R(G_1\,;G_2)$. Then, by Defn. 27, $\checkmark_R(G_1)$ and $\checkmark_R(G_2)$.
   - Recall $\checkmark_R(G_1)$ and $G_1\downarrow$. Then, by induction, $[\![\phi(G_1,\phi(G_2,\chi))]\!] \subseteq [\![\phi(G_2,\chi)]\!]$.
   - Recall $\checkmark_R(G_2)$ and $G_2\downarrow$. Then, by induction, $[\![\phi(G_2,\chi)]\!] \subseteq [\![\chi]\!]$.
   - Recall $[\![\phi(G_1,\phi(G_2,\chi))]\!] \subseteq [\![\phi(G_2,\chi)]\!] \subseteq [\![\chi]\!]$. Then, $[\![\phi(G_1,\phi(G_2,\chi))]\!] \subseteq [\![\chi]\!]$. Then, by Defn. 38, $[\![\phi(G_1\,;G_2,\chi)]\!] \subseteq [\![\chi]\!]$. Then, $[\![\phi(G,\chi)]\!] \subseteq [\![\chi]\!]$.

   – **Step:** [↓1-PAR], such that $G = G_1 \parallel G_2$ and $G_1\downarrow$ and $G_2\downarrow$, for some G_1, G_2.
   - **Case:** $G_1, G_2 \in \underline{\mathbb{G}}$ and $G_1 \# G_2$.
     * Recall $\checkmark_R(G)$. Then, $\checkmark_R(G_1 \parallel G_2)$. Then, by Defn. 27, $\checkmark_R(G_1)$ and $\checkmark_R(G_2)$.
     * Recall $\checkmark_R(G_1)$ and $G_1\downarrow$. Then, by induction, $[\![\phi(G_1,\phi(G_2,\chi))]\!] \subseteq [\![\phi(G_2,\chi)]\!]$.
     * Recall $\checkmark_R(G_2)$ and $G_2\downarrow$. Then, by induction, $[\![\phi(G_2,\chi)]\!] \subseteq [\![\chi]\!]$.
     * Recall $G_1, G_2 \in \underline{\mathbb{G}}$ and $G_1 \# G_2$. Then, by Defn. 38, $\phi(G_1 \parallel G_2,\chi) = \phi(G_1,\phi(G_2,\chi))$.
     * Recall $[\![\phi(G_1,\phi(G_2,\chi))]\!] \subseteq [\![\phi(G_2,\chi)]\!] \subseteq [\![\chi]\!]$. Then, $[\![\phi(G_1,\phi(G_2,\chi))]\!] \subseteq [\![\chi]\!]$. Then, $[\![\phi(G_1 \parallel G_2,\chi)]\!] \subseteq [\![\chi]\!]$. Then, $[\![\phi(G,\chi)]\!] \subseteq [\![\chi]\!]$.
   - **Case:** Not $G_1, G_2 \in \underline{\mathbb{G}}$, or not $G_1 \# G_2$.
     * Recall not $G_1, G_2 \in \underline{\mathbb{G}}$, or not $G_1 \# G_2$. Then, by Defn. 38, $\phi(G_1 \parallel G_2,\chi) = \mathtt{false}$.
     * Recall $[\![\mathtt{false}]\!] \subseteq [\![\chi]\!]$. Then, $[\![\phi(G_1 \parallel G_2,\chi)]\!] \subseteq [\![\chi]\!]$. Then, $[\![\phi(G,\chi)]\!] \subseteq [\![\chi]\!]$.

   – **Step:** [↓1-NIF], such that $G = \mathbf{if}\ \xi|_0\ G_1|_{R_1}\ G_2|_{R_2}$ and $\mathsf{subj}(\xi) = R_1 \cup R_2$, and $R_1 \neq \emptyset$ implies $G_1\downarrow$, and $R_2 \neq \emptyset$ implies $G_2\downarrow$, for some R_1, R_2, G_1, G_2, ξ.
   By Defn. 38:
   - **Case:** $R_1 = \emptyset = R_2$.
     * Recall $\checkmark_R(G)$. Then, $\checkmark_R(\mathbf{if}\ \xi|_0\ G_1|_{R_1}\ G_2|_{R_2})$. Then, by Defn. 27, $R = \mathsf{subj}(\xi)$.

  * Recall $\checkmark_R(G)$. Then, by Lem. 47, $R \neq \emptyset$. Then, $\mathsf{subj}(\xi) \neq \emptyset$. Then, $R_1 \cup R_2 \neq \emptyset$. Then, $\emptyset \cup \emptyset \neq \emptyset$. Then, **false**.
- **Case:** $\phi(\textbf{if } \xi|_0\, G_1|_{R_1}\, G_2|_{R_2}, \chi) = \phi(G_2, \chi) \wedge \bigwedge\{\xi^- \restriction r\}_{r \in \mathsf{subj}(\xi) \setminus R_2}$ and $R_1 = \emptyset \neq R_2$.
  * Recall $\checkmark_R(G)$. Then, $\checkmark_R(\textbf{if } \xi|_0\, G_1|_{R_1}\, G_2|_{R_2})$. Then, by Defn. 27, $\checkmark_R(G_2)$.
  * Recall $R_2 \neq \emptyset$. Then, $G_2\!\downarrow$.
  * Recall $\checkmark_R(G_2)$ and $G_2\!\downarrow$. Then, by induction, $[\![\phi(G_2, \chi)]\!] \subseteq [\![\chi]\!]$. Then, $[\![\phi(G_2, \chi)]\!] \cap [\![\bigwedge\{\xi^- \restriction r\}_{r \in \mathsf{subj}(\xi) \setminus R_2}]\!] \subseteq [\![\chi]\!]$. Then, by Defn. 13, $[\![\phi(G_2, \chi) \wedge \bigwedge\{\xi^- \restriction r\}_{r \in \mathsf{subj}(\xi) \setminus R_2}]\!] \subseteq [\![\chi]\!]$. Then, $[\![\phi(\textbf{if } \xi|_0\, G_1|_{R_1}\, G_2|_{R_2}, \chi)]\!] \subseteq [\![\chi]\!]$. Then, $[\![\phi(G, \chi)]\!] \subseteq [\![\chi]\!]$.
- **Case:** $\phi(\textbf{if } \xi|_0\, G_1|_{R_1}\, G_2|_{R_2}, \chi) = \phi(G_1, \chi) \wedge \bigwedge\{\xi^+ \restriction r\}_{r \in \mathsf{subj}(\xi) \setminus R_1}$ and $R_1 \neq \emptyset = R_2$. Similar to case "$\phi(\textbf{if } \xi|_0\, G_1|_{R_1}\, G_2|_{R_2}, \chi) = \phi(G_2, \chi) \wedge \bigwedge\{\xi^- \restriction r\}_{r \in \mathsf{subj}(\xi) \setminus R_2}$ and $R_1 = \emptyset \neq R_2$".
- **Case:** $\phi(\textbf{if } \xi|_0\, G_1|_{R_1}\, G_2|_{R_2}, \chi) = \mathtt{false}$.
  By Lem. 15, $[\![\mathtt{false}]\!] \subseteq [\![\chi]\!]$. Then, $[\![\phi(\textbf{if } \xi|_0\, G_1|_{R_1}\, G_2|_{R_2}, \chi)]\!] \subseteq [\![\chi]\!]$. Then, $[\![\phi(G, \chi)]\!] \subseteq [\![\chi]\!]$. □

2. – Recall $\checkmark_R(\mathcal{G})$. Then, by Defn. 27, $\mathcal{G} = \{G\}$ and $\checkmark_R(G)$, for some G.
   – Recall $\mathcal{G}\!\downarrow$. Then, $\{G\}\!\downarrow$. Then, by Defn. 29, $G\!\downarrow$.
   – Recall $\checkmark_R(G)$ and $G\!\downarrow$. Then, by 1, $[\![\phi(G, \chi)]\!] \subseteq [\![\chi]\!]$. Then, by Defn. 38, $[\![\phi(\{G\}, \chi)]\!] \subseteq [\![\chi]\!]$. Then, $[\![\phi(\mathcal{G}, \chi)]\!] \subseteq [\![\chi]\!]$. □
3. – Recall $(\mathcal{G}, \mathcal{S})\!\downarrow$. Then, by Defn. 30, $\mathcal{G}\!\downarrow$.
   – Recall $\checkmark_R(\mathcal{G})$ and $\mathcal{G}\!\downarrow$. Then, by 2, $[\![\phi(\mathcal{G}, \chi)]\!] \subseteq [\![\chi]\!]$.
   – Recall $\mathcal{S} \in [\![\phi(\mathcal{G}, \chi)]\!] \subseteq [\![\chi]\!]$. Then, $\mathcal{S} \in [\![\chi]\!]$. □

Proof (of Lem. 89).

1. Recall $G \xrightarrow{\xi,\gamma} G'$. Then, by Defn. 31:
   – **Base:** [$\rightarrow$1-Act], such that $G' = \textbf{skip}$.
   Recall $\checkmark_R(G)$. Then, by Lem. 47, $R \neq \emptyset$. Then, by [$\checkmark$1-Skip], $\checkmark_R(\textbf{skip})$. Then, $\checkmark_R(G')$.
   – **Base:** [$\rightarrow$1-If1], such that $G = \hat{R}.\textbf{if }\hat{\xi}\, G_1\, G_2$ and $G' = G_1$, for some $\hat{R}, G_1, G_2, \hat{\xi}$.
   Recall $\checkmark_R(G)$. Then, $\checkmark_R(\hat{R}.\textbf{if }\hat{\xi}\, G_1\, G_2)$. Then, by Defn. 27, $\checkmark_R(G_1)$. Then, $\checkmark_R(G')$.
   – **Base:** [$\rightarrow$1-If2]. Similar to case [$\rightarrow$1-If1].
   – **Base:** [$\rightarrow$1-While1], such that $G = \hat{R}.\textbf{while }\hat{\xi}\, \{\psi\}\, \hat{G}$ and $G' = \hat{G}\,;\hat{R}.\textbf{while }\hat{\xi}\, \{\psi\}\, \hat{G}$, for some $\hat{R}, \hat{G}, \psi, \hat{\xi}$.
   - Recall $\checkmark_R(G)$. Then, $\checkmark_R(\hat{R}.\textbf{while }\hat{\xi}\, \{\psi\}\, \hat{G})$.
   - Recall $\checkmark_R(\hat{R}.\textbf{while }\hat{\xi}\, \{\psi\}\, \hat{G})$. Then, by Defn. 27, $\checkmark_R(\hat{G})$.
   - Recall $\checkmark_R(\hat{G})$ and $\checkmark_R(\hat{R}.\textbf{while }\hat{\xi}\, \{\psi\}\, \hat{G})$. Then, by [$\checkmark$1-Seq], $\checkmark_R(\hat{G}\,;\hat{R}.\textbf{while }\hat{\xi}\, \{\psi\}\, \hat{G})$. Then, $\checkmark_R(G')$.
   – **Base:** [$\rightarrow$1-While2]. Similar to case [$\rightarrow$1-Act].
   – **Base:** [$\rightarrow$1-NIf1], such that $0 > 0$. Then, **false**.
   – **Base:** [$\rightarrow$1-NIf2], such that $G = \textbf{if } \hat{\xi}|_0\, G_1|_{R_1}\, G_2|_{R_2}$ and $G' = \textbf{if } \hat{\xi}|_0\, G_1|_{R_1 \cup \{r\}}\, G_2|_{R_2}$ and $\xi = \hat{\xi}^+ \restriction r$ and $r \in \mathsf{subj}(\hat{\xi}) \setminus (R_1 \cup R_2)$, for some $R_1, R_2, G_1, G_2, r, \hat{\xi}$.
   Recall:
   - **Case:** $R_1 = \emptyset = R_2$.
     * Recall $\checkmark_R(G)$. Then, $\checkmark_R(\textbf{if } \hat{\xi}|_0\, G_1|_{R_1}\, G_2|_{R_2})$. Then, by Defn. 27, $\checkmark_R(G_1)$ and $\checkmark_R(G_2)$ and $R = \mathsf{subj}(\xi)$ and $R_1, R_2 \subseteq \mathsf{subj}(\hat{\xi})$.
     * Recall $r \in \mathsf{subj}(\hat{\xi}) \setminus (R_1 \cup R_2)$. Then, $r \in \mathsf{subj}(\hat{\xi})$. Then, $\{r\} \subseteq \mathsf{subj}(\hat{\xi})$.
     * Recall $R_1 \subseteq \mathsf{subj}(\hat{\xi})$ and $\{r\} \subseteq \mathsf{subj}(\hat{\xi})$. Then, $R_1 \cup \{r\} \subseteq \mathsf{subj}(\hat{\xi})$.
     * Recall $R_2 = \emptyset$. Then, $R_1 \cup \{r\} \neq \emptyset$ implies $R_2 = \emptyset$.
     * Recall $R_2 = \emptyset$. Then, $R_2 \neq \emptyset$ implies $R_1 \cup \{r\} = \emptyset$.
     * Recall $\checkmark_R(G_1)$ and $\checkmark_R(G_2)$ and $R = \mathsf{subj}(\xi)$ and $R_1 \cup \{r\} \subseteq \mathsf{subj}(\hat{\xi})$ and $R_2 \subseteq \mathsf{subj}(\hat{\xi})$, and $R_1 \cup \{r\} \neq \emptyset$ implies $R_2 = \emptyset$, and $R_2 \neq \emptyset$ implies $R_1 \cup \{r\} = \emptyset$. Then, by [$\checkmark$1-NIf], $\checkmark_R(\textbf{if } \hat{\xi}|_0\, G_1|_{R_1 \cup \{r\}}\, G_2|_{R_2})$. Then, $\checkmark_R(G')$.
   - **Case:** $R_1 = \emptyset \neq R_2$.

- $*$ Recall $R_1 = \emptyset \neq R_2$. Then, by Defn. 38, $\phi(\textbf{if } \hat{\xi}|_0\, G_1|_{R_1}\, G_2|_{R_2}, \chi) = \phi(G_2, \chi) \wedge \bigwedge\{\hat{\xi}^- \upharpoonright \hat{r}\}_{\hat{r} \in \mathsf{subj}(\hat{\xi}) \setminus R_2}$.
- $*$ Recall $r \in \mathsf{subj}(\hat{\xi}) \setminus (R_1 \cup R_2)$. Then, $r \in \mathsf{subj}(\hat{\xi}) \setminus R_2$. Then, $r \in \mathsf{subj}(\hat{\xi}) \setminus R_2 \subseteq \mathsf{subj}(\hat{\xi})$. Then, by Lem. 29, $[\![\bigwedge\{\hat{\xi}^- \upharpoonright \hat{r}\}_{\hat{r} \in \mathsf{subj}(\hat{\xi}) \setminus R_2}]\!] \cap [\![\hat{\xi}^+ \upharpoonright r]\!] = [\![\mathtt{false}]\!]$.
- $*$ Recall $[\![\phi(G, \chi) \wedge \xi]\!] \neq \emptyset$. Then, $[\![\phi(\textbf{if } \hat{\xi}|_0\, G_1|_{R_1}\, G_2|_{R_2}, \chi) \wedge (\hat{\xi}^+ \upharpoonright \hat{r})]\!] \neq \emptyset$. Then, $[\![\phi(G_2, \chi) \wedge \bigwedge\{\hat{\xi}^- \upharpoonright r\}_{r \in \mathsf{subj}(\hat{\xi}) \setminus R_2} \wedge (\hat{\xi}^+ \upharpoonright \hat{r})]\!] \neq \emptyset$. Then, by Defn. 13, $[\![\phi(G_2, \chi)]\!] \cap [\![\bigwedge\{\hat{\xi}^- \upharpoonright r\}_{r \in \mathsf{subj}(\hat{\xi}) \setminus R_2}]\!] \cap [\![\hat{\xi}^+ \upharpoonright \hat{r}]\!] \neq \emptyset$. Then, $[\![\phi(G_2, \chi)]\!] \cap [\![\mathtt{false}]\!] \neq \emptyset$. Then, by Lem. 15, $[\![\phi(G_2, \chi)]\!] \cap \emptyset \neq \emptyset$. Then, $\emptyset \neq \emptyset$. Then, **false**.
- **Case:** $R_1 \neq \emptyset = R_2$. Similar to case $R_1 = \emptyset = R_2$.
- **Case:** $R_1 \neq \emptyset \neq R_2$.
  - $*$ Recall $R_1 \neq \emptyset \neq R_2$. Then, by Defn. 38, $\phi(\textbf{if } \hat{\xi}|_0\, G_1|_{R_1}\, G_2|_{R_2}, \chi) = \mathtt{false}$.
  - $*$ Recall $[\![\phi(G, \chi) \wedge \xi]\!] \neq \emptyset$. Then, $[\![\phi(\textbf{if } \hat{\xi}|_0\, G_1|_{R_1}\, G_2|_{R_2}, \chi) \wedge \xi]\!] \neq \emptyset$. Then, $[\![\mathtt{false} \wedge \xi]\!] \neq \emptyset$. Then, by Defn. 13, $[\![\mathtt{false}]\!] \cap [\![\xi]\!] \neq \emptyset$. Then, by Lem. 15, $\emptyset \cap [\![\xi]\!] \neq \emptyset$. Then, $\emptyset \neq \emptyset$. Then, **false**.

– **Base:** [$\to$1-NIF3]. Similar to case [$\to$1-NIF2].

– **Step:** [$\to$1-SEQ1], such that $G = G_1 \,;\, G_2$ and $G' = G'_1 \,;\, G_2$ and $G_1 \xrightarrow{\xi,\gamma} G'_1$, for some G_1, G'_1, G_2.
- Recall $\checkmark_R(G)$. Then, $\checkmark_R(G_1 \,;\, G_2)$. Then, by Defn. 27, $\checkmark_R(G_1)$ and $\checkmark_R(G_2)$.
- Recall $[\![\phi(G, \chi) \wedge \xi]\!] \neq \emptyset$. Then, $[\![\phi(G_1 \,;\, G_2, \chi) \wedge \xi]\!] \neq \emptyset$. Then, by Defn. 38, $[\![\phi(G_1, \phi(G_2, \chi)) \wedge \xi]\!] \neq \emptyset$.
- Recall $\checkmark_R(G_1)$ and $[\![\phi(G_1, \phi(G_2, \chi)) \wedge \xi]\!] \neq \emptyset$ and $G_1 \xrightarrow{\xi,\gamma} G'_1$. Then, by induction, $\checkmark_R(G'_1)$.
- Recall $\checkmark_R(G'_1)$ and $\checkmark_R(G_2)$. Then, by [$\checkmark$1-SEQ], $\checkmark_R(G'_1 \,;\, G_2)$. Then, $\checkmark_R(G')$.

– **Step:** [$\to$1-SEQ2], such that $G = G_1 \,;\, G_2$ and $G' = G_1 \,;\, G'_2$ and $\mathsf{subj}(G_1) \cap (\mathsf{subj}(\xi) \cup \mathsf{subj}(\gamma)) = \emptyset$ and $G_2 \xrightarrow{\xi,\gamma} G'_2$, for some G_1, G_2, G'_2.
- Recall $\checkmark_R(G)$. Then, $\checkmark_R(G_1 \,;\, G_2)$. Then, by Defn. 27, $\checkmark_R(G_1)$ and $\checkmark_R(G_2)$.
- Recall $[\![\phi(G, \chi) \wedge \xi]\!] \neq \emptyset$. Then, $[\![\phi(G_1 \,;\, G_2, \chi) \wedge \xi]\!] \neq \emptyset$. Then, by Defn. 38, $[\![\phi(G_1, \phi(G_2, \chi)) \wedge \xi]\!] \neq \emptyset$.
- Recall $\checkmark_R(G_1)$ and $[\![\phi(G_1, \phi(G_2, \chi)) \wedge \xi]\!] \neq \emptyset$. Then, by Lem. 87:
  - $*$ **Case:** $\mathsf{subj}(G_1) = R$.
    - · Recall $G \xrightarrow{\xi,\gamma} G'$ and $\checkmark_R(G)$. Then, by Lem. 58, $\mathsf{subj}(\gamma) \neq \emptyset$. Then, $\mathsf{subj}(\xi) \cup \mathsf{subj}(\gamma) \neq \emptyset$.
    - · Recall $G \xrightarrow{\xi,\gamma} G'$. Then, by Lem. 53, $\mathsf{subj}(\xi) \cup \mathsf{subj}(\gamma) \cup \mathsf{subj}(G') \subseteq \mathsf{subj}(G)$. Then, $\mathsf{subj}(\xi) \cup \mathsf{subj}(\gamma) \subseteq \mathsf{subj}(G)$.
    - · Recall $\checkmark_R(G)$. Then, by Lem. 47, $\mathsf{subj}(G) \subseteq R$.
    - · Recall $\emptyset \neq \mathsf{subj}(\xi) \cup \mathsf{subj}(\gamma) \subseteq \mathsf{subj}(G) \subseteq R$. Then, $\emptyset \neq \mathsf{subj}(\xi) \cup \mathsf{subj}(\gamma) \subseteq R$. Then, $\emptyset \neq \mathsf{subj}(\xi) \cup \mathsf{subj}(\gamma) \subseteq \mathsf{subj}(G_1)$. Then, $\mathsf{subj}(G_1) \cap (\mathsf{subj}(\xi) \cup \mathsf{subj}(\gamma)) \neq \emptyset$. Then, **false**.
  - $*$ **Case:** $G_1 \in \underline{\mathbb{G}}$.
    - · Recall $\mathsf{subj}(G_1) \cap (\mathsf{subj}(\xi) \cup \mathsf{subj}(\gamma)) = \emptyset$. Then, $(\mathsf{subj}(G_1) \cap \mathsf{subj}(\xi)) \cup (\mathsf{subj}(G_1) \cap \mathsf{subj}(\gamma)) = \emptyset$. Then, $\mathsf{subj}(G_1) \cap \mathsf{subj}(\xi) = \emptyset$. Then, by Lem. 41, $G_1 \mathbin{\#} \xi$.
    - · Recall $G_1 \in \underline{\mathbb{G}}$ and $G_1 \mathbin{\#} \xi$. Then, $[\![\phi(G_1, \phi(G_2, \chi)) \wedge \xi]\!] \subseteq [\![\phi(G_1, \xi)]\!]$. Then, by Defn. 13, $[\![\phi(G_1, \phi(G_2, \chi))]\!] \cap [\![\xi]\!] \subseteq [\![\phi(G_1, \xi)]\!]$. Then, $[\![\phi(G_1, \phi(G_2, \chi))]\!] \cap [\![\xi]\!] \subseteq [\![\phi(G_1, \phi(G_2, \chi))]\!] \cap [\![\phi(G_1, \xi)]\!]$. Then, by Defn. 13, $[\![\phi(G_1, \phi(G_2, \chi)) \wedge \xi]\!] \subseteq [\![\phi(G_1, \phi(G_2, \chi)) \wedge \phi(G_1, \xi)]\!]$. Then, by Lem. 82, $[\![\phi(G_1, \phi(G_2, \chi)) \wedge \xi]\!] \subseteq [\![\phi(G_1, \phi(G_2, \chi) \wedge \xi)]\!]$.
    - · Recall $\emptyset \neq [\![\phi(G_1, \phi(G_2, \chi)) \wedge \xi]\!] \subseteq [\![\phi(G_1, \phi(G_2, \chi) \wedge \xi)]\!]$. Then, $\emptyset \neq [\![\phi(G_1, \phi(G_2, \chi) \wedge \xi)]\!]$.
    - · Recall $G_1 \in \underline{\mathbb{G}}$ and $\emptyset \neq [\![\phi(G_1, \phi(G_2, \chi) \wedge \xi)]\!]$. Then, by Lem. 84, $[\![\phi(G_2, \chi) \wedge \xi]\!] \neq \emptyset$.

· Recall $\checkmark_R(G_2)$ and $[\![\phi(G_2,\chi) \wedge \xi]\!] \neq \emptyset$ and $G_2 \xrightarrow{\xi,\gamma} G'_2$. Then, by induction, $\checkmark_R(G'_2)$.
· Recall $\checkmark_R(G_1)$ and $\checkmark_R(G'_2)$. Then, by [✓1-Seq], $\checkmark_R(G_1 \,;G'_2)$. Then, $\checkmark_R(G')$.

– **Step:** [→1-Par1], such that $G = G_1 \parallel G_2$ and $G' = G'_1 \parallel G_2$ and $G_1 \xrightarrow{\xi,\gamma} G'_1$, for some G_1, G'_1, G_2.
By Defn. 38:
- **Case:** $\phi(G_1 \parallel G_2, \chi) = \phi(G_1, \phi(G_2, \chi))$.
  * Recall $\checkmark_R(G)$. Then, $\checkmark_R(G_1 \parallel G_2)$. Then, by Defn. 27, $\checkmark_R(G_1)$ and $\checkmark_R(G_2)$ and $\mathsf{chan}(G_1) \cap \mathsf{chan}(G_2) = \emptyset$.
  * Recall $[\![\phi(G,\chi) \wedge \xi]\!] \neq \emptyset$. Then, $[\![\phi(G_1 \parallel G_2,\chi) \wedge \xi]\!] \neq \emptyset$. Then, $[\![\phi(G_1,\phi(G_2,\chi)) \wedge \xi]\!] \neq \emptyset$.
  * Recall $\checkmark_R(G_1)$ and $[\![\phi(G_1,\phi(G_2,\chi)) \wedge \xi]\!] \neq \emptyset$ and $G_1 \xrightarrow{\xi,\gamma} G'_1$. Then, by induction, $\checkmark_R(G'_1)$.
  * Recall $G_1 \xrightarrow{\xi,\gamma} G'_1$. Then, by Lem. 54, $\mathsf{chan}(G'_1) \subseteq \mathsf{chan}(G_1)$.
  * Recall $\mathsf{chan}(G_1) \cap \mathsf{chan}(G_2) = \emptyset$ and $\mathsf{chan}(G'_1) \subseteq \mathsf{chan}(G_1)$. Then, $\mathsf{chan}(G'_1) \cap \mathsf{chan}(G_2) = \emptyset$.
  * Recall $\checkmark_R(G'_1)$ and $\checkmark_R(G_2)$ and $\mathsf{chan}(G'_1) \cap \mathsf{chan}(G_2) = \emptyset$. Then, by [✓1-Par], $\checkmark_R(G'_1 \parallel G_2)$. Then, $\checkmark_R(G')$.
- **Case:** $\phi(G_1 \parallel G_2, \chi) = \mathtt{false}$.
  Recall $\emptyset \cap [\![\xi]\!] = \emptyset$. Then, by Lem. 15, $[\![\mathtt{false}]\!] \cap [\![\xi]\!] = \emptyset$. Then, by Defn. 13, $[\![\mathtt{false} \wedge \xi]\!] = \emptyset$. Then, $[\![\phi(G_1 \parallel G_2,\chi) \wedge \xi]\!] = \emptyset$. Then, $[\![\phi(G,\chi) \wedge \xi]\!] = \emptyset$. Then, **false**.

– **Step:** [→1-Par2], such that $G = G_1 \parallel G_2$ and $G' = G_1 \parallel G'_2$ and $G_2 \xrightarrow{\xi,\gamma} G'_2$, for some G_1, G_2, G'_2.
By Defn. 38:
- **Case:** $\phi(G_1 \parallel G_2, \chi) = \phi(G_1, \phi(G_2, \chi))$ and $G_1, G_2 \in \underline{\mathbb{G}}$ and $G_1 \# G_2$.
  * Recall $\checkmark_R(G)$. Then, $\checkmark_R(G_1 \parallel G_2)$. Then, by Defn. 27, $\checkmark_R(G_1)$ and $\checkmark_R(G_2)$ and $\mathsf{chan}(G_1) \cap \mathsf{chan}(G_2) = \emptyset$.
  * Recall $G_1 \# G_2$ and $G_2 \xrightarrow{\xi,\gamma} G'_2$. Then, by Lem. 52, $G_1 \# \xi$.
  * Recall $G_1 \in \underline{\mathbb{G}}$ and $G_1 \# \xi$. Then, $[\![\phi(G_1,\phi(G_2,\chi)) \wedge \xi]\!] \subseteq [\![\phi(G_1,\xi)]\!]$. Then, by Defn. 13, $[\![\phi(G_1,\phi(G_2,\chi))]\!] \cap [\![\xi]\!] \subseteq [\![\phi(G_1,\xi)]\!]$. Then, $[\![\phi(G_1,\phi(G_2,\chi))]\!] \cap [\![\xi]\!] \subseteq [\![\phi(G_1,\phi(G_2,\chi))]\!] \cap [\![\phi(G_1,\xi)]\!]$. Then, by Defn. 13, $[\![\phi(G_1,\phi(G_2,\chi)) \wedge \xi]\!] \subseteq [\![\phi(G_1,\phi(G_2,\chi)) \wedge \phi(G_1,\xi)]\!]$. Then, by Lem. 82, $[\![\phi(G_1,\phi(G_2,\chi)) \wedge \xi]\!] \subseteq [\![\phi(G_1,\phi(G_2,\chi) \wedge \xi)]\!]$.
  * Recall $\emptyset \neq [\![\phi(G_1,\phi(G_2,\chi)) \wedge \xi]\!] \subseteq [\![\phi(G_1,\phi(G_2,\chi) \wedge \xi)]\!]$. Then, $\emptyset \neq [\![\phi(G_1,\phi(G_2,\chi) \wedge \xi)]\!]$.
  * Recall $G_1 \in \underline{\mathbb{G}}$ and $\emptyset \neq [\![\phi(G_1,\phi(G_2,\chi) \wedge \xi)]\!]$. Then, by Lem. 84, $[\![\phi(G_2,\chi) \wedge \xi]\!] \neq \emptyset$.
  * Recall $\checkmark_R(G_2)$ and $[\![\phi(G_2,\chi) \wedge \xi]\!] \neq \emptyset$ and $G_2 \xrightarrow{\xi,\gamma} G'_2$. Then, by induction, $\checkmark_R(G'_2)$.
  * Recall $G_2 \xrightarrow{\xi,\gamma} G'_2$. Then, by Lem. 54, $\mathsf{chan}(G'_2) \subseteq \mathsf{chan}(G_2)$.
  * Recall $\mathsf{chan}(G_1) \cap \mathsf{chan}(G_2) = \emptyset$ and $\mathsf{chan}(G'_2) \subseteq \mathsf{chan}(G_2)$. Then, $\mathsf{chan}(G_1) \cap \mathsf{chan}(G'_2) = \emptyset$.
  * Recall $\checkmark_R(G_1)$ and $\checkmark_R(G'_2)$ and $\mathsf{chan}(G_1) \cap \mathsf{chan}(G'_2) = \emptyset$. Then, by [✓1-Par], $\checkmark_R(G_1 \parallel G'_2)$. Then, $\checkmark_R(G')$.
- **Case:** $\phi(G_1 \parallel G_2, \chi) = \mathtt{false}$.
  Recall $\emptyset \cap [\![\xi]\!] = \emptyset$. Then, by Lem. 15, $[\![\mathtt{false}]\!] \cap [\![\xi]\!] = \emptyset$. Then, by Defn. 13, $[\![\mathtt{false} \wedge \xi]\!] = \emptyset$. Then, $[\![\phi(G_1 \parallel G_2,\chi) \wedge \xi]\!] = \emptyset$. Then, $[\![\phi(G,\chi) \wedge \xi]\!] = \emptyset$. Then, **false**.

– **Step:** [→1-NIf4], such that $G = \textbf{if } \hat{\xi}|_0\, G_1|_{R_1}\, G_2|_{R_2}$ and $G' = \textbf{if } \hat{\xi}|_0\, G'_1|_{R_1}\, G_2|_{R_2}$ and $G_1 \xrightarrow{\xi,\gamma} G'_1$, for some $R_1, R_2, G_1, G'_1, G_2, \hat{\xi}$.
- Recall $\checkmark_R(G)$. Then, $\checkmark_R(\textbf{if } \hat{\xi}|_0\, G_1|_{R_1}\, G_2|_{R_2})$. Then, by Defn. 27, $\checkmark_R(G_1)$ and $\checkmark_R(G_2)$ and $R = \mathsf{subj}(\xi)$ and $R_1, R_2 \subseteq \mathsf{subj}(\hat{\xi})$, and $R_1 \neq \emptyset$ implies $R_2 = \emptyset$, and $R_2 \neq \emptyset$ implies $R_1 = \emptyset$.
- Recall $\checkmark_R(G_1)$ and $G_1 \xrightarrow{\xi,\gamma} G'_1$. Then, by induction, $\checkmark_R(G'_1)$.

  - Recall $\checkmark_R(G_1)$ and $\checkmark_R(G_2)$ and $R = \mathsf{subj}(\xi)$ and $R_1, R_2 \subseteq \mathsf{subj}(\hat{\xi})$, and $R_1 \neq \emptyset$ implies $R_2 = \emptyset$, and $R_2 \neq \emptyset$ implies $R_1 = \emptyset$. Then, by [$\checkmark$1-NIf], $\checkmark_R(\textbf{if}\ G_1'|_0\ R_1|_{G_2}\ R_2|_))$. Then, $\checkmark_R(G')$.
- **Step:** [$\rightarrow$1-NIf5]. Similar to case [$\rightarrow$1-NIf4].
- **Step:** [$\rightarrow$1-NWhile], such that $G = \textbf{while}\ \hat{\xi}|_0\ \{\psi\}\ \hat{G}|_\emptyset$ and $\textbf{if}\ \hat{\xi}|_0\ (\hat{G}\,;\textbf{while}\ \hat{\xi}|_0\ \{\psi\}\ \hat{G}|_\emptyset)|_\emptyset\ \textbf{skip}|_\emptyset \xrightarrow{\xi,\gamma} G'$, for some $\hat{G}, \psi, \hat{\xi}$.
  - Recall $\checkmark_R(G)$. Then, $\checkmark_R(\textbf{while}\ \hat{\xi}|_0\ \{\psi\}\ \hat{G}|_\emptyset)$. Then, by Lem. 46, $\checkmark_R(\textbf{if}\ \hat{\xi}|_0\ (\hat{G}\,;\textbf{while}\ \hat{\xi}|_0\ \{\psi\}\ \hat{G}|_\emptyset)|_\emptyset\ \textbf{skip}|_\emptyset)$.
  - By Lem. 80, $[\![\phi(\textbf{while}\ \hat{\xi}|_0\ \{\psi\}\ \hat{G}|_\emptyset, \chi)]\!] \subseteq [\![\phi(\textbf{if}\ \hat{\xi}|_0\ (\hat{G}\,;\textbf{while}\ \hat{\xi}|_0\ \{\psi\}\ \hat{G}|_\emptyset)|_\emptyset\ \textbf{skip}|_\emptyset, \chi)]\!]$. Then, $[\![\phi(G,\chi)]\!] \subseteq [\![\phi(\textbf{if}\ \hat{\xi}|_0\ (\hat{G}\,;\textbf{while}\ \hat{\xi}|_0\ \{\psi\}\ \hat{G}|_\emptyset)|_\emptyset\ \textbf{skip}|_\emptyset, \chi)]\!]$. Then, $[\![\phi(G,\chi)]\!] \cap [\![\xi]\!] \subseteq [\![\phi(\textbf{if}\ \hat{\xi}|_0\ (\hat{G}\,;\textbf{while}\ \hat{\xi}|_0\ \{\psi\}\ \hat{G}|_\emptyset)|_\emptyset\ \textbf{skip}|_\emptyset, \chi)]\!] \cap [\![\xi]\!]$. Then, by Defn. 13, $[\![\phi(G,\chi) \wedge \xi]\!] \subseteq [\![\phi(\textbf{if}\ \hat{\xi}|_0\ (\hat{G}\,;\textbf{while}\ \hat{\xi}|_0\ \{\psi\}\ \hat{G}|_\emptyset)|_\emptyset\ \textbf{skip}|_\emptyset, \chi) \wedge \xi]\!]$.
  - Recall $\emptyset \neq [\![\phi(G,\chi) \wedge \xi]\!] \subseteq [\![\phi(\textbf{if}\ \hat{\xi}|_0\ (\hat{G}\,;\textbf{while}\ \hat{\xi}|_0\ \{\psi\}\ \hat{G}|_\emptyset)|_\emptyset\ \textbf{skip}|_\emptyset, \chi) \wedge \xi]\!]$. Then, $\emptyset \neq [\![\phi(\textbf{if}\ \hat{\xi}|_0\ (\hat{G}\,;\textbf{while}\ \hat{\xi}|_0\ \{\psi\}\ \hat{G}|_\emptyset)|_\emptyset\ \textbf{skip}|_\emptyset, \chi) \wedge \xi]\!]$.
  - Recall $\checkmark_R(\textbf{if}\ \hat{\xi}|_0\ (\hat{G}\,;\textbf{while}\ \hat{\xi}|_0\ \{\psi\}\ \hat{G}|_\emptyset)|_\emptyset\ \textbf{skip}|_\emptyset)$ and $[\![\phi(\textbf{if}\ \hat{\xi}|_0\ (\hat{G}\,;\textbf{while}\ \hat{\xi}|_0\ \{\psi\}\ \hat{G}|_\emptyset)|_\emptyset\ \textbf{skip}|_\emptyset, \chi) \wedge \xi]\!] \neq \emptyset$ and $\textbf{if}\ \hat{\xi}|_0\ (\hat{G}\,;\textbf{while}\ \hat{\xi}|_0\ \{\psi\}\ \hat{G}|_\emptyset)|_\emptyset\ \textbf{skip}|_\emptyset \xrightarrow{\xi,\gamma} G'$. Then, by induction, $\checkmark_R(G')$. □

2.
- Recall $\checkmark_R(\mathcal{G})$. Then, by Defn. 27, $\mathcal{G} = \{G\}$ and $\checkmark_R(G)$, for some G.
- Recall $[\![\phi(\mathcal{G},\chi) \wedge \xi]\!] \neq \emptyset$. Then, $[\![\phi(\{G\},\chi) \wedge \xi]\!] \neq \emptyset$. Then, by Defn. 38, $[\![\phi(G,\chi) \wedge \xi]\!] \neq \emptyset$.
- Recall $\mathcal{G} \xrightarrow{\xi,\gamma} \mathcal{G}'$. Then, $\{G\} \xrightarrow{\xi,\gamma} \mathcal{G}'$. Then, by Defn. 32, $\mathcal{G}' = \{G'\}$ and $G \xrightarrow{\xi,\gamma} G'$, for some G'.
- Recall $\checkmark_R(G)$ and $[\![\phi(G,\chi) \wedge \xi]\!] \neq \emptyset$ and $G \xrightarrow{\xi,\gamma} G'$. Then, by 1, $\checkmark_R(G')$. Then, by [$\checkmark$2], $\checkmark_R(\{G'\})$. Then, $\checkmark_R(\mathcal{G}')$. □

3.
- Recall $\mathcal{S} \in [\![\phi(\mathcal{G},\chi) \wedge \xi]\!]$. Then, $[\![\phi(\mathcal{G},\chi) \wedge \xi]\!] \neq \emptyset$.
- Recall $(\mathcal{G},\mathcal{S}) \xrightarrow{\gamma} (\mathcal{G}',\mathcal{S}')$. Then, by Defn. 33, $\mathcal{G} \xrightarrow{\xi,\gamma} \mathcal{G}'$, for some γ.
- Recall $\checkmark_R(\mathcal{G})$ and $[\![\phi(\mathcal{G},\chi) \wedge \xi]\!] \neq \emptyset$ and $\mathcal{G} \xrightarrow{\xi,\gamma} \mathcal{G}'$. Then, by 2, $\checkmark_R(\mathcal{G}')$. □

Proof (of Lem. 90). Recall $G \xrightarrow{\xi,\gamma} G'$. Then, by Defn. 31:

- **Base:** [$\rightarrow$1-Act], such that $G' = \textbf{skip}$.
  By Defn. 21, $\textbf{skip} \in \underline{\mathbb{G}}$. Then, $G' \in \underline{\mathbb{G}}$.
- **Base:** [$\rightarrow$1-If1], such that $G = \hat{R}.\textbf{if}\ \hat{\xi}\ G_1\ G_2$ and $G' = G_1$, for some $\hat{R}, G_1, G_2, \hat{\xi}$.
  Recall $G \in \underline{\mathbb{G}}$. Then, $\hat{R}.\textbf{if}\ \hat{\xi}\ G_1\ G_2 \in \underline{\mathbb{G}}$. Then, by Defn. 21, $G_1 \in \underline{\mathbb{G}}$. Then, $G' \in \underline{\mathbb{G}}$.
- **Base:** [$\rightarrow$1-If2]. Similar to case [$\rightarrow$1-If1].
- **Base:** [$\rightarrow$1-While1], such that $G = \hat{R}.\textbf{while}\ \hat{\xi}\ \{\psi\}\ \hat{G}$, for some $\hat{R}, \hat{G}, \psi, \hat{\xi}$.
  By Defn. 21, not $\hat{R}.\textbf{while}\ \hat{\xi}\ \{\psi\}\ \hat{G} \in \underline{\mathbb{G}}$. Then, not $G \in \underline{\mathbb{G}}$. Then, **false**.
- **Base:** [$\rightarrow$1-While2]. Similar to case [$\rightarrow$1-While1].
- **Base:** [$\rightarrow$1-NIf1], such that $0 > 0$. Then, **false**.
- **Base:** [$\rightarrow$1-NIf2], such that $G = \textbf{if}\ \hat{\xi}|_0\ G_1|_{R_1}\ G_2|_{R_2}$ and $G' = \textbf{if}\ \hat{\xi}|_0\ G_1|_{R_1 \cup \{\hat{r}\}}\ G_2|_{R_2}$ and $\xi = \hat{\xi}^+ \upharpoonright \hat{r}$ and $\hat{r} \in \mathsf{subj}(\hat{\xi}) \setminus (R_1 \cup R_2)$, for some $R_1, R_2, G_1, G_2, \hat{\xi}, \hat{r}$.
  Recall $G \in \underline{\mathbb{G}}$. Then, $\textbf{if}\ \hat{\xi}|_0\ G_1|_{R_1}\ G_2|_{R_2} \in \underline{\mathbb{G}}$. Then, by Defn. 21:
  - **Case:** $G_1, G_2 \in \underline{\mathbb{G}}$. Then, by Defn. 21, $\textbf{if}\ \hat{\xi}|_0\ G_1|_{R_1 \cup \{\hat{r}\}}\ G_2|_{R_2} \in \underline{\mathbb{G}}$. Then, $G' \in \underline{\mathbb{G}}$.
  - **Case:** $G_2 \in \underline{\mathbb{G}}$ and $R_1 = \emptyset \neq R_2$.
    - Recall $R_1 = \emptyset \neq R_2$. Then, by Defn. 38, $\phi(\textbf{if}\ \hat{\xi}|_0\ G_1|_{R_1}\ G_2|_{R_2}, \chi) = \phi(G_2,\chi) \wedge \bigwedge\{\hat{\xi}^- \upharpoonright r\}_{r \in \mathsf{subj}(\hat{\xi}) \setminus R_2}$.
    - Recall $\hat{r} \in \mathsf{subj}(\hat{\xi}) \setminus (R_1 \cup R_2)$. Then, $\hat{r} \in \mathsf{subj}(\hat{\xi}) \setminus R_2$. Then, $\hat{r} \in \mathsf{subj}(\hat{\xi}) \setminus R_2 \subseteq \mathsf{subj}(\hat{\xi})$. Then, by Lem. 29, $[\![\bigwedge\{\hat{\xi}^- \upharpoonright r\}_{r \in \mathsf{subj}(\hat{\xi}) \setminus R_2}]\!] \cap [\![\hat{\xi}^+ \upharpoonright \hat{r}]\!] = [\![\mathtt{false}]\!]$.
    - Recall $[\![\phi(G,\chi) \wedge \xi]\!] = [\![\phi(G,\chi) \wedge \xi]\!]$. Then, $[\![\phi(G,\chi) \wedge \xi]\!] = [\![\phi(\textbf{if}\ \hat{\xi}|_0\ G_1|_{R_1}\ G_2|_{R_2}, \chi) \wedge (\hat{\xi}^+ \upharpoonright \hat{r})]\!]$. Then, $[\![\phi(G,\chi) \wedge \xi]\!] = [\![\phi(G_2,\chi) \wedge \bigwedge\{\hat{\xi}^- \upharpoonright r\}_{r \in \mathsf{subj}(\hat{\xi}) \setminus R_2} \wedge (\hat{\xi}^+ \upharpoonright \hat{r})]\!]$. Then, by Defn. 13, $[\![\phi(G,\chi) \wedge \xi]\!] = [\![\phi(G_2,\chi)]\!] \cap [\![\bigwedge\{\hat{\xi}^- \upharpoonright r\}_{r \in \mathsf{subj}(\hat{\xi}) \setminus R_2}]\!] \cap [\![\hat{\xi}^+ \upharpoonright \hat{r}]\!]$. Then, $[\![\phi(G,\chi) \wedge \xi]\!] = [\![\phi(G_2,\chi)]\!] \cap [\![\mathtt{false}]\!]$. Then, by Lem. 15, $[\![\phi(G,\chi) \wedge \xi]\!] = [\![\phi(G_2,\chi)]\!] \cap \emptyset$. Then, $[\![\phi(G,\chi) \wedge \xi]\!] = \emptyset$.

- **Case:** $G_1 \in \underline{\mathbb{G}}$ and $R_2 = \emptyset$.
  Recall $G_1 \in \underline{\mathbb{G}}$. Then, by Defn. 21, $\textbf{if}\ \hat{\xi}|_0\ G_1|_{R_1 \cup \{\hat{r}\}}\ G_2|_\emptyset \in \underline{\mathbb{G}}$. Then, $\textbf{if}\ \hat{\xi}|_0\ G_1|_{R_1 \cup \{\hat{r}\}}\ G_2|_{R_2} \in \underline{\mathbb{G}}$. Then, $G' \in \underline{\mathbb{G}}$.

– **Base:** [$\to$1-NIF3]. Similar to case [$\to$1-NIF2].

– **Step:** [$\to$1-SEQ1], such that $G = G_1 \,;G_2$ and $G' = G'_1 \,; G_2$ and $G_1 \xrightarrow{\xi,\gamma} G'_1$, for some G_1, G'_1, G_2.
  - Recall $G \in \underline{\mathbb{G}}$. Then, $G_1 \,; G_2 \in \underline{\mathbb{G}}$. Then, by Defn. 21, $G_1, G_2 \in \underline{\mathbb{G}}$.
  - Recall $\checkmark_R(G)$. Then, $\checkmark_R(G_1 \,; G_2)$. Then, by Defn. 27, $\checkmark_R(G_1)$.
  - Recall $G_1 \in \underline{\mathbb{G}}$ and $\checkmark_R(G_1)$and $G_1 \xrightarrow{\xi,\gamma} G'_1$. Then, by induction:
    - ∗ **Case:** $G'_1 \in \underline{\mathbb{G}}$.
      Recall $G'_1, G_2 \in \underline{\mathbb{G}}$. Then, by Defn. 21, $G'_1 \,; G_2 \in \underline{\mathbb{G}}$. Then, $G' \in \underline{\mathbb{G}}$.
    - ∗ **Case:** $[\![\phi(G_1, \phi(G_2, \chi)) \wedge \xi]\!] = \emptyset$. Then, by Defn. 38, $[\![\phi(G_1 \,; G_2, \chi) \wedge \xi]\!] = \emptyset$. Then, $[\![\phi(G, \chi) \wedge \xi]\!] = \emptyset$.

– **Step:** [$\to$1-SEQ2], such that $G = G_1 \,; G_2$ and $G' = G_1 \,; G'_2$ and $\mathsf{subj}(G_1) \cap (\mathsf{subj}(\xi) \cup \mathsf{subj}(\gamma)) = \emptyset$ and $G_2 \xrightarrow{\xi,\gamma} G'_2$, for some G_1, G_2, G'_2.
  - Recall $G \in \underline{\mathbb{G}}$. Then, $G_1 \,; G_2 \in \underline{\mathbb{G}}$. Then, by Defn. 21, $G_1, G_2 \in \underline{\mathbb{G}}$.
  - Recall $\checkmark_R(G)$. Then, $\checkmark_R(G_1 \,; G_2)$. Then, by Defn. 27, $\checkmark_R(G_2)$.
  - Recall $G_2 \in \underline{\mathbb{G}}$ and $\checkmark_R(G_2)$ and $G_2 \xrightarrow{\xi,\gamma} G'_2$. Then, by induction:
    - ∗ **Case:** $G'_2 \in \underline{\mathbb{G}}$.
      Recall $G_1, G'_2 \in \underline{\mathbb{G}}$. Then, by Defn. 21, $G_1 \,; G'_2 \in \underline{\mathbb{G}}$. Then, $G' \in \underline{\mathbb{G}}$.
    - ∗ **Case:** $[\![\phi(G_2, \chi) \wedge \xi]\!] = \emptyset$.
      - · Recall $\mathsf{subj}(G_1) \cap (\mathsf{subj}(\xi) \cup \mathsf{subj}(\gamma)) = \emptyset$. Then, $\mathsf{subj}(G_1) \cap \mathsf{subj}(\xi) = \emptyset$. Then, by Lem. 41, $G_1 \,\#\, \xi$.
      - · Recall $[\![\phi(G_2, \chi) \wedge \xi]\!] = \emptyset$. Then, by Lem. 15, $[\![\phi(G_2, \chi) \wedge \xi]\!] = [\![\mathtt{false}]\!]$. Then, by Lem. 81, $[\![\phi(G_1, \phi(G_2, \chi) \wedge \xi)]\!] = [\![\phi(G_1, \mathtt{false})]\!]$.
      - · Recall $G_1 \in \underline{\mathbb{G}}$. Then,by Lem. 83, $[\![\phi(G_1, \mathtt{false})]\!] = [\![\mathtt{false}]\!]$.
      - · Recall $G_1 \in \underline{\mathbb{G}}$ and $G_1 \,\#\, \xi$. Then, by Lem. 85, $[\![\phi(G_1, \phi(G_2, \chi)) \wedge \xi]\!] \subseteq [\![\phi(G_1, \xi)]\!]$. Then, by Defn. 13, $[\![\phi(G_1, \phi(G_2, \chi))]\!] \cap [\![\xi]\!] \subseteq [\![\phi(G_1, \xi)]\!]$. Then, $[\![\phi(G_1, \phi(G_2, \chi))]\!] \cap [\![\xi]\!] \subseteq [\![\phi(G_1, \phi(G_2, \chi))]\!] \cap [\![\phi(G_1, \xi)]\!]$. Then, by Defn. 13, $[\![\phi(G_1, \phi(G_2, \chi)) \wedge \xi]\!] \subseteq [\![\phi(G_1, \phi(G_2, \chi)) \wedge \phi(G_1, \xi)]\!]$. Then, by Defn. 38, $[\![\phi(G_1 \,; G_2, \chi) \wedge \xi]\!] \subseteq [\![\phi(G_1, \phi(G_2, \chi)) \wedge \phi(G_1, \xi)]\!]$. Then, $[\![\phi(G, \chi) \wedge \xi]\!] \subseteq [\![\phi(G_1, \phi(G_2, \chi)) \wedge \phi(G_1, \xi)]\!]$. Then, by Lem. 82, $[\![\phi(G, \chi) \wedge \xi]\!] \subseteq [\![\phi(G_1, \phi(G_2, \chi) \wedge \xi)]\!]$. Then, $[\![\phi(G, \chi) \wedge \xi]\!] \subseteq [\![\phi(G_1, \mathtt{false})]\!]$. Then, $[\![\phi(G, \chi) \wedge \xi]\!] \subseteq [\![\mathtt{false}]\!]$. Then, by Lem. 15, $[\![\phi(G, \chi) \wedge \xi]\!] \subseteq \emptyset$. Then, $[\![\phi(G, \chi) \wedge \xi]\!] = \emptyset$.

– **Step:** [$\to$1-PAR1], such that $G = G_1 \parallel G_2$.
  By Defn. 38:
  - **Case:** $G_1, G_2 \in \underline{\mathbb{G}}$. Then, by Defn. 21, $G_1 \parallel G_2 \in \underline{\mathbb{G}}$. Then, $G \in \underline{\mathbb{G}}$.
  - **Case:** $\phi(G_1 \parallel G_2, \chi) = \mathtt{false}$.
    Recall $\emptyset \cap [\![\xi]\!] = \emptyset$. Then, by Lem. 15, $[\![\mathtt{false}]\!] \cap [\![\xi]\!] = \emptyset$. Then, by Defn. 13, $[\![\mathtt{false} \wedge \xi]\!] = \emptyset$. Then, $[\![\phi(G_1 \parallel G_2, \chi) \wedge \xi]\!] = \emptyset$. Then, $[\![\phi(G, \chi) \wedge \xi]\!] = \emptyset$.

– **Step:** [$\to$1-PAR2]. Similar to case [$\to$1-PAR1].

– **Step:** [$\to$1-NIF4], such that $G = \textbf{if}\ \hat{\xi}|_0\ G_1|_{R_1}\ G_2|_{R_2}$ and $G' = \textbf{if}\ \hat{\xi}|_0\ G'_1|_{R_1}\ G_2|_{R_2}$ and $G_1 \xrightarrow{\xi,\gamma} G'_1$ and $\mathsf{subj}(\xi) \cup \mathsf{subj}(\gamma) \subseteq R_1$, for some $R_1, R_2, G_1, G_2, \hat{\xi}$.
  - Recall $\checkmark_R(G)$. Then, $\checkmark_R(\textbf{if}\ \hat{\xi}|_0\ G_1|_{R_1}\ G_2|_{R_2})$. Then, by Defn. 27, $\checkmark_R(G_1)$.
  - Recall $G \xrightarrow{\xi,\gamma} G'$ and $\checkmark_R(G)$. Then, by Lem. 58, $\mathsf{subj}(\gamma) \neq \emptyset$.
  - Recall $\mathsf{subj}(\xi) \cup \mathsf{subj}(\gamma) \subseteq R_1 \setminus R_2$. Then, $\mathsf{subj}(\gamma) \subseteq R_1 \setminus R_2$. Then, $\mathsf{subj}(\gamma) \subseteq R_1$.
  - Recall $\mathsf{subj}(\gamma) \neq \emptyset$ and $\mathsf{subj}(\gamma) \subseteq R_1$. Then, $R_1 \neq \emptyset$.
  - Recall $G \in \underline{\mathbb{G}}$. Then, $\textbf{if}\ \hat{\xi}|_0\ G_1|_{R_1}\ G_2|_{R_2} \in \underline{\mathbb{G}}$. Then, by Defn. 21:
    - ∗ **Case:** $G_1, G_2 \in \underline{\mathbb{G}}$.
      Recall $G_1 \in \underline{\mathbb{G}}$ and $\checkmark_R(G_1)$ and $G_1 \xrightarrow{\xi,\gamma} G'_1$. Then, by induction:
      - · **Case:** $G'_1 \in \underline{\mathbb{G}}$.
        Recall $G'_1, G_2 \in \underline{\mathbb{G}}$. Then, by Defn. 21, $\textbf{if}\ \hat{\xi}|_0\ G'_1|_{R_1}\ G_2|_{R_2} \in \underline{\mathbb{G}}$. Then, $G' \in \underline{\mathbb{G}}$.

· **Case:** $[\![\phi(G_1,\chi) \wedge \xi]\!] = \emptyset$.
Recall:
· **Case:** $R_2 = \emptyset$.
Recall $R_1 \neq \emptyset = R_2$. Then, by Defn. 38,
$\phi(\textbf{if } \hat{\xi}|_0\, G_1|_{R_1}\, G_2|_{R_2}, \chi) = \phi(G_1,\chi) \wedge \bigwedge\{\xi^+ \restriction r\}_{r \in \mathsf{subj}(\xi)\setminus R_1}$. Then,
$[\![\phi(\textbf{if } \hat{\xi}|_0\, G_1|_{R_1}\, G_2|_{R_2}, \chi)]\!] = [\![\phi(G_1,\chi) \wedge \bigwedge\{\xi^+ \restriction r\}_{r \in \mathsf{subj}(\xi)\setminus R_1}]\!]$. Then, by Defn. 13, $[\![\phi(\textbf{if } \hat{\xi}|_0\, G_1|_{R_1}\, G_2|_{R_2}, \chi)]\!] = [\![\phi(G_1,\chi)]\!] \cap [\![\bigwedge\{\xi^+ \restriction r\}_{r \in \mathsf{subj}(\xi)\setminus R_1}]\!]$.
Then,
$[\![\phi(\textbf{if } \hat{\xi}|_0\, G_1|_{R_1}\, G_2|_{R_2}, \chi)]\!] \cap [\![\xi]\!] = [\![\phi(G_1,\chi)]\!] \cap [\![\bigwedge\{\xi^+ \restriction r\}_{r \in \mathsf{subj}(\xi)\setminus R_1}]\!] \cap [\![\xi]\!]$.
Then, by Defn. 13,
$[\![\phi(\textbf{if } \hat{\xi}|_0\, G_1|_{R_1}\, G_2|_{R_2}, \chi) \wedge \xi]\!] = [\![\phi(G_1,\chi) \wedge \xi]\!] \cap [\![\bigwedge\{\xi^+ \restriction r\}_{r \in \mathsf{subj}(\xi)\setminus R_1}]\!]$. Then,
$[\![\phi(G,\chi) \wedge \xi]\!] = [\![\phi(G_1,\chi) \wedge \xi]\!] \cap [\![\bigwedge\{\xi^+ \restriction r\}_{r \in \mathsf{subj}(\xi)\setminus R_1}]\!]$. Then,
$[\![\phi(G,\chi) \wedge \xi]\!] = \emptyset \cap [\![\bigwedge\{\xi^+ \restriction r\}_{r \in \mathsf{subj}(\xi)\setminus R_1}]\!]$. Then, $[\![\phi(G,\chi) \wedge \xi]\!] = \emptyset$.
· **Case:** $R_2 \neq \emptyset$.
Recall $R_1 \neq \emptyset \neq R_2$. Then, by Defn. 38, $\phi(\textbf{if } \hat{\xi}|_0\, G_1|_{R_1}\, G_2|_{R_2}, \chi) = \mathtt{false}$.
Then, $[\![\phi(\textbf{if } \hat{\xi}|_0\, G_1|_{R_1}\, G_2|_{R_2}, \chi)]\!] = [\![\mathtt{false}]\!]$. Then, $[\![\phi(G,\chi)]\!] = [\![\mathtt{false}]\!]$. Then, by Lem. 15, $[\![\phi(G,\chi)]\!] = \emptyset$.

∗ **Case:** $R_1 = \emptyset$.
Recall $R_1 \neq \emptyset$. Then, $\emptyset \neq \emptyset$. Then, **false**.

∗ **Case:** $G_1 \in \underline{\mathbb{G}}$ and $R_2 = \emptyset$.
Recall $G_1 \in \underline{\mathbb{G}}$ and $\checkmark_R(G_1)$ and $G_1 \xrightarrow{\xi,\gamma} G_1'$. Then, by induction:
· **Case:** $G_1' \in \underline{\mathbb{G}}$.
Recall $G_1' \in \underline{\mathbb{G}}$ and $R_1 \neq \emptyset$. Then, by Defn. 21, $\textbf{if } \hat{\xi}|_0\, G_1'|_{R_1}\, G_2|_{\emptyset} \in \underline{\mathbb{G}}$. Then, $\textbf{if } \hat{\xi}|_0\, G_1'|_{R_1}\, G_2|_{R_2} \in \underline{\mathbb{G}}$. Then, $G' \in \underline{\mathbb{G}}$.
· **Case:** $[\![\phi(G_1,\chi) \wedge \xi]\!] = \emptyset$.
Recall $R_1 \neq \emptyset = R_2$. Then, by Defn. 38,
$\phi(\textbf{if } \hat{\xi}|_0\, G_1|_{R_1}\, G_2|_{R_2}, \chi) = \phi(G_1,\chi) \wedge \bigwedge\{\xi^+ \restriction r\}_{r \in \mathsf{subj}(\xi)\setminus R_1}$. Then,
$[\![\phi(\textbf{if } \hat{\xi}|_0\, G_1|_{R_1}\, G_2|_{R_2}, \chi)]\!] = [\![\phi(G_1,\chi) \wedge \bigwedge\{\xi^+ \restriction r\}_{r \in \mathsf{subj}(\xi)\setminus R_1}]\!]$. Then, by Defn. 13, $[\![\phi(\textbf{if } \hat{\xi}|_0\, G_1|_{R_1}\, G_2|_{R_2}, \chi)]\!] = [\![\phi(G_1,\chi)]\!] \cap [\![\bigwedge\{\xi^+ \restriction r\}_{r \in \mathsf{subj}(\xi)\setminus R_1}]\!]$. Then,
$[\![\phi(\textbf{if } \hat{\xi}|_0\, G_1|_{R_1}\, G_2|_{R_2}, \chi)]\!] \cap [\![\xi]\!] = [\![\phi(G_1,\chi)]\!] \cap [\![\bigwedge\{\xi^+ \restriction r\}_{r \in \mathsf{subj}(\xi)\setminus R_1}]\!] \cap [\![\xi]\!]$.
Then, by Defn. 13,
$[\![\phi(\textbf{if } \hat{\xi}|_0\, G_1|_{R_1}\, G_2|_{R_2}, \chi) \wedge \xi]\!] = [\![\phi(G_1,\chi) \wedge \xi]\!] \cap [\![\bigwedge\{\xi^+ \restriction r\}_{r \in \mathsf{subj}(\xi)\setminus R_1}]\!]$. Then,
$[\![\phi(G,\chi) \wedge \xi]\!] = [\![\phi(G_1,\chi) \wedge \xi]\!] \cap [\![\bigwedge\{\xi^+ \restriction r\}_{r \in \mathsf{subj}(\xi)\setminus R_1}]\!]$. Then,
$[\![\phi(G,\chi) \wedge \xi]\!] = \emptyset \cap [\![\bigwedge\{\xi^+ \restriction r\}_{r \in \mathsf{subj}(\xi)\setminus R_1}]\!]$. Then, $[\![\phi(G,\chi) \wedge \xi]\!] = \emptyset$.

– **Step:** [→1-NIf5]. Similar to case [→1-NIf4].
– **Step:** [→1-NWhile]. Similar to case [→1-While1]. □

Proof (of Lem. 91).

1. Recall $G \xrightarrow{\xi,\gamma} G'$. Then, by Defn. 31:
   – **Base:** [→1-Act], such that $G = \gamma$ and $G' = \textbf{skip}$ and $\xi = \mathtt{true}$.
   By Lem. 15, $[\![\phi(\gamma,\chi)]\!] \subseteq [\![\mathtt{true}]\!]$. Then, $[\![\phi(\gamma,\chi)]\!] \cap [\![\mathtt{true}]\!] = [\![\phi(\gamma,\chi)]\!]$. Then, by Defn. 13, $[\![\phi(\gamma,\chi) \wedge \mathtt{true}]\!] = [\![\phi(\gamma,\chi)]\!]$. Then, by Defn. 38, $[\![\phi(\gamma,\chi) \wedge \mathtt{true}]\!] = [\![\phi(\gamma, \phi(\textbf{skip},\chi))]\!]$.
   Then, $[\![\phi(G,\chi) \wedge \xi]\!] = [\![\phi(\gamma, \phi(G',\chi))]\!]$. Then, $[\![\phi(G,\chi) \wedge \xi]\!] \subseteq [\![\phi(\gamma, \phi(G',\chi))]\!]$.
   – **Base:** [→1-If1], such that $G = \hat{R}.\textbf{if } \hat{\xi}\, G_1\, G_2$ and $G' = G_1$ and $\xi = \hat{\xi}^+$ and $\gamma = 1^{\hat{R}}_{\mathsf{subj}(\hat{\xi})}$, for some $\hat{R}, G_1, G_2, \hat{\xi}$.
   Recall $[\![\phi(G,\chi) \wedge \xi]\!] \subseteq [\![\phi(G,\chi) \wedge \xi]\!]$. Then, $[\![\phi(G,\chi) \wedge \xi]\!] \subseteq [\![\phi(\hat{R}.\textbf{if } \hat{\xi}\, G_1\, G_2, \chi) \wedge \hat{\xi}^+]\!]$.
   Then, by Defn. 38, $[\![\phi(G,\chi) \wedge \xi]\!] \subseteq [\![(\hat{\xi}^+ \Rightarrow \phi(G_1,\chi)) \wedge (\hat{\xi}^- \Rightarrow \phi(G_2,\chi)) \wedge \hat{\xi}^{\equiv} \wedge \hat{\xi}^+]\!]$.
   Then, by Defn. 13, $[\![\phi(G,\chi) \wedge \xi]\!] \subseteq [\![\hat{\xi}^+ \Rightarrow \phi(G_1,\chi)]\!] \cap [\![\hat{\xi}^- \Rightarrow \phi(G_2,\chi)]\!] \cap [\![\hat{\xi}^{\equiv}]\!] \cap [\![\hat{\xi}^+]\!]$.
   Then, by Lem. 19, $[\![\phi(G,\chi) \wedge \xi]\!] \subseteq [\![\phi(G_1,\chi)]\!] \cap [\![\hat{\xi}^- \Rightarrow \phi(G_2,\chi)]\!] \cap [\![\hat{\xi}^{\equiv}]\!] \cap [\![\hat{\xi}^+]\!]$. Then, $[\![\phi(G,\chi) \wedge \xi]\!] \subseteq [\![\phi(G_1,\chi)]\!]$. Then, by Defn. 37, $[\![\phi(G,\chi) \wedge \xi]\!] \subseteq [\![\phi(1^{\hat{R}}_{\mathsf{subj}(\hat{\xi})}, \phi(G_1,\chi))]\!]$.
   Then, $[\![\phi(G,\chi) \wedge \xi]\!] \subseteq [\![\phi(\gamma, \phi(G',\chi))]\!]$.
   – **Base:** [→1-If2]. Similar to case [→1-If1].

– **Base:** [→1-WHILE1], such that $G = \hat{R}.\textbf{while}\,\hat{\xi}\,\{\psi\}\,\hat{G}$ and $G' = \hat{G}\,;\hat{R}.\textbf{while}\,\hat{\xi}\,\{\psi\}\,\hat{G}$ and $\xi = \hat{\xi}^+$ and $\gamma = 1^{\hat{R}}_{\mathsf{subj}(\hat{\xi})}$, for some $\hat{R}, \hat{G}, \hat{\xi}, \psi$.
  By Defn. 13:
  - **Case:** $[\![\forall(\psi \Rightarrow ((\hat{\xi}^+ \Rightarrow \phi(\hat{G},\psi)) \wedge (\hat{\xi}^- \Rightarrow \chi) \wedge \hat{\xi}^{\equiv}))]\!] = [\![\mathtt{true}]\!]$.
    * Recall $[\![\phi(G,\chi) \wedge \xi]\!] \subseteq [\![\phi(G,\chi) \wedge \xi]\!]$. Then, $[\![\phi(G,\chi) \wedge \xi]\!] \subseteq [\![\phi(\hat{R}.\textbf{while}\,\hat{\xi}\,\{\psi\}\,\hat{G},\chi) \wedge \hat{\xi}^+]\!]$. Then, by Defn. 38, $[\![\phi(G,\chi) \wedge \xi]\!] \subseteq [\![\psi \wedge \forall(\psi \Rightarrow ((\hat{\xi}^+ \Rightarrow \phi(\hat{G},\psi)) \wedge (\hat{\xi}^- \Rightarrow \chi) \wedge \hat{\xi}^{\equiv})) \wedge \hat{\xi}^+]\!]$. Then, by Defn. 13, $[\![\phi(G,\chi) \wedge \xi]\!] \subseteq [\![\psi]\!] \cap [\![\forall(\psi \Rightarrow ((\hat{\xi}^+ \Rightarrow \phi(\hat{G},\psi)) \wedge (\hat{\xi}^- \Rightarrow \chi) \wedge \hat{\xi}^{\equiv}))]\!] \cap [\![\hat{\xi}^+]\!]$. Then, by Lem. 24, $[\![\phi(G,\chi) \wedge \xi]\!] \subseteq [\![\psi]\!] \cap [\![\psi \Rightarrow ((\hat{\xi}^+ \Rightarrow \phi(\hat{G},\psi)) \wedge (\hat{\xi}^- \Rightarrow \chi) \wedge \hat{\xi}^{\equiv})]\!] \cap [\![\hat{\xi}^+]\!]$. Then, by Lem. 19, $[\![\phi(G,\chi) \wedge \xi]\!] \subseteq [\![\psi]\!] \cap [\![(\hat{\xi}^+ \Rightarrow \phi(\hat{G},\psi)) \wedge (\hat{\xi}^- \Rightarrow \chi) \wedge \hat{\xi}^{\equiv}]\!] \cap [\![\hat{\xi}^+]\!]$. Then, by Defn. 13, $[\![\phi(G,\chi) \wedge \xi]\!] \subseteq [\![\psi]\!] \cap [\![\hat{\xi}^+ \Rightarrow \phi(\hat{G},\psi)]\!] \cap [\![\hat{\xi}^- \Rightarrow \chi]\!] \cap [\![\hat{\xi}^{\equiv}]\!] \cap [\![\hat{\xi}^+]\!]$. Then, by Lem. 19, $[\![\phi(G,\chi) \wedge \xi]\!] \subseteq [\![\psi]\!] \cap [\![\phi(\hat{G},\psi)]\!] \cap [\![\hat{\xi}^- \Rightarrow \chi]\!] \cap [\![\hat{\xi}^{\equiv}]\!] \cap [\![\hat{\xi}^+]\!]$. Then, $[\![\phi(G,\chi) \wedge \xi]\!] \subseteq [\![\phi(\hat{G},\psi)]\!]$.
    * By Lem. 15, $[\![\psi]\!] \subseteq [\![\mathtt{true}]\!]$. Then, $[\![\psi]\!] \subseteq [\![\psi]\!] \cap [\![\mathtt{true}]\!]$. Then, $[\![\psi]\!] \subseteq [\![\psi]\!] \cap [\![\forall(\psi \Rightarrow ((\hat{\xi}^+ \Rightarrow \phi(\hat{G},\psi)) \wedge (\hat{\xi}^- \Rightarrow \chi) \wedge \hat{\xi}^{\equiv}))]\!]$. Then, by Defn. 13, $[\![\psi]\!] \subseteq [\![\psi \wedge \forall(\psi \Rightarrow ((\hat{\xi}^+ \Rightarrow \phi(\hat{G},\psi)) \wedge (\hat{\xi}^- \Rightarrow \chi) \wedge \hat{\xi}^{\equiv}))]\!]$. Then, by Lem. 81, $[\![\phi(\hat{G},\psi)]\!] \subseteq [\![\phi(\hat{G},\psi \wedge \forall(\psi \Rightarrow ((\hat{\xi}^+ \Rightarrow \phi(\hat{G},\psi)) \wedge (\hat{\xi}^- \Rightarrow \chi) \wedge \hat{\xi}^{\equiv})))]\!]$. Then, by Defn. 38, $[\![\phi(\hat{G},\psi)]\!] \subseteq [\![\phi(\hat{G},\phi(\hat{R}.\textbf{while}\,\hat{\xi}\,\{\psi\}\,\hat{G},\chi))]\!]$. Then, by Defn. 38, $[\![\phi(\hat{G},\psi)]\!] \subseteq [\![\phi(\hat{G}\,;\hat{R}.\textbf{while}\,\hat{\xi}\,\{\psi\}\,\hat{G},\chi)]\!]$. Then, by Defn. 37, $[\![\phi(\hat{G},\psi)]\!] \subseteq [\![\phi(1^{\hat{R}}_{\mathsf{subj}(\hat{\xi})},\phi(\hat{G}\,;\hat{R}.\textbf{while}\,\hat{\xi}\,\{\psi\}\,\hat{G},\chi))]\!]$. Then, $[\![\phi(\hat{G},\psi)]\!] \subseteq [\![\phi(\gamma,\phi(G',\chi))]\!]$.
    * Recall $[\![\phi(G,\chi) \wedge \xi]\!] \subseteq [\![\phi(\hat{G},\psi)]\!] \subseteq [\![\phi(\gamma,\phi(G',\chi))]\!]$. Then, $[\![\phi(G,\chi) \wedge \xi]\!] \subseteq [\![\phi(\gamma,\phi(G',\chi))]\!]$.
  - **Case:** $[\![\forall(\psi \Rightarrow ((\hat{\xi}^+ \Rightarrow \phi(\hat{G},\psi)) \wedge (\hat{\xi}^- \Rightarrow \chi) \wedge \hat{\xi}^{\equiv}))]\!] = [\![\mathtt{false}]\!]$.
    Recall $\emptyset \subseteq [\![\phi(\gamma,\phi(G',\chi))]\!]$. Then, by Lem. 15, $[\![\mathtt{false}]\!] \subseteq [\![\phi(\gamma,\phi(G',\chi))]\!]$. Then, $[\![\forall(\psi \Rightarrow ((\hat{\xi}^+ \Rightarrow \phi(\hat{G},\psi)) \wedge (\hat{\xi}^- \Rightarrow \chi) \wedge \hat{\xi}^{\equiv}))]\!] \subseteq [\![\phi(\gamma,\phi(G',\chi))]\!]$. Then, $[\![\psi]\!] \cap [\![\forall(\psi \Rightarrow ((\hat{\xi}^+ \Rightarrow \phi(\hat{G},\psi)) \wedge (\hat{\xi}^- \Rightarrow \chi) \wedge \hat{\xi}^{\equiv}))]\!] \cap [\![\xi]\!] \subseteq [\![\phi(\gamma,\phi(G',\chi))]\!]$. Then, by Defn. 13, $[\![\psi \wedge \forall(\psi \Rightarrow ((\hat{\xi}^+ \Rightarrow \phi(\hat{G},\psi)) \wedge (\hat{\xi}^- \Rightarrow \chi) \wedge \hat{\xi}^{\equiv})) \wedge \xi]\!] \subseteq [\![\phi(\gamma,\phi(G',\chi))]\!]$. Then, by Defn. 38, $[\![\phi(G,\chi) \wedge \xi]\!] \subseteq [\![\phi(\gamma,\phi(G',\chi))]\!]$.

– **Base:** [→1-WHILE2], such that $G = \hat{R}.\textbf{while}\,\hat{\xi}\,\{\psi\}\,\hat{G}$ and $G' = \textbf{skip}$ and $\xi = \hat{\xi}^-$ and $\gamma = 2^{\hat{R}}_{\mathsf{subj}(\hat{\xi})}$, for some $\hat{R}, \hat{G}, \hat{\psi}, \xi$.
  Recall $[\![\phi(G,\chi) \wedge \xi]\!] \subseteq [\![\phi(G,\chi) \wedge \xi]\!]$. Then, $[\![\phi(G,\chi) \wedge \xi]\!] \subseteq [\![\phi(\hat{R}.\textbf{while}\,\hat{\xi}\,\{\psi\}\,\hat{G},\chi) \wedge \hat{\xi}^+]\!]$. Then, by Defn. 38, $[\![\phi(G,\chi) \wedge \xi]\!] \subseteq [\![\psi \wedge \forall(\psi \Rightarrow ((\hat{\xi}^+ \Rightarrow \phi(\hat{G},\psi)) \wedge (\hat{\xi}^- \Rightarrow \chi) \wedge \hat{\xi}^{\equiv})) \wedge \hat{\xi}^-]\!]$. Then, by Defn. 13, $[\![\phi(G,\chi) \wedge \xi]\!] \subseteq [\![\psi]\!] \cap [\![\forall(\psi \Rightarrow ((\hat{\xi}^+ \Rightarrow \phi(\hat{G},\psi)) \wedge (\hat{\xi}^- \Rightarrow \chi) \wedge \hat{\xi}^{\equiv}))]\!] \cap [\![\hat{\xi}^-]\!]$. Then, by Lem. 24, $[\![\phi(G,\chi) \wedge \xi]\!] \subseteq [\![\psi]\!] \cap [\![\psi \Rightarrow ((\hat{\xi}^+ \Rightarrow \phi(\hat{G},\psi)) \wedge (\hat{\xi}^- \Rightarrow \chi) \wedge \hat{\xi}^{\equiv})]\!] \cap [\![\hat{\xi}^-]\!]$. Then, by Lem. 19, $[\![\phi(G,\chi) \wedge \xi]\!] \subseteq [\![\psi]\!] \cap [\![(\hat{\xi}^+ \Rightarrow \phi(\hat{G},\psi)) \wedge (\hat{\xi}^- \Rightarrow \chi) \wedge \hat{\xi}^{\equiv}]\!] \cap [\![\hat{\xi}^-]\!]$. Then, by Defn. 13, $[\![\phi(G,\chi) \wedge \xi]\!] \subseteq [\![\psi]\!] \cap [\![\hat{\xi}^+ \Rightarrow \phi(\hat{G},\psi)]\!] \cap [\![\hat{\xi}^- \Rightarrow \chi]\!] \cap [\![\hat{\xi}^{\equiv}]\!] \cap [\![\hat{\xi}^-]\!]$. Then, by Lem. 19, $[\![\phi(G,\chi) \wedge \xi]\!] \subseteq [\![\psi]\!] \cap [\![\hat{\xi}^+ \Rightarrow \phi(\hat{G},\psi)]\!] \cap [\![\chi]\!] \cap [\![\hat{\xi}^{\equiv}]\!] \cap [\![\hat{\xi}^-]\!]$. Then, $[\![\phi(G,\chi) \wedge \xi]\!] \subseteq [\![\chi]\!]$. Then, by Defn. 38, $[\![\phi(G,\chi) \wedge \xi]\!] \subseteq [\![\phi(\textbf{skip},\chi)]\!]$. Then, by Defn. 37, $[\![\phi(G,\chi) \wedge \xi]\!] \subseteq [\![\phi(2^{\hat{R}}_{\mathsf{subj}(\hat{\xi})},\phi(\textbf{skip},\chi))]\!]$. Then, $[\![\phi(G,\chi) \wedge \xi]\!] \subseteq [\![\phi(\gamma,\phi(G',\chi))]\!]$.

– **Base:** [→1-NIF1], such that $0 > 0$. Then, **false**.

– **Base:** [→1-NIF2], such that $G = \textbf{if}\,\hat{\xi}|_0\,G_1|_{R_1}\,G_2|_{R_2}$ and $G' = \textbf{if}\,\hat{\xi}|_0\,G_1|_{R_1 \cup \{\hat{r}\}}\,G_2|_{R_2}$ and $\xi = \hat{\xi}^+ \upharpoonright \hat{r}$ and $\gamma = 1\{^{\hat{r}}_{\hat{r}}\}$ and $\hat{r} \in \mathsf{subj}(\hat{\xi}) \setminus (R_1 \cup R_2)$, for some $R_1, R_2, G_1, G_2, \hat{\xi}, \hat{r}$.
  Recall:
  - **Case:** $R_1 = \emptyset = R_2$.
    * Recall $R_1 = \emptyset = R_2$. Then, By Defn. 38, $\phi(\textbf{if}\,\hat{\xi}|_0\,G_1|_{R_1}\,G_2|_{R_2},\chi) = (\hat{\xi}^+ \Rightarrow \phi(G_1,\chi)) \wedge (\hat{\xi}^- \Rightarrow \phi(G_2,\chi)) \wedge \hat{\xi}^{\equiv}$.
    * Recall $\{\hat{r}\} \neq \emptyset = R_2$. Then, by Defn. 38, $\phi(\textbf{if}\,\hat{\xi}|_0\,G_1|_{\{\hat{r}\}}\,G_2|_{R_2},\chi) = \phi(G_1,\chi) \wedge \bigwedge\{\hat{\xi}^+ \upharpoonright r\}_{r \in \mathsf{subj}(\hat{\xi}) \setminus \{\hat{r}\}}$.
    * Recall $\hat{r} \in \mathsf{subj}(\hat{\xi}) \setminus (R_1 \cup R_2)$. Then, $\hat{r} \in \mathsf{subj}(\hat{\xi})$. Then, by 28, $[\![\hat{\xi}^{\equiv}]\!] \cap [\![\hat{\xi}^+ \upharpoonright \hat{r}]\!] \subseteq [\![\hat{\xi}^+]\!]$ and $[\![\hat{\xi}^{\equiv}]\!] \cap [\![\hat{\xi}^+ \upharpoonright \hat{r}]\!] \subseteq [\![\bigwedge\{\hat{\xi}^+ \upharpoonright r\}_{r \in \mathsf{subj}(\hat{\xi}) \setminus \{\hat{r}\}}]\!]$. Then,

$[\![\hat{\xi}^{\equiv}]\!] \cap [\![\hat{\xi}^{+} \upharpoonright \hat{r}]\!] \subseteq [\![\hat{\xi}^{+}]\!] \cap [\![\bigwedge\{\hat{\xi}^{+} \upharpoonright r\}_{r \in \mathsf{subj}(\hat{\xi}) \setminus \{\hat{r}\}}]\!]$. Then, $[\![\hat{\xi}^{+} \Rightarrow \phi(G_1, \chi)]\!] \cap [\![\hat{\xi}^{-} \Rightarrow \phi(G_2, \chi)]\!] \cap [\![\hat{\xi}^{\equiv}]\!] \cap [\![\hat{\xi}^{+} \upharpoonright \hat{r}]\!] \subseteq [\![\hat{\xi}^{+} \Rightarrow \phi(G_1, \chi)]\!] \cap [\![\hat{\xi}^{+}]\!] \cap [\![\bigwedge\{\hat{\xi}^{+} \upharpoonright r\}_{r \in \mathsf{subj}(\hat{\xi}) \setminus \{\hat{r}\}}]\!]$. Then, by Lem. 19, $[\![\hat{\xi}^{+} \Rightarrow \phi(G_1, \chi)]\!] \cap [\![\hat{\xi}^{-} \Rightarrow \phi(G_2, \chi)]\!] \cap [\![\hat{\xi}^{\equiv}]\!] \cap [\![\hat{\xi}^{+} \upharpoonright \hat{r}]\!] \subseteq [\![\phi(G_1, \chi)]\!] \cap [\![\hat{\xi}^{+}]\!] \cap [\![\bigwedge\{\hat{\xi}^{+} \upharpoonright r\}_{r \in \mathsf{subj}(\hat{\xi}) \setminus \{\hat{r}\}}]\!]$. Then, $[\![\hat{\xi}^{+} \Rightarrow \phi(G_1, \chi)]\!] \cap [\![\hat{\xi}^{-} \Rightarrow \phi(G_2, \chi)]\!] \cap [\![\hat{\xi}^{\equiv}]\!] \cap [\![\hat{\xi}^{+} \upharpoonright \hat{r}]\!] \subseteq [\![\phi(G_1, \chi)]\!] \cap [\![\bigwedge\{\hat{\xi}^{+} \upharpoonright r\}_{r \in \mathsf{subj}(\hat{\xi}) \setminus \{\hat{r}\}}]\!]$. Then, by Defn. 13, $[\![(\hat{\xi}^{+} \Rightarrow \phi(G_1, \chi)) \wedge (\hat{\xi}^{-} \Rightarrow \phi(G_2, \chi)) \wedge \hat{\xi}^{\equiv} \wedge (\hat{\xi}^{+} \upharpoonright \hat{r})]\!] \subseteq [\![\phi(G_1, \chi) \wedge \bigwedge\{\hat{\xi}^{+} \upharpoonright r\}_{r \in \mathsf{subj}(\hat{\xi}) \setminus \{\hat{r}\}}]\!]$. Then, $[\![\phi(\mathbf{if}\ \hat{\xi}|_0\ G_1|_{R_1}\ G_2|_{R_2}, \chi) \wedge (\hat{\xi}^{+} \upharpoonright \hat{r})]\!] \subseteq [\![\phi(\mathbf{if}\ \hat{\xi}|_0\ G_1|_{\{\hat{r}\}}\ G_2|_{R_2}, \chi)]\!]$. Then, $[\![\phi(\mathbf{if}\ \hat{\xi}|_0\ G_1|_{R_1}\ G_2|_{R_2}, \chi) \wedge (\hat{\xi}^{+} \upharpoonright \hat{r})]\!] \subseteq [\![\phi(\mathbf{if}\ \hat{\xi}|_0\ G_1|_{\emptyset \cup \{\hat{r}\}}\ G_2|_{R_2}, \chi)]\!]$. Then, $[\![\phi(\mathbf{if}\ \hat{\xi}|_0\ G_1|_{R_1}\ G_2|_{R_2}, \chi) \wedge (\hat{\xi}^{+} \upharpoonright \hat{r})]\!] \subseteq [\![\phi(\mathbf{if}\ \hat{\xi}|_0\ G_1|_{R_1 \cup \{\hat{r}\}}\ G_2|_{R_2}, \chi)]\!]$. Then, by Defn. 37, $[\![\phi(\mathbf{if}\ \hat{\xi}|_0\ G_1|_{R_1}\ G_2|_{R_2}, \chi) \wedge (\hat{\xi}^{+} \upharpoonright \hat{r})]\!] \subseteq [\![\phi(1\{^{\hat{r}}_{\hat{r}}\}, \phi(\mathbf{if}\ \hat{\xi}|_0\ G_1|_{R_1 \cup \{\hat{r}\}}\ G_2|_{R_2}, \chi))]\!]$. Then, $[\![\phi(G, \chi) \wedge \xi]\!] \subseteq [\![\phi(\gamma, \phi(G', \chi))]\!]$.

- **Case:** $R_1 = \emptyset \neq R_2$.
  - ∗ Recall $R_1 = \emptyset \neq R_2$. Then, by Defn. 38, $\phi(\mathbf{if}\ \hat{\xi}|_0\ G_1|_{R_1}\ G_2|_{R_2}, \chi) = \phi(G_2, \chi) \wedge \bigwedge\{\hat{\xi}^{-} \upharpoonright r\}_{r \in \mathsf{subj}(\hat{\xi}) \setminus R_2}$.
  - ∗ Recall $\hat{r} \in \mathsf{subj}(\hat{\xi}) \setminus (R_1 \cup R_2)$. Then, $\hat{r} \in \mathsf{subj}(\hat{\xi}) \setminus R_2$.
  - ∗ Recall $\hat{r} \in \mathsf{subj}(\hat{\xi}) \setminus R_2 \subseteq \mathsf{subj}(\hat{\xi})$. Then, by Lem. 29, $[\![\bigwedge\{\hat{\xi}^{-} \upharpoonright r\}_{r \in \mathsf{subj}(\hat{\xi}) \setminus R_2}]\!] \cap [\![\hat{\xi}^{+} \upharpoonright \hat{r}]\!] \subseteq [\![\mathtt{false}]\!]$. Then, by Lem. 15, $[\![\bigwedge\{\hat{\xi}^{-} \upharpoonright r\}_{r \in \mathsf{subj}(\hat{\xi}) \setminus R_2}]\!] \cap [\![\hat{\xi}^{+} \upharpoonright \hat{r}]\!] \subseteq \emptyset$. Then, $[\![\phi(G_2, \chi)]\!] \cap [\![\bigwedge\{\hat{\xi}^{-} \upharpoonright r\}_{r \in \mathsf{subj}(\hat{\xi}) \setminus R_2}]\!] \cap [\![\hat{\xi}^{+} \upharpoonright \hat{r}]\!] \subseteq \emptyset$. Then, by Defn. 13, $[\![\phi(G_2, \chi) \wedge \bigwedge\{\hat{\xi}^{-} \upharpoonright r\}_{r \in \mathsf{subj}(\hat{\xi}) \setminus R_2} \wedge (\hat{\xi}^{+} \upharpoonright \hat{r})]\!] \subseteq \emptyset$. Then, $[\![\phi(\mathbf{if}\ \hat{\xi}|_0\ G_1|_{R_1}\ G_2|_{R_2}, \chi) \wedge (\hat{\xi}^{+} \upharpoonright \hat{r})]\!] \subseteq \emptyset$. Then, $[\![\phi(G, \chi) \wedge \xi]\!] \subseteq \emptyset$.
  - ∗ Recall $[\![\phi(G, \chi) \wedge \xi]\!] \subseteq \emptyset \subseteq [\![\phi(\gamma, \phi(G', \chi))]\!]$. Then, $[\![\phi(G, \chi) \wedge \xi]\!] \subseteq [\![\phi(\gamma, \phi(G', \chi))]\!]$.
- **Case:** $R_1 \neq \emptyset = R_2$.
  - ∗ Recall $R_1 \neq \emptyset = R_2$. Then, by Defn. 38, $\phi(\mathbf{if}\ \hat{\xi}|_0\ G_1|_{R_1}\ G_2|_{R_2}, \chi) = \phi(G_1, \chi) \wedge \bigwedge\{\hat{\xi}^{+} \upharpoonright r\}_{r \in \mathsf{subj}(\hat{\xi}) \setminus R_1}$.
  - ∗ Recall $R_1 \cup \{\hat{r}\} \neq \emptyset = R_2$. Then, by Defn. 38, $\phi(\mathbf{if}\ \hat{\xi}|_0\ G_1|_{R_1 \cup \{\hat{r}\}}\ G_2|_{R_2}, \chi) = \phi(G_1, \chi) \wedge \bigwedge\{\hat{\xi}^{+} \upharpoonright r\}_{r \in \mathsf{subj}(\hat{\xi}) \setminus (R_1 \cup \{\hat{r}\})}$.
  - ∗ Recall $\hat{r} \in \mathsf{subj}(\hat{\xi}) \setminus (R_1 \cup R_2)$. Then, $\hat{r} \in \mathsf{subj}(\hat{\xi}) \setminus R_1$. Then, $\mathsf{subj}(\hat{\xi}) \setminus R_1 = ((\mathsf{subj}(\hat{\xi}) \setminus R_1) \setminus \{\hat{r}\}) \cup \{\hat{r}\}$.
  - ∗ Recall $[\![\phi(G, \chi) \wedge \xi]\!] \subseteq [\![\phi(G, \chi) \wedge \xi]\!]$. Then, $[\![\phi(G, \chi) \wedge \xi]\!] \subseteq [\![\phi(\mathbf{if}\ \hat{\xi}|_0\ G_1|_{R_1}\ G_2|_{R_2}, \chi) \wedge (\hat{\xi}^{+} \upharpoonright \hat{r})]\!]$. Then, $[\![\phi(G, \chi) \wedge \xi]\!] \subseteq [\![\phi(G_1, \chi) \wedge \bigwedge\{\hat{\xi}^{+} \upharpoonright r\}_{r \in \mathsf{subj}(\hat{\xi}) \setminus R_1} \wedge (\hat{\xi}^{+} \upharpoonright \hat{r})]\!]$. Then, by Defn. 13, $[\![\phi(G, \chi) \wedge \xi]\!] \subseteq [\![\phi(G_1, \chi)]\!] \cap \bigcap\{[\![\hat{\xi}^{+} \upharpoonright r]\!]\}_{r \in \mathsf{subj}(\hat{\xi}) \setminus R_1} \cap [\![\hat{\xi}^{+} \upharpoonright \hat{r}]\!]$. Then, $[\![\phi(G, \chi) \wedge \xi]\!] \subseteq [\![\phi(G_1, \chi)]\!] \cap \bigcap\{[\![\hat{\xi}^{+} \upharpoonright r]\!]\}_{r \in ((\mathsf{subj}(\hat{\xi}) \setminus R_1) \setminus \{\hat{r}\}) \cup \{\hat{r}\}} \cap [\![\hat{\xi}^{+} \upharpoonright \hat{r}]\!]$. Then, $[\![\phi(G, \chi) \wedge \xi]\!] \subseteq [\![\phi(G_1, \chi)]\!] \cap \bigcap\{[\![\hat{\xi}^{+} \upharpoonright r]\!]\}_{r \in (\mathsf{subj}(\hat{\xi}) \setminus R_1) \setminus \{\hat{r}\}} \cap [\![\hat{\xi}^{+} \upharpoonright \hat{r}]\!] \cap [\![\hat{\xi}^{+} \upharpoonright \hat{r}]\!]$. Then, $[\![\phi(G, \chi) \wedge \xi]\!] \subseteq [\![\phi(G_1, \chi)]\!] \cap \bigcap\{[\![\hat{\xi}^{+} \upharpoonright r]\!]\}_{r \in (\mathsf{subj}(\hat{\xi}) \setminus R_1) \setminus \{\hat{r}\}}$. Then, $[\![\phi(G, \chi) \wedge \xi]\!] \subseteq [\![\phi(G_1, \chi)]\!] \cap \bigcap\{[\![\hat{\xi}^{+} \upharpoonright r]\!]\}_{r \in \mathsf{subj}(\hat{\xi}) \setminus (R_1 \cup \{\hat{r}\})}$. Then, by Defn. 13, $[\![\phi(G, \chi) \wedge \xi]\!] \subseteq [\![\phi(G_1, \chi) \wedge \bigwedge\{\hat{\xi}^{+} \upharpoonright r\}_{r \in \mathsf{subj}(\hat{\xi}) \setminus (R_1 \cup \{\hat{r}\})}]\!]$. Then, $[\![\phi(G, \chi) \wedge \xi]\!] \subseteq [\![\phi(\mathbf{if}\ \hat{\xi}|_0\ G_1|_{R_1 \cup \{\hat{r}\}}\ G_2|_{R_2}, \chi)]\!]$. Then, by Defn. 37, $[\![\phi(G, \chi) \wedge \xi]\!] \subseteq [\![\phi(1\{^{\hat{r}}_{\hat{r}}\}, \phi(\mathbf{if}\ \hat{\xi}|_0\ G_1|_{R_1 \cup \{\hat{r}\}}\ G_2|_{R_2}, \chi))]\!]$. Then, $[\![\phi(G, \chi) \wedge \xi]\!] \subseteq [\![\phi(\gamma, \phi(G', \chi))]\!]$.
- **Case:** $R_1 \neq \emptyset \neq R_2$.
  - ∗ Recall $R_1 \neq \emptyset \neq R_2$. Then, by Defn. 38, $\phi(\mathbf{if}\ \hat{\xi}|_0\ G_1|_{R_1}\ G_2|_{R_2}, \chi) = \mathtt{false}$.
  - ∗ Recall $\emptyset \subseteq [\![\phi(\gamma, \phi(G', \chi))]\!]$. Then, by Lem. 15, $[\![\mathtt{false}]\!] \subseteq [\![\phi(\gamma, \phi(G', \chi))]\!]$. Then, $[\![\phi(\mathbf{if}\ \hat{\xi}|_0\ G_1|_{R_1}\ G_2|_{R_2}, \chi)]\!] \subseteq [\![\phi(\gamma, \phi(G', \chi))]\!]$. Then, $[\![\phi(G, \chi)]\!] \subseteq [\![\phi(\gamma, \phi(G', \chi))]\!]$.

– **Base:** [→1-NIf3]. Similar to case [→1-NIf2].

– **Step:** [→1-Seq1], such that $G = G_1\,;G_2$ and $G' = G_1'\,;G_2$ and $G_1 \xrightarrow{\xi, \gamma} G_1'$, for some G_1, G_1', G_2.
- Recall $\checkmark_R(G)$. Then, $\checkmark_R(G_1\,;G_2)$. Then, by Defn. 27, $\checkmark_R(G_1)$.

- Recall $\checkmark_R(G_1)$ and $G_1 \xrightarrow{\xi,\gamma} G_1'$. Then, by induction, $[\![\phi(G_1, \phi(G_2, \chi)) \wedge \xi]\!] \subseteq [\![\phi(\gamma, \phi(G_1', \phi(G_2, \chi)))]\!]$. Then, by Defn. 38, $[\![\phi(G_1 \,;\, G_2, \chi) \wedge \xi]\!] \subseteq [\![\phi(\gamma, \phi(G_1' \,;\, G_2, \chi))]\!]$. Then, $[\![\phi(G, \chi) \wedge \xi]\!] \subseteq [\![\phi(\gamma, \phi(G', \chi))]\!]$.

– **Step:** [→1-Seq2], such that $G = G_1 \,;\, G_2$ and $G' = G_1 \,;\, G_2'$ and $\mathsf{subj}(G_1) \cap (\mathsf{subj}(\xi) \cup \mathsf{subj}(\gamma)) = \emptyset$ and $G_2 \xrightarrow{\xi,\gamma} G_2'$, for some G_1, G_2, G_2'.
Recall:
- **Case:** $[\![\phi(G_1, \phi(G_2, \chi))]\!] = \emptyset$.
Recall $\emptyset \subseteq [\![\phi(\gamma, \phi(G', \chi))]\!]$. Then, $\emptyset \cap [\![\xi]\!] \subseteq [\![\phi(\gamma, \phi(G', \chi))]\!]$. Then, $[\![\phi(G_1, \phi(G_2, \chi))]\!] \cap [\![\xi]\!] \subseteq [\![\phi(\gamma, \phi(G', \chi))]\!]$. Then, by Defn. 13, $[\![\phi(G_1, \phi(G_2, \chi)) \wedge \xi]\!] \subseteq [\![\phi(\gamma, \phi(G', \chi))]\!]$. Then, by Defn. 38, $[\![\phi(G_1 \,;\, G_2, \chi) \wedge \xi]\!] \subseteq [\![\phi(\gamma, \phi(G', \chi))]\!]$. Then, $[\![\phi(G, \chi) \wedge \xi]\!] \subseteq [\![\phi(\gamma, \phi(G', \chi))]\!]$.
- **Case:** $[\![\phi(G_1, \phi(G_2, \chi))]\!] \neq \emptyset$.
  * Recall $\checkmark_R(G)$. Then, $\checkmark_R(G_1 \,;\, G_2)$. Then, by Defn. 27, $\checkmark_R(G_1)$ and $\checkmark_R(G_2)$.
  * Recall $\checkmark_R(G_1)$ and $[\![\phi(G_1, \phi(G_2, \chi))]\!] \neq \emptyset$. Then, by Lem. 87:
    · **Case:** $\mathsf{subj}(G_1) = R$.
      · Recall $G_2 \xrightarrow{\xi,\gamma} G_2'$. Then, by Lem. 53, $\mathsf{subj}(\xi) \cup \mathsf{subj}(\gamma) \cup \mathsf{subj}(G_2') \subseteq G_2$. Then, $\mathsf{subj}(\xi) \cup \mathsf{subj}(\gamma) \subseteq G_2$.
      · Recall $\checkmark_R(G_2)$. Then, by Lem. 47, $\mathsf{subj}(G_2) \subseteq R \neq \emptyset$.
      · Recall $\mathsf{subj}(\xi) \cup \mathsf{subj}(\gamma) \subseteq G_2 \subseteq R$. Then, $\mathsf{subj}(\xi) \cup \mathsf{subj}(\gamma) \subseteq R$.
      · Recall $G_2 \xrightarrow{\xi,\gamma} G_2'$ and $\checkmark_R(G_2)$. Then, by Lem. 58, $\mathsf{subj}(\gamma) \neq \emptyset$. Then, $\mathsf{subj}(\xi) \cup \mathsf{subj}(\gamma) \neq \emptyset$.
      · Recall $\mathsf{subj}(\xi) \cup \mathsf{subj}(\gamma) \subseteq R$ and $\mathsf{subj}(\xi) \cup \mathsf{subj}(\gamma) \neq \emptyset$ and $R \neq \emptyset$. Then, $R \cap (\mathsf{subj}(\xi) \cup \mathsf{subj}(\gamma)) \neq \emptyset$. Then, $\mathsf{subj}(G_1) \cap (\mathsf{subj}(\xi) \cup \mathsf{subj}(\gamma)) \neq \emptyset$. Then, $\emptyset \neq \emptyset$. Then, **false**.
    · **Case:** $G_1 \in \underline{\mathbb{G}}$.
      · Recall $G_1 \in \underline{\mathbb{G}}$. Then, by Defn. 21, $\underline{G}_1 = G_1$, for some $\underline{G}_1$.
      · Recall $\mathsf{subj}(G_1) \cap (\mathsf{subj}(\xi) \cup \mathsf{subj}(\gamma)) = \emptyset$. Then, $\mathsf{subj}(\underline{G}_1) \cap (\mathsf{subj}(\xi) \cup \mathsf{subj}(\gamma)) = \emptyset$. Then, $\mathsf{subj}(\underline{G}_1) \cap \mathsf{subj}(\xi) = \emptyset$ and $\mathsf{subj}(\underline{G}_1) \cap \mathsf{subj}(\gamma) = \emptyset$. Then, by Lem. 41, $\underline{G}_1 \# \xi$ and $\underline{G}_1 \# \gamma$.
      · Recall $\underline{G}_1 \# \xi$. Then, by Lem. 85, $[\![\phi(\underline{G}_1, \phi(G_2, \chi)) \wedge \xi]\!] \subseteq [\![\phi(\underline{G}_1, \xi)]\!]$. Then, by Defn. 13, $[\![\phi(\underline{G}_1, \phi(G_2, \chi))]\!] \cap [\![\xi]\!] \subseteq [\![\phi(\underline{G}_1, \xi)]\!]$. Then, $[\![\phi(\underline{G}_1, \phi(G_2, \chi))]\!] \cap [\![\xi]\!] \subseteq [\![\phi(\underline{G}_1, \phi(G_2, \chi))]\!] \cap [\![\phi(\underline{G}_1, \xi)]\!]$. Then, by Defn. 13, $[\![\phi(\underline{G}_1, \phi(G_2, \chi)) \wedge \xi]\!] \subseteq [\![\phi(\underline{G}_1, \phi(G_2, \chi)) \wedge \phi(\underline{G}_1, \xi)]\!]$. Then, by Lem. 82, $[\![\phi(\underline{G}_1, \phi(G_2, \chi)) \wedge \xi]\!] \subseteq [\![\phi(\underline{G}_1, \phi(G_2, \chi) \wedge \xi)]\!]$.
      · Recall $\checkmark_R(G_2)$ and $G_2 \xrightarrow{\xi,\gamma} G_2'$. Then, by induction, $[\![\phi(G_2, \chi) \wedge \xi]\!] \subseteq [\![\phi(\gamma, \phi(G_2', \chi))]\!]$. Then, by Lem. 81, $[\![\phi(\underline{G}_1, \phi(G_2, \chi) \wedge \xi)]\!] \subseteq [\![\phi(\underline{G}_1, \phi(\gamma, \phi(G_2', \chi)))]\!]$.
      · Recall $\underline{G}_1 \# \gamma$. Then, by Lem. 86, $[\![\phi(\underline{G}_1, \phi(\gamma, \phi(G_2', \chi)))]\!] = [\![\phi(\gamma, \phi(\underline{G}_1, \phi(G_2', \chi)))]\!]$.
      · Recall $[\![\phi(\underline{G}_1, \phi(G_2, \chi)) \wedge \xi]\!] \subseteq [\![\phi(\underline{G}_1, \phi(G_2, \chi) \wedge \xi)]\!] \subseteq [\![\phi(\underline{G}_1, \phi(\gamma, \phi(G_2', \chi)))]\!]$. Then, $[\![\phi(\underline{G}_1, \phi(G_2, \chi)) \wedge \xi]\!] \subseteq [\![\phi(\underline{G}_1, \phi(\gamma, \phi(G_2', \chi)))]\!]$. Then, $[\![\phi(\underline{G}_1, \phi(G_2, \chi)) \wedge \xi]\!] \subseteq [\![\phi(\gamma, \phi(\underline{G}_1, \phi(G_2', \chi)))]\!]$. Then, by Defn. 38, $[\![\phi(\underline{G}_1 \,;\, G_2, \chi) \wedge \xi]\!] \subseteq [\![\phi(\gamma, \phi(\underline{G}_1 \,;\, G_2', \chi))]\!]$. Then, $[\![\phi(G_1 \,;\, G_2, \chi) \wedge \xi]\!] \subseteq [\![\phi(\gamma, \phi(G_1 \,;\, G_2', \chi))]\!]$. Then, $[\![\phi(G, \chi) \wedge \xi]\!] \subseteq [\![\phi(\gamma, \phi(G', \chi))]\!]$.

– **Step:** [→1-Par1], such that $G = G_1 \parallel G_2$ and $G' = G_1' \parallel G_2$ and $G_1 \xrightarrow{\xi,\gamma} G_1'$, for some G_1, G_1', G_2.
Recall:
- **Case:** $G_1, G_2 \in \underline{\mathbb{G}}$ and $G_1 \# G_2$.
  * Recall $G_1, G_2 \in \underline{\mathbb{G}}$. Then, by Defn. 21, $G_1 \parallel G_2 \in \underline{\mathbb{G}}$. Then, $G \in \underline{\mathbb{G}}$.
  * Recall $G \in \underline{\mathbb{G}}$ and $\checkmark_R(G)$ and $G \xrightarrow{\xi,\gamma} G'$. Then, by Lem. 90:
    · **Case:** $G' \in \underline{\mathbb{G}}$.
      · Recall $\checkmark_R(G)$. Then, $\checkmark_R(G_1 \parallel G_2)$. Then, by Defn. 27, $\checkmark_R(G_1)$.

- $\cdot$ Recall $G_1, G_2 \in \underline{\mathbb{G}}$ and $G_1 \# G_2$. Then, by Defn. 38, $\phi(G_1 \parallel G_2, \chi) = \phi(G_1, \phi(G_2, \chi))$.
- $\cdot$ Recall $G' \in \underline{\mathbb{G}}$. Then, $G'_1 \parallel G_2 \in \underline{\mathbb{G}}$. Then, by Defn. 21, $G'_1 \in \underline{\mathbb{G}}$.
- $\cdot$ Recall $G_1 \# G_2$ and $G_1 \xrightarrow{\xi,\gamma} G'_1$. Then, by Lem. 52, $G'_1 \# G_2$.
- $\cdot$ Recall $G'_1, G_2 \in \underline{\mathbb{G}}$ and $G'_1 \# G_2$. Then, by Defn. 38, $\phi(G'_1 \parallel G_2, \chi) = \phi(G'_1, \phi(G_2, \chi))$.
- $\cdot$ Recall $\checkmark_R(G_1)$ and $G_1 \xrightarrow{\xi,\gamma} G'_1$. Then, by induction, $[\![\phi(G_1, \phi(G_2, \chi)) \wedge \xi]\!] \subseteq [\![\phi(\gamma, \phi(G'_1, \phi(G_2, \chi)))]\!]$. Then, $[\![\phi(G_1 \parallel G_2, \chi) \wedge \xi]\!] \subseteq [\![\phi(\gamma, \phi(G'_1 \parallel G_2, \chi))]\!]$. Then, $[\![\phi(G, \chi) \wedge \xi]\!] \subseteq [\![\phi(\gamma, \phi(G', \chi))]\!]$.
- $\cdot$ **Case:** $[\![\phi(G, \chi) \wedge \xi]\!] = \emptyset$.
  Recall $\emptyset \subseteq [\![\phi(\gamma, \phi(G', \chi))]\!]$. Then, $[\![\phi(G, \chi) \wedge \xi]\!] \subseteq [\![\phi(\gamma, \phi(G', \chi))]\!]$.

- **Case:** Not $G_1, G_2 \in \underline{\mathbb{G}}$, or not $G_1 \# G_2$.
  - $*$ Recall $G_1, G_2 \in \underline{\mathbb{G}}$ and $G_1 \# G_2$. Then, by Defn. 38, $\phi(G_1 \parallel G_2, \chi) = \texttt{false}$.
  - $*$ Recall $\emptyset \subseteq [\![\phi(\gamma, \phi(G', \chi))]\!]$. Then, by Lem. 15, $[\![\texttt{false}]\!] \subseteq [\![\phi(\gamma, \phi(G', \chi))]\!]$. Then, $[\![\phi(G_1 \parallel G_2, \chi)]\!] \subseteq [\![\phi(\gamma, \phi(G', \chi))]\!]$. Then, $[\![\phi(G, \chi)]\!] \subseteq [\![\phi(\gamma, \phi(G', \chi))]\!]$.

– **Step:** [$\to$1-PAR2], such that $G = G_1 \parallel G_2$ and $G' = G_1 \parallel G'_2$ and $G_2 \xrightarrow{\xi,\gamma} G'_2$, for some G_1, G_2, G'_2.
Recall:

- **Case:** $G_1, G_2 \in \underline{\mathbb{G}}$ and $G_1 \# G_2$.
  - $*$ Recall $G_1, G_2 \in \underline{\mathbb{G}}$. Then, by Defn. 21, $G_1 \parallel G_2 \in \underline{\mathbb{G}}$. Then, $G \in \underline{\mathbb{G}}$.
  - $*$ Recall $G \in \underline{\mathbb{G}}$ and $\checkmark_R(G)$ and $G \xrightarrow{\xi,\gamma} G'$. Then, by Lem. 90:
    - $\cdot$ **Case:** $G' \in \underline{\mathbb{G}}$.
      - $\cdot$ Recall $\checkmark_R(G)$. Then, $\checkmark_R(G_1 \parallel G_2)$. Then, by Defn. 27, $\checkmark_R(G_2)$.
      - $\cdot$ Recall $G_1 \# G_2$ and $G_2 \xrightarrow{\xi,\gamma} G'_2$. Then, by Lem. 52, $G_1 \# \xi$ and $G_1 \# \gamma$ and $G_1 \# G'_2$.
      - $\cdot$ Recall $G_1 \# \xi$ and $G_1 \in \underline{\mathbb{G}}$. Then, by Lem. 85, $[\![\phi(G_1, \phi(G_2, \chi)) \wedge \xi]\!] \subseteq [\![\phi(G_1, \xi)]\!]$. Then, by Defn. 13, $[\![\phi(G_1, \phi(G_2, \chi))]\!] \cap [\![\xi]\!] \subseteq [\![\phi(G_1, \xi)]\!]$. Then, $[\![\phi(G_1, \phi(G_2, \chi))]\!] \cap [\![\xi]\!] \subseteq [\![\phi(G_1, \phi(G_2, \chi))]\!] \cap [\![\phi(G_1, \xi)]\!]$. Then, by Defn. 13, $[\![\phi(G_1, \phi(G_2, \chi)) \wedge \xi]\!] \subseteq [\![\phi(G_1, \phi(G_2, \chi)) \wedge \phi(G_1, \xi)]\!]$. Then, by Lem. 82, $[\![\phi(G_1, \phi(G_2, \chi)) \wedge \xi]\!] \subseteq [\![\phi(G_1, \phi(G_2, \chi) \wedge \xi)]\!]$.
      - $\cdot$ Recall $\checkmark_R(G_2)$ and $G_2 \xrightarrow{\xi,\gamma} G'_2$. Then, by induction, $[\![\phi(G_2, \chi) \wedge \xi]\!] \subseteq [\![\phi(\gamma, \phi(G'_2, \chi))]\!]$. Then, by Lem. 81, $[\![\phi(G_1, \phi(G_2, \chi) \wedge \xi)]\!] \subseteq [\![\phi(G_1, \phi(\gamma, \phi(G'_2, \chi)))]\!]$.
      - $\cdot$ Recall $G_1 \# \gamma$ and $G_1 \in \underline{\mathbb{G}}$. Then, by Lem. 86, $[\![\phi(G_1, \phi(\gamma, \phi(G'_2, \chi)))]\!] = [\![\phi(\gamma, \phi(G_1, \phi(G'_2, \chi)))]\!]$.
      - $\cdot$ Recall $G_1, G_2 \in \underline{\mathbb{G}}$ and $G_1 \# G_2$. Then, by Defn. 38, $\phi(G_1 \parallel G_2, \chi) = \phi(G_1, \phi(G_2, \chi))$.
      - $\cdot$ Recall $G' \in \underline{\mathbb{G}}$. Then, $G_1 \parallel G'_2 \in \underline{\mathbb{G}}$. Then, by Defn. 21, $G'_2 \in \underline{\mathbb{G}}$.
      - $\cdot$ Recall $G_1, G'_2 \in \underline{\mathbb{G}}$ and $G_1 \# G'_2$. Then, by Defn. 38, $\phi(G_1 \parallel G'_2, \chi) = \phi(G_1, \phi(G'_2, \chi))$.
      - $\cdot$ Recall $[\![\phi(G_1, \phi(G_2, \chi)) \wedge \xi]\!] \subseteq [\![\phi(G_1, \phi(G_2, \chi) \wedge \xi)]\!] \subseteq [\![\phi(G_1, \phi(\gamma, \phi(G'_2, \chi)))]\!]$. Then, $[\![\phi(G_1, \phi(G_2, \chi)) \wedge \xi]\!] \subseteq [\![\phi(G_1, \phi(\gamma, \phi(G'_2, \chi)))]\!]$. Then, $[\![\phi(G_1, \phi(G_2, \chi)) \wedge \xi]\!] \subseteq [\![\phi(\gamma, \phi(G_1, \phi(G'_2, \chi)))]\!]$. Then, by Defn. 38, $[\![\phi(G_1 \parallel G_2, \chi) \wedge \xi]\!] \subseteq [\![\phi(\gamma, \phi(G_1 \parallel G'_2, \chi))]\!]$. Then, $[\![\phi(G, \chi) \wedge \xi]\!] \subseteq [\![\phi(\gamma, \phi(G', \chi))]\!]$.
    - $\cdot$ **Case:** $[\![\phi(G, \chi) \wedge \xi]\!] = \emptyset$.
      Recall $\emptyset \subseteq [\![\phi(\gamma, \phi(G', \chi))]\!]$. Then, $[\![\phi(G, \chi) \wedge \xi]\!] \subseteq [\![\phi(\gamma, \phi(G', \chi))]\!]$.
- **Case:** Not $G_1, G_2 \in \underline{\mathbb{G}}$, or not $G_1 \# G_2$.
  - $*$ Recall $G_1, G_2 \in \underline{\mathbb{G}}$ and $G_1 \# G_2$. Then, by Defn. 38, $\phi(G_1 \parallel G_2, \chi) = \texttt{false}$.
  - $*$ Recall $\emptyset \subseteq [\![\phi(\gamma, \phi(G', \chi))]\!]$. Then, by Lem. 15, $[\![\texttt{false}]\!] \subseteq [\![\phi(\gamma, \phi(G', \chi))]\!]$. Then, $[\![\phi(G_1 \parallel G_2, \chi)]\!] \subseteq [\![\phi(\gamma, \phi(G', \chi))]\!]$. Then, $[\![\phi(G, \chi)]\!] \subseteq [\![\phi(\gamma, \phi(G', \chi))]\!]$.

– **Step:** [→1-NIf4], such that $G = \textbf{if}\ \hat{\xi}|_0\ G_1|_{R_1}\ G_2|_{R_2}$ and $G' = \textbf{if}\ \hat{\xi}|_0\ G_1'|_{R_1}\ G_2|_{R_2}$ and $G_1 \xrightarrow{\xi,\gamma} G_1'$ and $\mathsf{subj}(\xi) \cup \mathsf{subj}(\gamma) \subseteq R_1 \setminus R_2$, for some $R_1, R_2, G_1, G_1', G_2, \hat{\xi}$.
Recall:
  - **Case:** $R_1 = \emptyset$.
    - ∗ Recall $\mathsf{subj}(\gamma) \subseteq R_1$. Then, $\mathsf{subj}(\gamma) \subseteq \emptyset$. Then, $\mathsf{subj}(\gamma) = \emptyset$.
    - ∗ Recall $\checkmark_R(G)$. Then, $\checkmark_R(\textbf{if}\ \hat{\xi}|_0\ G_1|_{R_1}\ G_2|_{R_2})$. Then, by Defn. 27, $\checkmark_R(G_1)$.
    - ∗ Recall $G_1 \xrightarrow{\xi,\gamma} G_1'$ and $\checkmark_R(G_1)$. Then, by Lem. 58, $\mathsf{subj}(\gamma) \neq \emptyset$. Then, $\emptyset \neq \emptyset$. Then, **false**.
  - **Case:** $R_1 \neq \emptyset = R_2$.
    - ∗ Recall $\checkmark_R(G)$. Then, $\checkmark_R(\textbf{if}\ \hat{\xi}|_0\ G_1|_{R_1}\ G_2|_{R_2})$. Then, by Defn. 27, $\checkmark_R(G_1)$.
    - ∗ By Lem. 9, $\mathsf{subj}(\hat{\xi} \restriction r) = \{r\}$ for every $r \in \mathsf{subj}(\hat{\xi})$. Then, $\mathsf{subj}(\hat{\xi} \restriction r) = \{r\}$ for every $r \in \mathsf{subj}(\hat{\xi}) \setminus R_1$. Then, $\bigcup\{\mathsf{subj}(\hat{\xi} \restriction r)\}_{r\in\mathsf{subj}(\hat{\xi})\setminus R_1} = \bigcup\{\{r\}\}_{r\in\mathsf{subj}(\hat{\xi})\setminus R_1}$.
    - ∗ Recall $\mathsf{subj}(\xi) \cup \mathsf{subj}(\gamma) \subseteq R_1 \setminus R_2$. Then, $\mathsf{subj}(\xi) \cup \mathsf{subj}(\gamma) \subseteq R_1$. Then, $\mathsf{subj}(\gamma) \subseteq R_1$. Then, $\mathsf{subj}(\gamma) \cap (\mathsf{subj}(\hat{\xi}) \setminus R_1) = \emptyset$. Then, $\mathsf{subj}(\gamma) \cap \{r \mid r \in \mathsf{subj}(\hat{\xi}) \setminus R_1\} = \emptyset$. Then, $\mathsf{subj}(\gamma) \cap \bigcup\{\{r\}\}_{r\in\mathsf{subj}(\hat{\xi})\setminus R_1} = \emptyset$. Then, $\mathsf{subj}(\gamma) \cap \bigcup\{\mathsf{subj}(\hat{\xi} \restriction r)\}_{r\in\mathsf{subj}(\hat{\xi})\setminus R_1} = \emptyset$. Then, by Lem. 13, $\mathsf{subj}(\gamma) \cap \bigcup\{\mathsf{subj}((\hat{\xi} \restriction r)^+)\}_{r\in\mathsf{subj}(\hat{\xi})\setminus R_1} = \emptyset$. Then, by Lem. 11, $\mathsf{subj}(\gamma) \cap \bigcup\{\mathsf{subj}(\hat{\xi}^+ \restriction r)\}_{r\in\mathsf{subj}(\hat{\xi})\setminus R_1} = \emptyset$. Then, by Defn. 10, $\mathsf{subj}(\gamma) \cap \mathsf{subj}(\bigwedge\{\hat{\xi}^+ \restriction r\}_{r\in\mathsf{subj}(\hat{\xi})\setminus R_1}) = \emptyset$. Then, by Lem. 41, $\gamma \mathrel{\#} \bigwedge\{\hat{\xi}^+ \restriction r\}_{r\in\mathsf{subj}(\hat{\xi})\setminus R_1}$. Then, by Defn. 23, $\mathsf{write}(\gamma) \cap \mathsf{read}(\bigwedge\{\hat{\xi}^+ \restriction r\}_{r\in\mathsf{subj}(\hat{\xi})\setminus R_1}) = \emptyset$. Then, by Lem. 76, $\phi(\gamma, \bigwedge\{\hat{\xi}^+ \restriction r\}_{r\in\mathsf{subj}(\hat{\xi})\setminus R_1}) = \bigwedge\{\hat{\xi}^+ \restriction r\}_{r\in\mathsf{subj}(\hat{\xi})\setminus R_1}$.
    - ∗ Recall $R_1 \neq \emptyset = R_2$. Then, by Defn. 38, $\phi(\textbf{if}\ \hat{\xi}|_0\ G_1|_{R_1}\ G_2|_{R_2}, \chi) = \phi(G_1, \chi) \wedge \bigwedge\{\hat{\xi}^+ \restriction r\}_{r\in\mathsf{subj}(\hat{\xi})\setminus R_1}$ and $\phi(\textbf{if}\ \hat{\xi}|_0\ G_1'|_{R_1}\ G_2|_{R_2}, \chi) = \phi(G_1', \chi) \wedge \bigwedge\{\hat{\xi}^+ \restriction r\}_{r\in\mathsf{subj}(\hat{\xi})\setminus R_1}$.
    - ∗ Recall $\checkmark_R(G_1)$ and $G_1 \xrightarrow{\xi,\gamma} G_1'$. Then, by induction, $[\![\phi(G_1, \chi) \wedge \xi]\!] \subseteq [\![\phi(\gamma, \phi(G_1', \chi))]\!]$. Then, $[\![\phi(G_1, \chi) \wedge \xi]\!] \cap [\![\bigwedge\{\hat{\xi}^+ \restriction r\}_{r\in\mathsf{subj}(\hat{\xi})\setminus R_1}]\!] \subseteq [\![\phi(\gamma, \phi(G_1', \chi))]\!] \cap [\![\bigwedge\{\hat{\xi}^+ \restriction r\}_{r\in\mathsf{subj}(\hat{\xi})\setminus R_1}]\!]$. Then, $[\![\phi(G_1, \chi) \wedge \xi]\!] \cap [\![\bigwedge\{\hat{\xi}^+ \restriction r\}_{r\in\mathsf{subj}(\hat{\xi})\setminus R_1}]\!] \subseteq [\![\phi(\gamma, \phi(G_1', \chi))]\!] \cap [\![\phi(\gamma, \bigwedge\{\hat{\xi}^+ \restriction r\}_{r\in\mathsf{subj}(\hat{\xi})\setminus R_1})]\!]$. Then, by Defn. 13, $[\![\phi(G_1, \chi) \wedge \xi \wedge \bigwedge\{\hat{\xi}^+ \restriction r\}_{r\in\mathsf{subj}(\hat{\xi})\setminus R_1}]\!] \subseteq [\![\phi(\gamma, \phi(G_1', \chi)) \wedge \phi(\gamma, \bigwedge\{\hat{\xi}^+ \restriction r\}_{r\in\mathsf{subj}(\hat{\xi})\setminus R_1})]\!]$. Then, by Lem. 82, $[\![\phi(G_1, \chi) \wedge \xi \wedge \bigwedge\{\hat{\xi}^+ \restriction r\}_{r\in\mathsf{subj}(\hat{\xi})\setminus R_1}]\!] \subseteq [\![\phi(\gamma, \phi(G_1', \chi) \wedge \bigwedge\{\hat{\xi}^+ \restriction r\}_{r\in\mathsf{subj}(\hat{\xi})\setminus R_1})]\!]$. Then, $[\![\phi(\textbf{if}\ \hat{\xi}|_0\ G_1|_{R_1}\ G_2|_{R_2}, \chi) \wedge \xi]\!] \subseteq [\![\phi(\gamma, \phi(\textbf{if}\ \hat{\xi}|_0\ G_1'|_{R_1}\ G_2|_{R_2}, \chi))]\!]$. Then, $[\![\phi(G, \chi) \wedge \xi]\!] \subseteq [\![\phi(\gamma, \phi(G', \chi))]\!]$.
  - **Case:** $R_1 \neq \emptyset \neq R_2$.
    - ∗ Recall $R_1 \neq \emptyset \neq R_2$. Then, by Defn. 38, $\phi(\textbf{if}\ \hat{\xi}|_0\ G_1|_{R_1}\ G_2|_{R_2}, \chi) = \texttt{false}$.
    - ∗ Recall $\emptyset \subseteq [\![\phi(\gamma, \phi(G', \chi))]\!]$. Then, by Lem. 15, $[\![\texttt{false}]\!] \subseteq [\![\phi(\gamma, \phi(G', \chi))]\!]$. Then, $[\![\phi(\textbf{if}\ \hat{\xi}|_0\ G_1|_{R_1}\ G_2|_{R_2}, \chi)]\!] \subseteq [\![\phi(\gamma, \phi(G', \chi))]\!]$. Then, $[\![\phi(G, \chi)]\!] \subseteq [\![\phi(\gamma, \phi(G', \chi))]\!]$.

– **Step:** [→1-NIf5]. Similar to case [→1-NIf4].

– **Step:** [→1-NWhile], such that $G = \textbf{while}\ \hat{\xi}|_0\ \{\psi\}\ \hat{G}|_\emptyset$ and $\textbf{if}\ \hat{\xi}|_0\ (\hat{G}\,;\textbf{while}\ \hat{\xi}|_0\ \{\psi\}\ \hat{G}|_\emptyset)|_\emptyset\ \textbf{skip}|_\emptyset \xrightarrow{\xi,\gamma} G'$, for some $\hat{G}, \psi, \hat{\xi}$.
  - By Lem. 80, $[\![\phi(\textbf{while}\ \hat{\xi}|_0\ \{\psi\}\ \hat{G}|_\emptyset, \chi)]\!] \subseteq [\![\phi(\textbf{if}\ \hat{\xi}|_0\ (\hat{G}\,;\textbf{while}\ \hat{\xi}|_0\ \{\psi\}\ \hat{G}|_\emptyset)|_\emptyset\ \textbf{skip}|_\emptyset, \chi)]\!]$. Then, $[\![\phi(G, \chi)]\!] \subseteq [\![\phi(\textbf{if}\ \hat{\xi}|_0\ (\hat{G}\,;\textbf{while}\ \hat{\xi}|_0\ \{\psi\}\ \hat{G}|_\emptyset)|_\emptyset\ \textbf{skip}|_\emptyset, \chi)]\!]$. Then, $[\![\phi(G, \chi)]\!] \cap [\![\xi]\!] \subseteq [\![\phi(\textbf{if}\ \hat{\xi}|_0\ (\hat{G}\,;\textbf{while}\ \hat{\xi}|_0\ \{\psi\}\ \hat{G}|_\emptyset)|_\emptyset\ \textbf{skip}|_\emptyset, \chi)]\!] \cap [\![\xi]\!]$. Then, by Defn. 13, $[\![\phi(G, \chi) \wedge \xi]\!] \subseteq [\![\phi(\textbf{if}\ \hat{\xi}|_0\ (\hat{G}\,;\textbf{while}\ \hat{\xi}|_0\ \{\psi\}\ \hat{G}|_\emptyset)|_\emptyset\ \textbf{skip}|_\emptyset, \chi) \wedge \xi]\!]$.
  - Recall $\checkmark_R(G)$. Then, $\checkmark_R(\textbf{while}\ \hat{\xi}|_0\ \{\psi\}\ \hat{G}|_\emptyset)$. Then, by Lem. 46, $\checkmark_R(\textbf{if}\ \hat{\xi}|_0\ (\hat{G}\,;\textbf{while}\ \hat{\xi}|_0\ \{\psi\}\ \hat{G}|_\emptyset)|_\emptyset\ \textbf{skip}|_\emptyset)$.
  - Recall $\checkmark_R(\textbf{if}\ \hat{\xi}|_0\ (\hat{G}\,;\textbf{while}\ \hat{\xi}|_0\ \{\psi\}\ \hat{G}|_\emptyset)|_\emptyset\ \textbf{skip}|_\emptyset)$ and $\textbf{if}\ \hat{\xi}|_0\ (\hat{G}\,;\textbf{while}\ \hat{\xi}|_0\ \{\psi\}\ \hat{G}|_\emptyset)|_\emptyset\ \textbf{skip}|_\emptyset \xrightarrow{\xi,\gamma} G'$. Then, by induction, $[\![\phi(\textbf{if}\ \hat{\xi}|_0\ (\hat{G}\,;\textbf{while}\ \hat{\xi}|_0\ \{\psi\}\ \hat{G}|_\emptyset)|_\emptyset\ \textbf{skip}|_\emptyset, \chi) \wedge \xi]\!] \subseteq [\![\phi(\gamma, \phi(G', \chi))]\!]$.

- Recall $[\![\phi(G,\chi) \wedge \xi]\!] \subseteq [\![\phi(\mathbf{if}\ \hat{\xi}|_0\ (\hat{G}\,;\mathbf{while}\ \hat{\xi}|_0\ \{\psi\}\ \hat{G}|_\emptyset)|_\emptyset\ \mathbf{skip}|_\emptyset,\chi) \wedge \xi]\!] \subseteq [\![\phi(\gamma,\phi(G',\chi))]\!]$.
  Then, $[\![\phi(G,\chi) \wedge \xi]\!] \subseteq [\![\phi(\gamma,\phi(G',\chi))]\!]$. □

2. – Recall $\checkmark_R(\mathcal{G})$. Then, by Defn. 27, $\mathcal{G} = \{G\}$ and $\checkmark_R(G)$, for some G.
   – Recall $\mathcal{G} \xrightarrow{\xi,\gamma} \mathcal{G}'$. Then, $\{G\} \xrightarrow{\xi,\gamma} \mathcal{G}'$. Then, by Defn. 32, $\mathcal{G}' = \{G'\}$ and $G \xrightarrow{\xi,\gamma} G'$, for some G'.
   – Recall $\checkmark_R(G)$ and $G \xrightarrow{\xi,\gamma} G'$. Then, by 1, $[\![\phi(G,\chi) \wedge \xi]\!] \subseteq [\![\phi(\gamma,\phi(G',\chi))]\!]$. Then, by Defn. 38, $[\![\phi(\{G\},\chi) \wedge \xi]\!] \subseteq [\![\phi(\gamma,\phi(\{G'\},\chi))]\!]$. Then, $[\![\phi(\mathcal{G},\chi) \wedge \xi]\!] \subseteq [\![\phi(\gamma,\phi(\mathcal{G}',\chi))]\!]$. □
3. – Recall $(\mathcal{G},\mathcal{S}) \xrightarrow{\dot{\gamma}} (\mathcal{G}',\mathcal{S}')$. Then, by Defn. 33, $\mathcal{S}' = \mathsf{effect}(\dot{\gamma},\mathcal{S})$ and $\mathcal{G} \xrightarrow{\xi,\gamma} \mathcal{G}'$ and $\mathcal{S} \in [\![\xi]\!]$ and $\dot{\gamma} = \mathsf{con}_{\mathcal{S}}(\gamma)$, for some ξ,γ.
   – Recall $\mathcal{S} \in [\![\phi(\mathcal{G},\chi)]\!]$ and $\mathcal{S} \in [\![\xi]\!]$. Then, $\mathcal{S} \in [\![\phi(\mathcal{G},\chi)]\!] \cap [\![\xi]\!]$. Then, by Defn. 13, $\mathcal{S} \in [\![\phi(\mathcal{G},\chi) \wedge \xi]\!]$.
   – Recall $\checkmark_R(\mathcal{G})$ and $\mathcal{G} \xrightarrow{\xi,\gamma} \mathcal{G}'$. Then, by 2, $[\![\phi(\mathcal{G},\chi) \wedge \xi]\!] \subseteq [\![\phi(\gamma,\phi(\mathcal{G}',\chi))]\!]$.
   – Recall $\mathcal{S} \in [\![\phi(\mathcal{G},\chi) \wedge \xi]\!] \subseteq [\![\phi(\gamma,\phi(\mathcal{G}',\chi))]\!]$. Then, $\mathcal{S} \in [\![\phi(\gamma,\phi(\mathcal{G}',\chi))]\!]$. Then, by Lem. 79, $\mathsf{effect}(\mathsf{con}_{\mathcal{S}}(\gamma),\mathcal{S}) \in [\![\phi(\mathcal{G}',\chi)]\!]$. Then, $\mathsf{effect}(\dot{\gamma},\mathcal{S}) \in [\![\phi(\mathcal{G}',\chi)]\!]$. Then, $\mathcal{S}' \in [\![\phi(\mathcal{G}',\chi)]\!]$. □

Proof (of Lem. 92). Recall $(\mathcal{G},\mathcal{S}) \xrightarrow{\dot{\Gamma}}{}^* (\mathcal{G}^\dagger,\mathcal{S}^\dagger)$. Then, by Defn. 36:

– **Base:** [$\rightarrow^*$3-Base], such that $\mathcal{G} = \mathcal{G}^\dagger$ and $\mathcal{S} = \mathcal{S}^\dagger$.
  - Recall $\checkmark_R(\mathcal{G})$. Then, $\checkmark_R(\mathcal{G}^\dagger)$.
  - Recall $\mathcal{S} \in [\![\phi(\mathcal{G},\chi)]\!]$. Then, $\mathcal{S}^\dagger \in [\![\phi(\mathcal{G}^\dagger,\chi)]\!]$.

– **Step:** [$\rightarrow^*$3-Step], such that $(\mathcal{G},\mathcal{S}) \xrightarrow{\dot{\Gamma}}{}^* (\mathcal{G}^*,\mathcal{S}^*) \xrightarrow{\pi} (\mathcal{G}^\dagger,\mathcal{S}^\dagger)$ and $\pi \in \dot{\Gamma}$.
  - Recall $\checkmark_R(\mathcal{G})$ and $\mathcal{S} \in [\![\phi(\mathcal{G},\chi)]\!]$ and $(\mathcal{G},\mathcal{S}) \xrightarrow{\dot{\Gamma}}{}^* (\mathcal{G}^*,\mathcal{S}^*)$. Then, by induction, $\checkmark_R(\mathcal{G}^*)$ and $\mathcal{S}^* \in [\![\phi(\mathcal{G}^*,\chi)]\!]$.
  - Recall $\pi \in \dot{\Gamma}$. Then, $\pi = \dot{\gamma}$, for some $\dot{\gamma}$.
  - Recall $(\mathcal{G}^*,\mathcal{S}^*) \xrightarrow{\pi} (\mathcal{G}^\dagger,\mathcal{S}^\dagger)$. Then, $(\mathcal{G}^*,\mathcal{S}^*) \xrightarrow{\dot{\gamma}} (\mathcal{G}^\dagger,\mathcal{S}^\dagger)$.
  - Recall $\checkmark_R(\mathcal{G}^*)$ and $\mathcal{S}^* \in [\![\phi(\mathcal{G}^*,\chi)]\!]$ and $(\mathcal{G}^*,\mathcal{S}^*) \xrightarrow{\dot{\gamma}} (\mathcal{G}^\dagger,\mathcal{S}^\dagger)$. Then, by Lem. 89, $\checkmark_R(\mathcal{G}^\dagger)$.
  - Recall $\checkmark_R(\mathcal{G}^*)$ and $\mathcal{S}^* \in [\![\phi(\mathcal{G}^*,\chi)]\!]$ and $(\mathcal{G}^*,\mathcal{S}^*) \xrightarrow{\dot{\gamma}} (\mathcal{G}^\dagger,\mathcal{S}^\dagger)$. Then, by Lem. 91, $\mathcal{S}^\dagger \in [\![\phi(\mathcal{G}^\dagger,\chi)]\!]$. □

Proof (of Lem. 93).

1. Recall $\checkmark_R(G)$. Then, by Defn. 27:
   – **Base:** [✓1-Act1], such that $G = q.y{:=}e$, for some q, y, e.
     - By Defn. 4, $\mathsf{eval}_{\mathcal{S}}(\mathtt{true}) = \mathtt{true}$. Then, $\mathcal{S} \in \{\hat{\mathcal{S}} \mid \mathsf{eval}_{\hat{\mathcal{S}}}(\mathtt{true}) = \mathtt{true}\}$. Then, by Defn. 13, $\mathcal{S} \in [\![\mathtt{true}]\!]$. Then, $\xi = \mathtt{true}$ and $\mathcal{S} \in [\![\xi]\!]$, for some ξ.
     - By [$\rightarrow$1-Act], $q.y{:=}e \xrightarrow{\mathtt{true},q.y:=e} \mathbf{skip}$. Then, $G \xrightarrow{\xi,q.y:=e} \mathbf{skip}$. Then, $G' = \mathbf{skip}$ and $\gamma = q.y{:=}e$ and $G \xrightarrow{\xi,\gamma} G'$, for some G',γ.

   – **Base:** [✓1-Act2]. Similar to case [✓1-Act1].
   – **Base:** [✓1-Skip], such that $G = \mathbf{skip}$.
     By [↓1-Skip], $\mathbf{skip}\downarrow$. Then, $G\downarrow$.
   – **Step:** [✓1-Seq], such that $G = G_1\,;G_2$ and $\checkmark_R(G_1)$ and $\checkmark_R(G_2)$, for some G_1, G_2.
     - Recall $\mathcal{S} \in [\![\phi(G,\chi)]\!]$. Then, $\mathcal{S} \in [\![\phi(G_1\,;G_2,\chi)]\!]$. Then, by Defn. 38, $\mathcal{S} \in [\![\phi(G_1,\phi(G_2,\chi))]\!]$.
     - Recall $\checkmark_R(G_1)$ and $\mathcal{S} \in [\![\phi(G_1,\phi(G_2,\chi))]\!]$. Then, by induction:
       * **Case:** $G_1\downarrow$.
         · Recall $\checkmark_R(G_1)$ and $G_1\downarrow$. Then, by Lem. 88, $[\![\phi(G_1,\phi(G_2,\chi))]\!] \subseteq [\![\phi(G_2,\chi)]\!]$.
         · Recall $\mathcal{S} \in [\![\phi(G_1,\phi(G_2,\chi))]\!] \subseteq [\![\phi(G_2,\chi)]\!]$. Then, $\mathcal{S} \in [\![\phi(G_2,\chi)]\!]$.
         · Recall $\checkmark_R(G_2)$ and $\mathcal{S} \in [\![\phi(G_2,\chi)]\!]$. Then, by induction:
           · **Case:** $G_2\downarrow$.
             Recall $G_1\downarrow$ and $G_2\downarrow$. Then, by [↓1-Seq], $G_1\,;G_2\downarrow$. Then, $G\downarrow$.

· **Case:** $G_2 \xrightarrow{\xi,\gamma} G_2'$ and $\mathcal{S} \in [\![\xi]\!]$, for some G_2', for some ξ, γ.
  · Recall $G_1\downarrow$ and $\checkmark_R(G_1)$. Then, by Lem. 48, $\mathsf{subj}(G_1) = \emptyset$.
  · Recall $\emptyset \cap (\mathsf{subj}(\xi) \cup \mathsf{subj}(\gamma)) = \emptyset$. Then, $\mathsf{subj}(G_1) \cap (\mathsf{subj}(\xi) \cup \mathsf{subj}(\gamma)) = \emptyset$.
  · Recall $\mathsf{subj}(G_1) \cap (\mathsf{subj}(\xi) \cup \mathsf{subj}(\gamma)) = \emptyset$ and $G_2 \xrightarrow{\xi,\gamma} G_2'$. Then, by [→1-SEQ2], $G_1 \,;\, G_2 \xrightarrow{\xi,\gamma} G_1 \,;\, G_2'$. Then, $G \xrightarrow{\xi,\gamma} G_1 \,;\, G_2'$. Then, $G' = G_1 \,;\, G_2'$ and $G \xrightarrow{\xi,\gamma} G'$, for some G'.

* **Case:** $G_1 \xrightarrow{\xi,\gamma} G_1'$ and $\mathcal{S} \in [\![\xi]\!]$, for some G_1', for some ξ, γ.
  Recall $G_1 \xrightarrow{\xi,\gamma} G_1'$. Then, by [→1-SEQ1], $G_1 \,;\, G_2 \xrightarrow{\xi,\gamma} G_1' \,;\, G_2$. Then, $G \xrightarrow{\xi,\gamma} G_1' \,;\, G_2$. Then, $G' = G_1' \,;\, G_2$ and $G \xrightarrow{\xi,\gamma} G'$, for some G'.

– **Step:** [✓1-PAR], such that $G = G_1 \parallel G_2$ and $\checkmark_R(G_1)$ and $\checkmark_R(G_2)$, for some G_1, G_2.
By Defn. 38:

- **Case:** $\phi(G_1 \parallel G_2, \chi) = \phi(G_1, \phi(G_2, \chi))$.
  * Recall $\mathcal{S} \in [\![\phi(G,\chi)]\!]$. Then, $\mathcal{S} \in [\![\phi(G_1 \parallel G_2,\chi)]\!]$. Then, $\mathcal{S} \in [\![\phi(G_1, \phi(G_2,\chi))]\!]$.
  * Recall $\checkmark_R(G_1)$ and $\mathcal{S} \in [\![\phi(G_1, \phi(G_2,\chi))]\!]$. Then, by induction:
    · **Case:** $G_1\downarrow$.
      · Recall $\checkmark_R(G_1)$ and $G_1\downarrow$. Then, by Lem. 88, $[\![\phi(G_1, \phi(G_2,\chi))]\!] \subseteq [\![\phi(G_2,\chi)]\!]$.
      · Recall $\mathcal{S} \in [\![\phi(G_1, \phi(G_2,\chi))]\!] \subseteq [\![\phi(G_2,\chi)]\!]$. Then, $\mathcal{S} \in [\![\phi(G_2,\chi)]\!]$.
      · Recall $\checkmark_R(G_2)$ and $\mathcal{S} \in [\![\phi(G_2,\chi)]\!]$. Then, by induction:
        · **Case:** $G_2\downarrow$.
          Recall $G_1\downarrow$ and $G_2\downarrow$. Then, by [↓1-PAR], $G_1 \parallel G_2\downarrow$. Then, $G\downarrow$.
        · **Case:** $G_2 \xrightarrow{\xi,\gamma} G_2'$ and $\mathcal{S} \in [\![\xi]\!]$, for some G_2', for some ξ, γ.
          Recall $G_2 \xrightarrow{\xi,\gamma} G_2'$. Then, by [→1-PAR2], $G_1 \parallel G_2 \xrightarrow{\xi,\gamma} G_1 \parallel G_2'$. Then, $G \xrightarrow{\xi,\gamma} G_1 \parallel G_2'$. Then, $G' = G_1 \parallel G_2'$ and $G \xrightarrow{\xi,\gamma} G'$, for some G'.
    · **Case:** $G_1 \xrightarrow{\xi,\gamma} G_1'$ and $\mathcal{S} \in [\![\xi]\!]$, for some G_1', for some ξ, γ.
      Recall $G_1 \xrightarrow{\xi,\gamma} G_1'$. Then, by [→1-PAR1], $G_1 \parallel G_2 \xrightarrow{\xi,\gamma} G_1' \parallel G_2$. Then, $G \xrightarrow{\xi,\gamma} G_1' \parallel G_2$. Then, $G' = G_1' \parallel G_2$ and $G \xrightarrow{\xi,\gamma} G'$, for some G'.
- **Case:** $\phi(G_1 \parallel G_2, \chi) = \mathtt{false}$.
  Recall $\mathcal{S} \in [\![\phi(G,\chi)]\!]$. Then, $\mathcal{S} \in [\![\phi(G_1 \parallel G_2,\chi)]\!]$. Then, $\mathcal{S} \in [\![\mathtt{false}]\!]$. Then, by Lem. 15, $\mathcal{S} \in \emptyset$. Then, **false**.

– **Step:** [✓1-IF], such that $G = R.\mathbf{if}\ \hat{\xi}\ G_1\ G_2$, for some $G_1, G_2, \hat{\xi}$.
Recall $\mathcal{S} \in [\![\phi(G,\chi)]\!]$. Then, $\mathcal{S} \in [\![\phi(R.\mathbf{if}\ \hat{\xi}\ G_1\ G_2,\chi)]\!]$. Then, by Defn. 38, $\mathcal{S} \in [\![(\hat{\xi}^+ \Rightarrow \phi(G_1,\chi)) \wedge (\hat{\xi}^- \Rightarrow \phi(G_2,\chi)) \wedge \hat{\xi}^{\equiv}]\!]$. Then, by Defn. 13, $\mathcal{S} \in [\![\hat{\xi}^+ \Rightarrow \phi(G_1,\chi)]\!] \cap [\![\hat{\xi}^- \Rightarrow \phi(G_2,\chi)]\!] \cap [\![\hat{\xi}^{\equiv}]\!]$. Then, $\mathcal{S} \in [\![\hat{\xi}^{\equiv}]\!]$. Then, by Lem. 27:

- **Case:** $\mathcal{S} \in [\![\hat{\xi}^+]\!]$.
  * By [→1-IF1], $R.\mathbf{if}\ \hat{\xi}\ G_1\ G_2 \xrightarrow{\hat{\xi}^+, 1^R_{\mathsf{subj}(\hat{\xi})}} G_1$. Then, $G \xrightarrow{\hat{\xi}^+, 1^R_{\mathsf{subj}(\hat{\xi})}} G_1$. Then, $G' = G_1$ and $\xi = \hat{\xi}^+$ and $\gamma = 1^R_{\mathsf{subj}(\hat{\xi})}$ and $G \xrightarrow{\xi,\gamma} G'$, for some G', ξ, γ.
  * Recall $\mathcal{S} \in [\![\hat{\xi}^+]\!]$. Then, $\mathcal{S} \in [\![\xi]\!]$.
- **Case:** $\mathcal{S} \in [\![\xi^-]\!]$. Similar to case $\mathcal{S} \in [\![\xi^+]\!]$.

– **Step:** [✓1-WHILE], such that $G = R.\mathbf{while}\ \hat{\xi}\ \{\psi\}\ \hat{G}$, for some $\hat{G}, \hat{\psi}, \xi$.

- Recall $\mathcal{S} \in [\![\phi(G,\chi)]\!]$. Then, $\mathcal{S} \in [\![\phi(R.\mathbf{while}\ \hat{\xi}\ \{\psi\}\ \hat{G},\chi)]\!]$. Then, by Defn. 38, $\mathcal{S} \in [\![\psi \wedge \forall(\psi \Rightarrow ((\hat{\xi}^+ \Rightarrow \phi(\hat{G},\psi)) \wedge (\hat{\xi}^- \Rightarrow \chi) \wedge \hat{\xi}^{\equiv}))]\!]$. Then, by Defn. 13, $\mathcal{S} \in [\![\psi]\!] \cap [\![\forall(\psi \Rightarrow ((\hat{\xi}^+ \Rightarrow \phi(\hat{G},\psi)) \wedge (\hat{\xi}^- \Rightarrow \chi) \wedge \hat{\xi}^{\equiv}))]\!]$. Then, $\mathcal{S} \in [\![\psi]\!]$ and $\mathcal{S} \in [\![\forall(\psi \Rightarrow ((\hat{\xi}^+ \Rightarrow \phi(\hat{G},\psi)) \wedge (\hat{\xi}^- \Rightarrow \chi) \wedge \hat{\xi}^{\equiv}))]\!]$.
- By Lem. 24, $[\![\forall(\psi \Rightarrow ((\hat{\xi}^+ \Rightarrow \phi(\hat{G},\psi)) \wedge (\hat{\xi}^- \Rightarrow \chi) \wedge \hat{\xi}^{\equiv}))]\!] \subseteq [\![\psi \Rightarrow ((\hat{\xi}^+ \Rightarrow \phi(\hat{G},\psi)) \wedge (\hat{\xi}^- \Rightarrow \chi) \wedge \hat{\xi}^{\equiv})]\!]$.
- Recall $\mathcal{S} \in [\![\forall(\psi \Rightarrow ((\hat{\xi}^+ \Rightarrow \phi(\hat{G},\psi)) \wedge (\hat{\xi}^- \Rightarrow \chi) \wedge \hat{\xi}^{\equiv}))]\!] \subseteq [\![\psi \Rightarrow ((\hat{\xi}^+ \Rightarrow \phi(\hat{G},\psi)) \wedge (\hat{\xi}^- \Rightarrow \chi) \wedge \hat{\xi}^{\equiv})]\!]$. Then, $\mathcal{S} \in [\![\psi \Rightarrow ((\hat{\xi}^+ \Rightarrow \phi(\hat{G},\psi)) \wedge (\hat{\xi}^- \Rightarrow \chi) \wedge \hat{\xi}^{\equiv})]\!]$.
- Recall $\mathcal{S} \in [\![\psi]\!]$ and $\mathcal{S} \in [\![\psi \Rightarrow ((\hat{\xi}^+ \Rightarrow \phi(\hat{G},\psi)) \wedge (\hat{\xi}^- \Rightarrow \chi) \wedge \hat{\xi}^{\equiv})]\!]$. Then, $\mathcal{S} \in [\![\psi]\!] \cap [\![\psi \Rightarrow ((\hat{\xi}^+ \Rightarrow \phi(\hat{G},\psi)) \wedge (\hat{\xi}^- \Rightarrow \chi) \wedge \hat{\xi}^{\equiv})]\!]$. Then, by Lem. 19, $\mathcal{S} \in [\![\psi]\!] \cap [\![(\hat{\xi}^+ \Rightarrow \phi(\hat{G},\psi)) \wedge (\hat{\xi}^- \Rightarrow \chi) \wedge \hat{\xi}^{\equiv}]\!]$. Then, $\mathcal{S} \in [\![(\hat{\xi}^+ \Rightarrow \phi(\hat{G},\psi)) \wedge (\hat{\xi}^- \Rightarrow \chi) \wedge \hat{\xi}^{\equiv}]\!]$. Then, by Defn. 13, $\mathcal{S} \in [\![\hat{\xi}^+ \Rightarrow \phi(\hat{G},\psi)]\!] \cap [\![\hat{\xi}^- \Rightarrow \chi]\!] \cap [\![\hat{\xi}^{\equiv}]\!]$. Then, $\mathcal{S} \in [\![\hat{\xi}^{\equiv}]\!]$. Then, by Lem. 27:
  * **Case:** $\mathcal{S} \in [\![\hat{\xi}^+]\!]$.

· By [$\rightarrow$1-WHILE1], $R.\mathbf{while}\ \hat{\xi}\ \{\psi\}\ \hat{G} \xrightarrow{\hat{\xi}^+, 1^R_{\mathsf{subj}(\hat{\xi})}} \hat{G}\,;R.\mathbf{while}\ \hat{\xi}\ \{\psi\}\ \hat{G}$. Then, $G \xrightarrow{\hat{\xi}^+, 1^R_{\mathsf{subj}(\hat{\xi})}} \hat{G}\,;R.\mathbf{while}\ \hat{\xi}\ \{\psi\}\ \hat{G}$. Then, $G' = \hat{G}\,;R.\mathbf{while}\ \hat{\xi}\ \{\psi\}\ \hat{G}$ and $\xi = \hat{\xi}^+$ and $\gamma = 1^R_{\mathsf{subj}(\hat{\xi})}$ and $G \xrightarrow{\xi,\gamma} G'$, for some G', ξ, γ.
· Recall $\mathcal{S} \in [\![\hat{\xi}^+]\!]$. Then, $\mathcal{S} \in [\![\xi]\!]$.

∗ **Case:** $\mathcal{S} \in [\![\xi^-]\!]$. Similar to case $\mathcal{S} \in [\![\xi^+]\!]$. □

– **Step:** [✓1-NIF], such that $G = \mathbf{if}\ \hat{\xi}|_0\ G_1|_{R_1}\ G_2|_{R_2}$ and $\checkmark_R(G_1)$ and $\checkmark_R(G_2)$ and $R = \mathsf{subj}(\hat{\xi})$ and $R_1, R_2 \subseteq \mathsf{subj}(\hat{\xi})$, for some $R_1, R_2, G_1, G_2, \hat{\xi}$.
Recall:

• **Case:** $R_1 = \emptyset = R_2$.

∗ Recall $\checkmark_R(G)$. Then, by Lem. 47, $R \neq \emptyset$. Then, $\mathsf{subj}(\hat{\xi}) \neq \emptyset$. Then, $r \in \mathsf{subj}(\hat{\xi})$, for some r.

∗ Recall $R_1 = \emptyset = R_2$. Then, by Defn. 38,
$\phi(\mathbf{if}\ \hat{\xi}|_0\ G_1|_{R_1}\ G_2|_{R_2}, \chi) = (\hat{\xi}^+ \Rightarrow \phi(G_1,\chi)) \wedge (\hat{\xi}^- \Rightarrow \phi(G_2,\chi)) \wedge \hat{\xi}^\equiv$.

∗ Recall $\mathcal{S} \in [\![\phi(G,\chi)]\!]$. Then, $\mathcal{S} \in [\![\phi(\mathbf{if}\ \hat{\xi}|_0\ G_1|_{R_1}\ G_2|_{R_2}, \chi)]\!]$. Then, $\mathcal{S} \in [\![(\hat{\xi}^+ \Rightarrow \phi(G_1,\chi)) \wedge (\hat{\xi}^- \Rightarrow \phi(G_2,\chi)) \wedge \hat{\xi}^\equiv]\!]$. Then, by Defn. 13, $\mathcal{S} \in [\![\hat{\xi}^+ \Rightarrow \phi(G_1,\chi)]\!] \cap [\![\hat{\xi}^- \Rightarrow \phi(G_2,\chi)]\!] \cap [\![\hat{\xi}^\equiv]\!]$. Then, $\mathcal{S} \in [\![\hat{\xi}^\equiv]\!]$. Then, by Lem. 27:

· **Case:** $\mathcal{S} \in [\![\hat{\xi}^+ \restriction r]\!]$.

· Recall $r \in \mathsf{subj}(\hat{\xi})$. Then, $r \in \mathsf{subj}(\hat{\xi}) \setminus (\emptyset \cup \emptyset)$. Then, $r \in \mathsf{subj}(\hat{\xi}) \setminus (R_1 \cup R_2)$. Then, by [$\rightarrow$1-NIF2], $\mathbf{if}\ \hat{\xi}|_0\ G_1|_{R_1}\ G_2|_{R_2} \xrightarrow{\hat{\xi}^+ \restriction r, 1^{\{r\}}_{\{r\}}} \mathbf{if}\ \hat{\xi}|_0\ G_1|_{R_1 \cup \{r\}}\ G_2|_{R_2}$. Then, $G \xrightarrow{\hat{\xi}^+ \restriction r, 1^{\{r\}}_{\{r\}}} \mathbf{if}\ \hat{\xi}|_0\ G_1|_{R_1 \cup \{r\}}\ G_2|_{R_2}$. Then, $G' = \mathbf{if}\ \hat{\xi}|_0\ G_1|_{R_1 \cup \{r\}}\ G_2|_{R_2}$ and $\xi = \hat{\xi}^+ \restriction r$ and $\gamma = 1^{\{r\}}_{\{r\}}$ and $G \xrightarrow{\xi,\gamma} G'$, for some G', ξ, γ.
· Recall $\mathcal{S} \in [\![\hat{\xi}^+ \restriction r]\!]$. Then, $\mathcal{S} \in [\![\xi]\!]$.

· **Case:** $\mathcal{S} \in [\![\hat{\xi}^- \restriction r]\!]$. Similar to case $\mathcal{S} \in [\![\hat{\xi}^+ \restriction r]\!]$.

• **Case:** $R_1 = \emptyset \neq R_2$.

∗ Recall $R_1 = \emptyset \neq R_2$. Then, by Defn. 38,
$\phi(\mathbf{if}\ \hat{\xi}|_0\ G_1|_{R_1}\ G_2|_{R_2}, \chi) = \phi(G_2,\chi) \wedge \bigwedge\{\hat{\xi}^- \restriction r\}_{r \in \mathsf{subj}(\hat{\xi}) \setminus R_2}$.

∗ Recall $\mathcal{S} \in [\![\phi(G,\chi)]\!]$. Then, $\mathcal{S} \in [\![\phi(G_2,\chi) \wedge \bigwedge\{\hat{\xi}^- \restriction r\}_{r \in \mathsf{subj}(\hat{\xi}) \setminus R_2}]\!]$. Then, by Defn. 13, $\mathcal{S} \in [\![\phi(G_2,\chi)]\!] \cap \bigcap\{[\![\hat{\xi}^- \restriction r]\!]\}_{r \in \mathsf{subj}(\hat{\xi}) \setminus R_2}$. Then, $\mathcal{S} \in [\![\phi(G_2,\chi)]\!]$, and $\mathcal{S} \in [\![\hat{\xi}^- \restriction r]\!]$ for every $r \in \mathsf{subj}(\hat{\xi}) \setminus R_2$.

∗ Recall:

· **Case:** $R_2 = \mathsf{subj}(\hat{\xi})$.
Recall $\checkmark_R(G_2)$ and $\mathcal{S} \in [\![\phi(G_2,\chi)]\!]$. Then, by induction:

· **Case:** $G_2\downarrow$.

· Recall $R_2 = \mathsf{subj}(\hat{\xi})$. Then, $\emptyset \cup R_2 = \mathsf{subj}(\hat{\xi})$. Then, $R_1 \cup R_2 = \mathsf{subj}(\hat{\xi})$.
· Recall $R_1 = \emptyset$. Then, $R_1 \neq \emptyset$ implies $G_1\downarrow$.
· Recall $G_2\downarrow$. Then, $R_2 \neq \emptyset$ implies $G_2\downarrow$.
· Recall $R_1 \cup R_2 = \mathsf{subj}(\hat{\xi})$, and $R_1 \neq \emptyset$ implies $G_1\downarrow$, and $R_2 \neq \emptyset$ implies $G_2\downarrow$. Then, by [↓1-NIF], $\mathbf{if}\ \hat{\xi}|_0\ G_1|_{R_1}\ G_2|_{R_2}\downarrow$. Then, $G\downarrow$.

· **Case:** $G_2 \xrightarrow{\xi,\gamma} G_2'$ and $\mathcal{S} \in [\![\xi]\!]$, for some G_2', for some ξ, γ.

· Recall $G_2 \xrightarrow{\xi,\gamma} G_2'$. Then, by Lem. 53,
$\mathsf{subj}(\xi) \cup \mathsf{subj}(\gamma) \cup \mathsf{subj}(G_2') \subseteq \mathsf{subj}(G_2)$. Then, $\mathsf{subj}(\xi) \cup \mathsf{subj}(\gamma) \subseteq \mathsf{subj}(G_2)$.
· Recall $\checkmark_R(G_2)$. Then, by Lem. 47, $\mathsf{subj}(G_2) \subseteq R$.
· Recall $\mathsf{subj}(\xi) \cup \mathsf{subj}(\gamma) \subseteq \mathsf{subj}(G_2) \subseteq R$. Then, $\mathsf{subj}(\xi) \cup \mathsf{subj}(\gamma) \subseteq R$. Then, $\mathsf{subj}(\xi) \cup \mathsf{subj}(\gamma) \subseteq \mathsf{subj}(\hat{\xi})$. Then, $\mathsf{subj}(\xi) \cup \mathsf{subj}(\gamma) \subseteq R_2$. Then, $\mathsf{subj}(\xi) \cup \mathsf{subj}(\gamma) \subseteq R_2 \setminus \emptyset$. Then, $\mathsf{subj}(\xi) \cup \mathsf{subj}(\gamma) \subseteq R_2 \setminus R_1$.
· Recall $G_2 \xrightarrow{\xi,\gamma} G_2'$ and $\mathsf{subj}(\xi) \cup \mathsf{subj}(\gamma) \subseteq R_2 \setminus R_1$. Then, by [$\rightarrow$1-NIF5], $\mathbf{if}\ \hat{\xi}|_0\ G_1|_{R_1}\ G_2|_{R_2} \xrightarrow{\xi,\gamma} \mathbf{if}\ \hat{\xi}|_0\ G_1|_{R_1}\ G_2'|_{R_2}$. Then, $G \xrightarrow{\xi,\gamma} \mathbf{if}\ \hat{\xi}|_0\ G_1|_{R_1}\ G_2'|_{R_2}$. Then, $G' = \mathbf{if}\ \hat{\xi}|_0\ G_1|_{R_1}\ G_2'|_{R_2}$ and $G \xrightarrow{\xi,\gamma} G'$, for some G'.

· **Case:** $R_2 \neq \mathsf{subj}(\hat{\xi})$.

· Recall $R_2 \subseteq \mathsf{subj}(\hat{\xi})$ and $R_2 \neq \mathsf{subj}(\hat{\xi})$. Then, $\mathsf{subj}(\hat{\xi}) \setminus R_2 \neq \emptyset$. Then, $r \in \mathsf{subj}(\hat{\xi}) \setminus R_2$, for some r.

· Recall $r \in \mathsf{subj}(\hat{\xi}) \setminus R_2$. Then, $r \in \mathsf{subj}(\hat{\xi}) \setminus (\emptyset \cup R_2)$. Then, $r \in \mathsf{subj}(\hat{\xi}) \setminus (R_1 \cup R_2)$. Then, by [$\rightarrow$1-NIf3], $\textbf{if}\ \hat{\xi}|_0\ G_1|_{R_1}\ G_2|_{R_2} \xrightarrow{\hat{\xi}^- \restriction r, 2\{^r_r\}} \textbf{if}\ \hat{\xi}|_0\ G_1|_{R_1 \cup \{r\}}\ G_2|_{R_2}$. Then, $G \xrightarrow{\hat{\xi}^- \restriction r, 2\{^r_r\}} \textbf{if}\ \hat{\xi}|_0\ G_1|_{R_1 \cup \{r\}}\ G_2|_{R_2}$. Then, $G' = \textbf{if}\ \hat{\xi}|_0\ G_1|_{R_1 \cup \{r\}}\ G_2|_{R_2}$ and $\xi = \hat{\xi}^- \restriction r$ and $\gamma = 2\{^r_r\}$ and $G \xrightarrow{\xi,\gamma} G'$, for some G', ξ, γ.
· Recall $r \in \mathsf{subj}(\hat{\xi}) \setminus R_2$. Then, $\mathcal{S} \in [\![\hat{\xi}^- \restriction r]\!]$. Then, $\mathcal{S} \in [\![\xi]\!]$.

- **Case:** $R_1 \neq \emptyset = R_2$. Similar to case $R_1 = \emptyset \neq R_2$.
- **Case:** $R_1 \neq \emptyset \neq R_2$.
  * Recall $R_1 \neq \emptyset \neq R_2$. Then, by Defn. 38, $\phi(\textbf{if}\ \hat{\xi}|_0\ G_1|_{R_1}\ G_2|_{R_2}, \chi) = \mathtt{false}$.
  * Recall $\mathcal{S} \in [\![\phi(G, \chi)]\!]$. Then, $\mathcal{S} \in [\![\phi(\textbf{if}\ \hat{\xi}|_0\ G_1|_{R_1}\ G_2|_{R_2}, \chi)]\!]$. Then, $\mathcal{S} \in [\![\mathtt{false}]\!]$. Then, by Lem. 15, $\mathcal{S} \in \emptyset$. Then, **false**.

– **Step:** [✓1-NWhile], such that $G = \textbf{while}\ \hat{\xi}|_0\ \{\psi\}\ \hat{G}|_\emptyset$ and $R = \mathsf{subj}(\hat{\xi})$, for some $\hat{G}, \psi, \hat{\xi}$.

- Recall $\checkmark_R(G)$. Then, by Lem. 47, $R \neq \emptyset$. Then, $\mathsf{subj}(\hat{\xi}) \neq \emptyset$. Then, $r \in \mathsf{subj}(\hat{\xi})$, for some r.
- By Lem. 80, $[\![\phi(\textbf{while}\ \hat{\xi}|_0\ \{\psi\}\ \hat{G}|_\emptyset, \chi)]\!] \subseteq [\![\phi(\textbf{if}\ \hat{\xi}|_0\ (\hat{G}\,;\textbf{while}\ \hat{\xi}|_0\ \{\psi\}\ \hat{G}|_\emptyset)|_\emptyset\ \textbf{skip}|_\emptyset, \chi)]\!]$. Then, $[\![\phi(G, \chi)]\!] \subseteq [\![\phi(\textbf{if}\ \hat{\xi}|_0\ (\hat{G}\,;\textbf{while}\ \hat{\xi}|_0\ \{\psi\}\ \hat{G}|_\emptyset)|_\emptyset\ \textbf{skip}|_\emptyset, \chi)]\!]$. Then, by Defn. 38, $[\![\phi(G, \chi)]\!] \subseteq [\![(\hat{\xi}^+ \Rightarrow \phi(\hat{G}\,;\textbf{while}\ \hat{\xi}|_0\ \{\psi\}\ \hat{G}|_\emptyset, \chi)) \wedge (\hat{\xi}^- \Rightarrow \phi(\textbf{skip}, \chi)) \wedge \hat{\xi}^\equiv]\!]$. Then, by Defn. 13, $[\![\phi(G, \chi)]\!] \subseteq [\![\hat{\xi}^+ \Rightarrow \phi(\hat{G}\,;\textbf{while}\ \hat{\xi}|_0\ \{\psi\}\ \hat{G}|_\emptyset, \chi)]\!] \cap [\![\hat{\xi}^- \Rightarrow \phi(\textbf{skip}, \chi)]\!] \cap [\![\hat{\xi}^\equiv]\!]$. Then, $[\![\phi(G, \chi)]\!] \subseteq [\![\hat{\xi}^\equiv]\!]$.
- Recall $\mathcal{S} \in [\![\phi(G, \chi)]\!] \subseteq [\![\hat{\xi}^\equiv]\!]$. Then, $\mathcal{S} \in [\![\hat{\xi}^\equiv]\!]$. Then, by Lem. 27:
  * **Case:** $\mathcal{S} \in [\![\hat{\xi}^+ \restriction r]\!]$.
    · Recall $r \in \mathsf{subj}(\hat{\xi})$. Then, $r \in \mathsf{subj}(\hat{\xi}) \setminus (\emptyset \cup \emptyset)$. Then, by [$\rightarrow$1-NIf2], $\textbf{if}\ \hat{\xi}|_0\ (\hat{G}\,;\textbf{while}\ \hat{\xi}|_0\ \{\psi\}\ \hat{G}|_\emptyset)|_\emptyset\ \textbf{skip}|_\emptyset \xrightarrow{\hat{\xi}^+ \restriction r, 1\{^r_r\}}$ $\textbf{if}\ \hat{\xi}|_0\ (\hat{G}\,;\textbf{while}\ \hat{\xi}|_0\ \{\psi\}\ \hat{G}|_\emptyset)|_{\emptyset \cup \{r\}}\ \textbf{skip}|_\emptyset$. Then, by [$\rightarrow$1-NWhile], $\textbf{while}\ \hat{\xi}|_0\ \{\psi\}\ \hat{G}|_\emptyset \xrightarrow{\hat{\xi}^+ \restriction r, 1\{^r_r\}} \textbf{if}\ \hat{\xi}|_0\ (\hat{G}\,;\textbf{while}\ \hat{\xi}|_0\ \{\psi\}\ \hat{G}|_\emptyset)|_{\emptyset \cup \{r\}}\ \textbf{skip}|_\emptyset$. Then, $G \xrightarrow{\hat{\xi}^+ \restriction r, 1\{^r_r\}} \textbf{if}\ \hat{\xi}|_0\ (\hat{G}\,;\textbf{while}\ \hat{\xi}|_0\ \{\psi\}\ \hat{G}|_\emptyset)|_{\emptyset \cup \{r\}}\ \textbf{skip}|_\emptyset$. Then, $G' = \textbf{if}\ \hat{\xi}|_0\ (\hat{G}\,;\textbf{while}\ \hat{\xi}|_0\ \{\psi\}\ \hat{G}|_\emptyset)|_{\emptyset \cup \{r\}}\ \textbf{skip}|_\emptyset$ and $\xi = \hat{\xi}^+ \restriction r$ and $\gamma = 1\{^r_r\}$ and $G \xrightarrow{\xi,\gamma} G'$, for some G', ξ, γ.
    · Recall $\mathcal{S} \in [\![\hat{\xi}^+ \restriction r]\!]$. Then, $\mathcal{S} \in [\![\xi]\!]$.
  * **Case:** $\mathcal{S} \in [\![\hat{\xi}^- \restriction r]\!]$. Similar to case $\mathcal{S} \in [\![\hat{\xi}^+ \restriction r]\!]$. □

2. – Recall $\checkmark_R(\mathcal{G})$. Then, by Defn. 27, $\mathcal{G} = \{G\}$ and $\checkmark_R(G)$, for some G.
   – Recall $\mathcal{S} \in [\![\phi(\mathcal{G}, \chi)]\!]$. Then, $\mathcal{S} \in [\![\phi(\{G\}, \chi)]\!]$. Then, by Defn. 13, $\mathcal{S} \in [\![\phi(G, \chi)]\!]$.
   – Recall $\checkmark_R(G)$ and $\mathcal{S} \in [\![\phi(G, \chi)]\!]$. Then, by 1:
   - **Case:** $G\downarrow$. Then, by [$\downarrow$2-Glob], $\{G\}\downarrow$. Then, $\mathcal{G}\downarrow$.
   - **Case:** $G \xrightarrow{\xi,\gamma} G'$ and $\mathcal{S} \in [\![\xi]\!]$, for some G', for some ξ, γ. Recall $G \xrightarrow{\xi,\gamma} G'$. Then, by [$\rightarrow$2-Glob], $\{G\} \xrightarrow{\xi,\gamma} \{G'\}$. Then, $\mathcal{G} \xrightarrow{\xi,\gamma} \{G'\}$. Then, $\mathcal{G}' = \{G'\}$ and $\mathcal{G} \xrightarrow{\xi,\gamma} \mathcal{G}'$, for some $\mathcal{G}'$. □

3. Recall $\checkmark_R(\mathcal{G})$ and $\mathcal{S} \in [\![\phi(\mathcal{G}, \chi)]\!]$. Then, by 2:
   – **Case:** $\mathcal{G}\downarrow$. Then, by [$\downarrow$3], $(\mathcal{G}, \mathcal{S})\downarrow$.
   – **Case:** $\mathcal{G} \xrightarrow{\xi,\gamma} \mathcal{G}'$ and $\mathcal{S} \in [\![\xi]\!]$, for some ξ, γ, for some $\mathcal{G}'$.
   - By Defn. 19, $\dot{\gamma} = \mathsf{con}_{\mathcal{S}}(\gamma)$, for some $\dot{\gamma}$.
   - Recall $\mathcal{G} \xrightarrow{\xi,\gamma} \mathcal{G}'$ and $\mathcal{S} \in [\![\xi]\!]$ and $\dot{\gamma} = \mathsf{con}_{\mathcal{S}}(\gamma)$. Then, by [$\rightarrow$3], $(\mathcal{G}, \mathcal{S}) \xrightarrow{\dot{\gamma}} (\mathcal{G}', \mathsf{effect}(\dot{\gamma}, \mathcal{S}))$. Then, $\mathcal{S}' = \mathsf{effect}(\dot{\gamma}, \mathcal{S})$ and $(\{G\}, \mathcal{S}) \xrightarrow{\dot{\gamma}} (\{G'\}, \mathcal{S}')$, for some $\mathcal{S}'$. □

Proof (of Thm. 1). Recall $\checkmark_R(\mathcal{G})$ and $\mathcal{S} \in [\![\phi(\mathcal{G}, \chi)]\!]$ and $(\mathcal{G}, \mathcal{S}) \xrightarrow{\dot{\Gamma}}{}^* (\mathcal{G}^\dagger, \mathcal{S}^\dagger)$. Then, by Lem. 92, $\checkmark_R(\mathcal{G}^\dagger)$ and $\mathcal{S}^\dagger \in [\![\phi(\mathcal{G}^\dagger, \chi)]\!]$. Then, by Lem. 93, $(\mathcal{G}^\dagger, \mathcal{S}^\dagger)\downarrow$, or $(\mathcal{G}^\dagger, \mathcal{S}^\dagger) \xrightarrow{\dot{\gamma}} (\mathcal{G}^\ddagger, \mathcal{S}^\ddagger)$, for some $\mathcal{G}^\ddagger, \mathcal{S}^\ddagger, \dot{\gamma}$. □

Proof (of Thm. 2).

– Recall $\checkmark_R(\mathcal{G})$ and $\mathcal{S} \in [\![\phi(\mathcal{G}, \chi)]\!]$ and $(\mathcal{G}, \mathcal{S}) \xrightarrow{\dot{\Gamma}}{}^* (\mathcal{G}^\dagger, \mathcal{S}^\dagger)$. Then, by Lem. 92, $\checkmark_R(\mathcal{G}^\dagger)$ and $\mathcal{S}^\dagger \in [\![\phi(\mathcal{G}^\dagger, \chi)]\!]$.
– Recall $\checkmark_R(\mathcal{G}^\dagger)$ and $\mathcal{S}^\dagger \in [\![\phi(\mathcal{G}^\dagger, \chi)]\!]$ and $(\mathcal{G}^\dagger, \mathcal{S}^\dagger)\downarrow$. Then, by Lem. 88, $\mathcal{S}^\dagger \in [\![\chi]\!]$. □

I Theorem: Operational Equivalence

Proof (of Lem. 94). Recall $L_1 \Rightarrow\!\Leftarrow L_2$. Then, by Defn. 40, $L_1 \Longrightarrow L^\dagger \Longleftarrow L_2$, for some $L^\dagger$. Then, $L_2 \Longrightarrow L^\dagger \Longleftarrow L_1$, for some $L^\dagger$. Then, by [$\Rightarrow\!\Leftarrow$], $L_2 \Rightarrow\!\Leftarrow L_1$. □

Proof (of Lem. 95).

1. – Recall $L_1 \Rightarrow\!\Leftarrow L_2$. Then, by Defn. 40, $L_1 \Longrightarrow L^\dagger$ and $L^\dagger \Longleftarrow L_2$, for some $L^\dagger$.
   – Recall $L^\dagger \Longleftarrow L_2\downarrow$. Then, by Lem. 70, $L^\dagger\downarrow$. □
2. – Recall $L_1 \Rightarrow\!\Leftarrow L_2$. Then, by Defn. 40, $L_1 \Longrightarrow L^\dagger \Longleftarrow L_2$, for some $L^\dagger$.
   – Recall $L_1 \Longrightarrow L^\dagger \Longleftarrow L_2 \xrightarrow{\texttt{true},\tau} L_2'$. Then, by Lem. 70, $L_1 \Longrightarrow L^\ddagger \Longleftarrow L_2'$, for some $L^\ddagger$. Then, by [$\Rightarrow\!\Leftarrow$], $L_1 \Rightarrow\!\Leftarrow L_2'$. □
3. – Recall $L_1 \Rightarrow\!\Leftarrow L_2$. Then, by Defn. 40, $L_1 \Longrightarrow L^\dagger \Longleftarrow L_2$, for some $L^\dagger$.
   – Recall $L_1 \Longrightarrow L^\dagger \Longleftarrow L_2 \xrightarrow{\xi,\lambda} L_2'$. Then, by Lem. 70, $L_1 \xrightarrow{\xi,\lambda} L_1'$ and $L_1' \Longrightarrow L^\ddagger \Longleftarrow L_2'$, for some $L^\ddagger$, for some L_1'.
   – Recall $L_1' \Longrightarrow L^\ddagger \Longleftarrow L_2'$. Then, by [$\Rightarrow\!\Leftarrow$], $L_1' \Rightarrow\!\Leftarrow L_2'$. □

Proof (of Lem. 96).

– Recall $\checkmark_R(\mathcal{G})$. Then, by Defn. 27, $\mathcal{G} = \{G\}$ and $\checkmark_R(G)$, for some G.
– Recall $\mathcal{S} \in [\![\phi(\mathcal{G},\texttt{true})]\!]$. Then, $\mathcal{S} \in [\![\phi(\{G\},\texttt{true})]\!]$. Then, by Defn. 38, $\mathcal{S} \in [\![\phi(G,\texttt{true})]\!]$.
– By [$\to^*$1-Base], $(G \restriction r) \xrightarrow{\{(\texttt{true},\tau)\}}{}^* (G \restriction r)$ for every $r \in R$. Then, by [$\Rightarrow$1], $(G \restriction r) \Longrightarrow (G \restriction r)$ for every $r \in R$. Then, by [$\Rightarrow\!\Leftarrow$], $(G \restriction r) \Rightarrow\!\Leftarrow (G \restriction r)$ for every $r \in R$.
– Recall $\checkmark_R(G)$ and $\mathcal{S} \in [\![\phi(G,\texttt{true})]\!]$, and $(G \restriction r) \Rightarrow\!\Leftarrow (G \restriction r)$ for every $r \in R$. Then, by [$\bowtie$], $(\{G\},\mathcal{S}) \bowtie (\{G \restriction r\}_{r\in R},\mathcal{S})$. Then, by Defn. 26, $(\{G\},\mathcal{S}) \bowtie (\{G\} \restriction R,\mathcal{S})$. Then, $(\mathcal{G},\mathcal{S}) \bowtie (\mathcal{G} \restriction R,\mathcal{S})$. □

Proof (of Lem. 97).

1. – Recall $(\mathcal{G},\mathcal{S}) \bowtie (\mathcal{L},\mathcal{S})$. Then, by Defn. 41, $\mathcal{G} = \{G\}$ and $\mathcal{L} = \{L_r\}_{r\in R}$ and $\checkmark_R(G)$, and $(G \restriction r) \Rightarrow\!\Leftarrow L_r$ for every $r \in R$, for some $G, \{L_r\}_{r\in R}$.
   – Recall $(\mathcal{G},\mathcal{S})\downarrow$. Then, $(\{G\},\mathcal{S})\downarrow$. Then, by Defn. 30, $\{G\}\downarrow$. Then, by Defn. 29, $G\downarrow$ and $\checkmark_R(G)$. Then, by Lem. 49, $(G \restriction r)\downarrow$ for every $r \in R$.
   – Recall $(G \restriction r) \Rightarrow\!\Leftarrow L_r$ for every $r \in R$. Then, by Lem. 94, $L_r \Rightarrow\!\Leftarrow (G \restriction r)$ for every $r \in R$.
   – Recall $L_r \Rightarrow\!\Leftarrow (G \restriction r)\downarrow$ for every $r \in R$. Then, by Lem. 95, $L_r \Longrightarrow L_r^\dagger\downarrow$ for every $r \in R$, for some $\{L_r^\dagger\}_{r\in R}$.
   – Recall $L_r \Longrightarrow L_r^\dagger$ for every $r \in R$. Then, by Lem. 72, $\{L_r\}_{r\in R} \Longrightarrow \{L_r^\dagger\}_{r\in R}$. Then, by Lem. 74, $(\{L_r\}_{r\in R},\mathcal{S}) \Longrightarrow (\{L_r^\dagger\}_{r\in R},\mathcal{S})$. Then, $(\mathcal{L},\mathcal{S}) \Longrightarrow (\{L_r^\dagger\}_{r\in R},\mathcal{S})$. Then, $\mathcal{L}^\dagger = \{L_r^\dagger\}_{r\in R}$ and $\mathcal{S}^\dagger = \mathcal{S}$ and $(\mathcal{L},\mathcal{S}) \Longrightarrow (\mathcal{L}^\dagger,\mathcal{S}^\dagger)$, for some $\mathcal{L}^\dagger,\mathcal{S}^\dagger$.
   – Recall $L_r^\dagger\downarrow$ for every $r \in R$. Then, by [$\downarrow$2-Locs], $\{L_r^\dagger\}_{r\in R}\downarrow$. Then, by [$\downarrow$3], $(\{L_r^\dagger\}_{r\in R},\mathcal{S})\downarrow$. Then, $(\mathcal{L}^\dagger,\mathcal{S}^\dagger)\downarrow$. □
2. – Recall $(\mathcal{G},\mathcal{S}) \bowtie (\mathcal{L},\mathcal{S})$. Then, by Defn. 41, $\mathcal{G} = \{G\}$ and $\checkmark_R(G)$, for some G.
   – Recall $(\mathcal{G},\mathcal{S}) \xrightarrow{\tau} (\mathcal{G}',\mathcal{S}')$. Then, $(\{G\},\mathcal{S}) \xrightarrow{\tau} (\mathcal{G}',\mathcal{S}')$. Then, by Defn. 33, $\{G\} \xrightarrow{\xi,\gamma} \mathcal{G}'$ and $\tau = \mathsf{con}_{\mathcal{S}}(\gamma)$, for some ξ,γ.
   – Recall $\{G\} \xrightarrow{\xi,\gamma} \mathcal{G}'$. Then, by Defn. 32, $\mathcal{G}' = \{G'\}$ and $G \xrightarrow{\xi,\gamma} G'$, for some G'.
   – Recall $G \xrightarrow{\xi,\gamma} G'$ and $\checkmark_R(G)$. Then, by Lem. 58, $\gamma \neq \tau$.
   – Recall $\tau = \mathsf{con}_{\mathcal{S}}(\gamma)$. Then, by Defn. 19, $\gamma = \tau$. Then, **false**. □
3. – Recall $(\mathcal{G},\mathcal{S}) \bowtie (\mathcal{L},\mathcal{S})$. Then, by Defn. 41, $\mathcal{G} = \{G\}$ and $\mathcal{L} = \{L_r\}_{r\in R}$ and $\checkmark_R(G)$ and $\mathcal{S} \in [\![\phi(G,\texttt{true})]\!]$, and $(G \restriction r) \Rightarrow\!\Leftarrow L_r$ for every $r \in R$, for some $G, \{L_r\}_{r\in R}$.
   – Recall $(\mathcal{G},\mathcal{S}) \xrightarrow{\dot\gamma} (\mathcal{G}',\mathcal{S}')$. Then, $(\{G\},\mathcal{S}) \xrightarrow{\dot\gamma} (\mathcal{G}',\mathcal{S}')$. Then, by Defn. 33, $\mathcal{S}' = \mathsf{effect}(\dot\gamma,\mathcal{S})$ and $\{G\} \xrightarrow{\xi,\gamma} \mathcal{G}'$ and $\mathcal{S} \in [\![\xi]\!]$ and $\dot\gamma = \mathsf{con}_{\mathcal{S}}(\gamma)$, for some ξ,γ.
   – Recall $\{G\} \xrightarrow{\xi,\gamma} \mathcal{G}'$. Then, by Defn. 32, $\mathcal{G}' = \{G'\}$ and $G \xrightarrow{\xi,\gamma} G'$, for some G'.
   – Recall $\mathcal{S} \in [\![\phi(G,\texttt{true})]\!]$ and $\mathcal{S} \in [\![\xi]\!]$. Then, $\mathcal{S} \in [\![\phi(G,\texttt{true})]\!] \cap [\![\xi]\!]$. Then, by Defn. 13, $\mathcal{S} \in [\![\phi(G,\texttt{true}) \wedge \xi]\!]$.
   – Recall $\mathcal{S} \in [\![\phi(G,\texttt{true}) \wedge \xi]\!]$. Then, $[\![\phi(G,\texttt{true}) \wedge \xi]\!] \neq \emptyset$.

– Recall $\checkmark_R(G)$ and $[\![\phi(G, \mathtt{true}) \wedge \xi]\!] \neq \emptyset$ and $G \xrightarrow{\xi,\gamma} G'$. Then, by Lem. 89, $\checkmark_R(G')$.
– Recall $\checkmark_R(G)$ and $G \xrightarrow{\xi,\gamma} G'$. Then, by Lem. 91, $[\![\phi(G, \mathtt{true}) \wedge \xi]\!] \subseteq [\![\phi(\gamma, \phi(G', \mathtt{true}))]\!]$.
– Recall $\mathcal{S} \in [\![\phi(G, \mathtt{true}) \wedge \xi]\!] \subseteq [\![\phi(\gamma, \phi(G', \mathtt{true}))]\!]$. Then, $\mathcal{S} \in [\![\phi(\gamma, \phi(G', \mathtt{true}))]\!]$. Then, by Lem. 79, $\mathsf{effect}(\mathsf{con}_{\mathcal{S}}(\gamma), \mathcal{S}) \in [\![\phi(G', \mathtt{true})]\!]$. Then, $\mathsf{effect}(\dot{\gamma}, \mathcal{S}) \in [\![\phi(G', \mathtt{true})]\!]$.
– Recall $G \xrightarrow{\xi,\gamma} G'$. Then, by Lem. 53, $\mathsf{subj}(\xi) \cup \mathsf{subj}(\gamma) \cup \mathsf{subj}(G') \subseteq \mathsf{subj}(G)$. Then, $\mathsf{subj}(\gamma) \subseteq \mathsf{subj}(G)$.
– Recall $\checkmark_R(G)$. Then, by Lem. 47, $\mathsf{subj}(G) \subseteq R$.
– Recall $\mathsf{subj}(\gamma) \subseteq \mathsf{subj}(G) \subseteq R$. Then, $\mathsf{subj}(\gamma) \subseteq R$.
– Recall $\mathsf{subj}(\gamma) \subseteq R$. Then, $\mathsf{subj}(\gamma) = R \cap \mathsf{subj}(\gamma)$ and $R = (R \setminus \mathsf{subj}(\gamma)) \cup \mathsf{subj}(\gamma)$.
– Recall $(G \upharpoonright r) \Rrightarrow\!\Lleftarrow L_r$ for every $r \in R$. Then, $(G \upharpoonright r) \Rrightarrow\!\Lleftarrow L_r$ for every $r \in R \cap \mathsf{subj}(\gamma)$. Then, $(G \upharpoonright r) \Rrightarrow\!\Lleftarrow L_r$ for every $r \in \mathsf{subj}(\gamma)$. Then, by Lem. 94, $L_r \Rrightarrow\!\Lleftarrow (G \upharpoonright r)$ for every $r \in R \cap \mathsf{subj}(\gamma)$.
– Recall $G \xrightarrow{\xi,\gamma} G'$ and $\checkmark_R(G)$. Then, by Lem. 59, $(G \upharpoonright r) \xrightarrow{\xi \upharpoonright r, \gamma \upharpoonright r} (G' \upharpoonright r)$ for every $r \in R \cap \mathsf{subj}(\gamma)$. Then, $(G \upharpoonright r) \xrightarrow{\xi \upharpoonright r, \gamma \upharpoonright r} (G' \upharpoonright r)$ for every $r \in \mathsf{subj}(\gamma)$.
– Recall $L_r \Rrightarrow\!\Lleftarrow (G \upharpoonright r) \xrightarrow{\xi \upharpoonright r, \gamma \upharpoonright r} (G' \upharpoonright r)$ for every $r \in \mathsf{subj}(\gamma)$. Then, by Lem. 95, $L_r \xrightarrow{\xi \upharpoonright r, \gamma \upharpoonright r} L'_r \Rrightarrow\!\Lleftarrow (G' \upharpoonright r)$ for every $r \in \mathsf{subj}(\gamma)$, for some $\{L'_r\}_{r \in \mathsf{subj}(\gamma)}$.
– Recall $(G \upharpoonright r) \Rrightarrow\!\Lleftarrow L_r$ for every $r \in R$. Then, $(G \upharpoonright r) \Rrightarrow\!\Lleftarrow L_r$ for every $r \in R \setminus \mathsf{subj}(\gamma)$. Then, by Lem. 94, $L_r \Rrightarrow\!\Lleftarrow (G \upharpoonright r)$ for every $r \in R \setminus \mathsf{subj}(\gamma)$.
– Recall $G \xrightarrow{\xi,\gamma} G'$ and $\checkmark_R(G)$. Then, by Lem. 59, $(G \upharpoonright r) \xrightarrow{\mathtt{true}, \tau} (G' \upharpoonright r)$ for every $r \in R \setminus \mathsf{subj}(\gamma)$.
– Recall $L_r \Rrightarrow\!\Lleftarrow (G \upharpoonright r) \xrightarrow{\mathtt{true}, \tau} (G' \upharpoonright r)$ for every $r \in R \setminus \mathsf{subj}(\gamma)$. Then, by Lem. 95, $L_r \Rrightarrow\!\Lleftarrow (G' \upharpoonright r)$ for every $r \in R \setminus \mathsf{subj}(\gamma)$. Then, $L'_r = L_r$ and $L'_r \Rrightarrow\!\Lleftarrow (G' \upharpoonright r)$, for every $r \in R \setminus \mathsf{subj}(\gamma)$, for some $\{L'_r\}_{r \in R \setminus \mathsf{subj}(\gamma)}$.
– Recall $L'_r \Rrightarrow\!\Lleftarrow (G' \upharpoonright r)$ for every $r \in \mathsf{subj}(\gamma)$, and $L'_r \Rrightarrow\!\Lleftarrow (G' \upharpoonright r)$ for every $r \in R \setminus \mathsf{subj}(\gamma)$. Then, $L'_r \Rrightarrow\!\Lleftarrow (G' \upharpoonright r)$ for every $r \in (R \setminus \mathsf{subj}(\gamma)) \cup \mathsf{subj}(\gamma)$. Then, $L'_r \Rrightarrow\!\Lleftarrow (G' \upharpoonright r)$ for every $r \in R$. Then, by Lem. 94, $(G' \upharpoonright r) \Rrightarrow\!\Lleftarrow L'_r$ for every $r \in R$.
– Recall $\checkmark_R(G')$ and $\mathsf{effect}(\dot{\gamma}, \mathcal{S}) \in [\![\phi(G', \mathtt{true})]\!]$, and $(G' \upharpoonright r) \Rrightarrow\!\Lleftarrow L'_r$ for every $r \in R$. Then, by [$\bowtie$], $(\{G'\}, \mathsf{effect}(\dot{\gamma}, \mathcal{S})) \bowtie (\{L'_r\}_{r \in R}, \mathsf{effect}(\dot{\gamma}, \mathcal{S}))$. Then, $(\mathcal{G}', \mathcal{S}') \bowtie (\{L'_r\}_{r \in R}, \mathsf{effect}(\dot{\gamma}, \mathcal{S}))$. Then, $\mathcal{L}^{¶} = \{L'_r\}_{r \in R}$ and $\mathcal{S}^{¶} = \mathsf{effect}(\dot{\gamma}, \mathcal{S})$ and $(\mathcal{G}', \mathcal{S}') \bowtie (\mathcal{L}^{¶}, \mathcal{S}^{¶})$, for some $\mathcal{L}^{¶}, \mathcal{S}^{¶}$.
– Recall $G \xrightarrow{\xi,\gamma} G'$ and $\checkmark_R(G)$. Then, by Lem. 58, $\gamma \neq \tau$.
– Recall $L_r \xrightarrow{\xi \upharpoonright r, \gamma \upharpoonright r} L'_r$ for every $r \in \mathsf{subj}(\gamma) \subseteq R$, and $L_r = L'_r$ for every $r \in R \setminus \mathsf{subj}(\gamma)$, and $\gamma \neq \tau$. Then, by Lem. 68, $\{L_r\}_{r \in R} \xrightarrow{\bigwedge\{\xi \upharpoonright r\}_{r \in \mathsf{subj}(\gamma)}, \gamma} \{L'_r\}_{r \in R}$.
– Recall $G \xrightarrow{\xi,\gamma} G'$. Then, by Lem. 53, $\mathsf{subj}(\xi) \subseteq \mathsf{subj}(\gamma)$. Then, by Lem. 26, $[\![\xi]\!] = [\![\bigwedge\{\xi \upharpoonright r\}_{r \in \mathsf{subj}(\gamma)}]\!]$.
– Recall $\mathcal{S} \in [\![\xi]\!]$. Then, $\mathcal{S} \in [\![\bigwedge\{\xi \upharpoonright r\}_{r \in \mathsf{subj}(\gamma)}]\!]$.
– Recall $\{L_r\}_{r \in R} \xrightarrow{\bigwedge\{\xi \upharpoonright r\}_{r \in \mathsf{subj}(\gamma)}, \gamma} \{L'_r\}_{r \in R}$ and $\mathcal{S} \in [\![\bigwedge\{\xi \upharpoonright r\}_{r \in \mathsf{subj}(\gamma)}]\!]$ and $\dot{\gamma} = \mathsf{con}_{\mathcal{S}}(\gamma)$. Then, by [$\rightarrow$3], $(\{L_r\}_{r \in R}, \mathcal{S}) \xrightarrow{\dot{\gamma}} (\{L'_r\}_{r \in R}, \mathsf{effect}(\dot{\gamma}, \mathcal{S}))$. Then $\mathcal{L}^{\dagger} = \{L_r\}_{r \in R}$ and $\mathcal{L}^{\ddagger} = \{L'_r\}_{r \in R}$ and $\mathcal{S}^{\dagger} = \mathcal{S}$ and $\mathcal{S}^{\ddagger} = \mathsf{effect}(\dot{\gamma}, \mathcal{S})$ and $(\mathcal{L}^{\dagger}, \mathcal{S}^{\dagger}) \xrightarrow{\dot{\gamma}} (\mathcal{L}^{\ddagger}, \mathcal{S}^{\ddagger})$, for some $\mathcal{L}^{\dagger}, \mathcal{L}^{\ddagger}, \mathcal{S}^{\dagger}, \mathcal{S}^{\ddagger}$.
– By [$\rightarrow^*$3-Base], $(\{L_r\}_{r \in R}, \mathcal{S}) \xrightarrow{\{\tau\}}{}^* (\{L_r\}_{r \in R}, \mathcal{S})$ and $(\{L'_r\}_{r \in R}, \mathsf{effect}(\dot{\gamma}, \mathcal{S})) \xrightarrow{\{\tau\}}{}^* (\{L'_r\}_{r \in R}, \mathsf{effect}(\dot{\gamma}, \mathcal{S}))$. Then, by [$\Rightarrow$3], $(\{L_r\}_{r \in R}, \mathcal{S}) \Longrightarrow (\{L_r\}_{r \in R}, \mathcal{S})$ and $(\{L'_r\}_{r \in R}, \mathsf{effect}(\dot{\gamma}, \mathcal{S})) \Longrightarrow (\{L'_r\}_{r \in R}, \mathsf{effect}(\dot{\gamma}, \mathcal{S}))$. Then, $(\mathcal{L}, \mathcal{S}) \Longrightarrow (\mathcal{L}^{\dagger}, \mathcal{S}^{\dagger})$ and $(\mathcal{L}^{\ddagger}, \mathcal{S}^{\ddagger}) \Longrightarrow (\mathcal{L}^{¶}, \mathcal{S}^{¶})$. □

4. – Recall $(\mathcal{G}, \mathcal{S}) \bowtie (\mathcal{L}, \mathcal{S})$. Then, by Defn. 41, $\mathcal{G} = \{G\}$ and $\mathcal{L} = \{L_r\}_{r \in R}$ and $\checkmark_R(G)$, and $(G \upharpoonright r) \Rrightarrow\!\Lleftarrow L_r$ for every $r \in R$, for some $G, \{L_r\}_{r \in R}$.
   – Recall $\checkmark_R(G)$. Then, by Lem. 57:
     - **Case:** $G\!\downarrow$.
       * By [$\rightarrow^*$3-Base], $(\{G\}, \mathcal{S}) \xrightarrow{\{\tau\}}{}^* (\{G\}, \mathcal{S})$. Then, by [$\Rightarrow$3], $(\{G\}, \mathcal{S}) \Longrightarrow (\{G\}, \mathcal{S})$. Then, $(\mathcal{G}, \mathcal{S}) \Longrightarrow (\{G\}, \mathcal{S})$. Then, $\mathcal{G}^{\dagger} = \{G\}$ and $\mathcal{S}^{\dagger} = \mathcal{S}$ and $(\mathcal{G}, \mathcal{S}) \Longrightarrow (\mathcal{G}^{\dagger}, \mathcal{S}^{\dagger})$, for some $\mathcal{G}^{\dagger}, \mathcal{S}^{\dagger}$.
       * Recall $G\!\downarrow$. Then, by [$\downarrow$2-Glob], $\{G\}\!\downarrow$. Then, by [$\downarrow$3], $(\{G\}, \mathcal{S})\!\downarrow$. Then, $(\mathcal{G}^{\dagger}, \mathcal{S}^{\dagger})\!\downarrow$.
     - **Case:** $G \xrightarrow{\xi,\gamma} G'$, for some G', ξ, γ.

* Recall $G \xrightarrow{\xi,\gamma} G'$ and $\checkmark_R(G)$. Then, by Lem. 58, $\mathsf{subj}(\gamma) \neq \emptyset$. Then, $\hat{r} \in \mathsf{subj}(\gamma)$, for some $\hat{r}$.
* Recall $G \xrightarrow{\xi,\gamma} G'$. Then, by Lem. 53, $\mathsf{subj}(\xi) \cup \mathsf{subj}(\gamma) \cup \mathsf{subj}(G') \subseteq \mathsf{subj}(G)$. Then, $\mathsf{subj}(\gamma) \subseteq \mathsf{subj}(G)$.
* Recall $\checkmark_R(G)$. Then, by Lem. 47, $\mathsf{subj}(G) \subseteq R$.
* Recall $\hat{r} \in \mathsf{subj}(\gamma) \subseteq \mathsf{subj}(G) \subseteq R$. Then, $\hat{r} \in R$.
* Recall $(\mathcal{L}, \mathcal{S}){\downarrow}$. Then, $(\{L_r\}_{r \in R}, \mathcal{S}){\downarrow}$ Then, by Defn. 30, $\{L_r\}_{r\in R}{\downarrow}$. Then, by Defn. 29, $L_r{\downarrow}$ for every $r \in R$.
* Recall $\hat{r} \in R$. Then, $L_{\hat{r}}{\downarrow}$.
* Recall $\hat{r} \in R$. Then, $(G \upharpoonright \hat{r}) \Rrightarrow\!\Lleftarrow L_{\hat{r}}$. Then, by Lem. 94, $L_{\hat{r}} \Rrightarrow\!\Lleftarrow (G \upharpoonright \hat{r})$.
* Recall $\hat{r} \in R$ and $\hat{r} \in \mathsf{subj}(\gamma)$. Then, $\hat{r} \in R \cap \mathsf{subj}(\gamma)$.
* Recall $G \xrightarrow{\xi,\gamma} G'$ and $\checkmark_R(G)$ and $\hat{r} \in R \cap \mathsf{subj}(\gamma)$. Then, by Lem. 59, $(G \upharpoonright \hat{r}) \xrightarrow{\xi\upharpoonright\hat{r},\gamma\upharpoonright\hat{r}} (G' \upharpoonright \hat{r})$.
* Recall $L_{\hat{r}} \Rrightarrow\!\Lleftarrow (G \upharpoonright \hat{r}) \xrightarrow{\xi\upharpoonright\hat{r},\gamma\upharpoonright\hat{r}} (G' \upharpoonright \hat{r})$. Then, by Lem. 95, $L_{\hat{r}} \xrightarrow{\xi\upharpoonright\hat{r},\gamma\upharpoonright\hat{r}} L'_{\hat{r}}$, for some $L'_{\hat{r}}$.
* Recall $L_{\hat{r}}{\downarrow}$ and $L_{\hat{r}} \xrightarrow{\xi\upharpoonright\hat{r},\gamma\upharpoonright\hat{r}} L'_{\hat{r}}$. Then, by Lem. 55, $\gamma \upharpoonright \hat{r} = \tau$.
* Recall $\hat{r} \in \mathsf{subj}(\gamma)$. Then, by Lem. 33, $\mathsf{subj}(\gamma \upharpoonright \hat{r}) = \{\hat{r}\}$. Then, $\mathsf{subj}(\tau) = \{\hat{r}\}$. Then, by Defn. 16, $\emptyset = \{\hat{r}\}$. Then, **false**. □

5. – Recall $(\mathcal{G}, \mathcal{S}) \bowtie (\mathcal{L}, \mathcal{S})$. Then, by Defn. 41, $\mathcal{G} = \{G\}$ and $\mathcal{L} = \{L_r\}_{r\in R}$ and $\checkmark_R(G)$ and $\mathcal{S} \in [\![\phi(G, \mathtt{true})]\!]$, and $(G \upharpoonright r) \Rrightarrow\!\Lleftarrow L_r$ for every $r \in R$, for some $G, \{L_r\}_{r\in R}$.
   – Recall $(\mathcal{L}, \mathcal{S}) \xrightarrow{\tau} (\mathcal{L}', \mathcal{S}')$. Then, $(\{L_r\}_{r\in R}, \mathcal{S}) \xrightarrow{\tau} (\mathcal{L}', \mathcal{S}')$. Then, by Defn. 33, $\{L_r\}_{r\in R} \xrightarrow{\xi,\gamma} \mathcal{L}'$ and $\tau = \mathsf{con}_{\mathcal{S}}(\gamma)$, for some ξ, γ.
   – Recall $\tau = \mathsf{con}_{\mathcal{S}}(\gamma)$. Then, by Defn. 19, $\gamma = \tau$.
   – Recall $\{L_r\}_{r\in R} \xrightarrow{\xi,\gamma} \mathcal{L}'$. Then, $\{L_r\}_{r\in R} \xrightarrow{\xi,\tau} \mathcal{L}'$. Then, by Defn. 32, $\mathcal{L}' = \{L'_r\}_{r\in R}$ and $\{L_r\}_{r\in R} \xrightarrow{\xi,\tau} \{L'_r\}_{r\in R}$, for some $\{L'_r\}_{r\in R}$.
   – Recall $\{L_r\}_{r\in R} \xrightarrow{\xi,\tau} \{L'_r\}_{r\in R}$. Then, by Lem. 67, $L_{\hat{r}} \xrightarrow{\xi,\tau} L'_{\hat{r}}$ and $\hat{r} \in R$, and $L_r = L'_r$ for every $r \in R \setminus \{\hat{r}\}$, for some $\hat{r}$.
   – Recall $\hat{r} \in R$. Then, $(G \upharpoonright \hat{r}) \Rrightarrow\!\Lleftarrow L_{\hat{r}}$.
   – Recall $L_{\hat{r}} \xrightarrow{\xi,\tau} L'_{\hat{r}}$ and $\gamma = \tau$. Then, by Lem. 50, $\xi = \mathtt{true}$.
   – Recall $L_{\hat{r}} \xrightarrow{\xi,\tau} L'_{\hat{r}}$. Then, $L_{\hat{r}} \xrightarrow{\mathtt{true},\tau} L'_{\hat{r}}$.
   – Recall $(G \upharpoonright \hat{r}) \Rrightarrow\!\Lleftarrow L_{\hat{r}} \xrightarrow{\mathtt{true},\tau} L'_{\hat{r}}$. Then, by Lem. 95, $(G \upharpoonright \hat{r}) \Rrightarrow\!\Lleftarrow L'_{\hat{r}}$.
   – Recall $(G \upharpoonright r) \Rrightarrow\!\Lleftarrow L_r$ for every $r \in R$. Then, $(G \upharpoonright r) \Rrightarrow\!\Lleftarrow L_r$ for every $r \in R \setminus \{\hat{r}\}$. Then, $(G \upharpoonright r) \Rrightarrow\!\Lleftarrow L'_r$ for every $r \in R \setminus \{\hat{r}\}$.
   – Recall $\hat{r} \in R$. Then, $R = (R \setminus \{\hat{r}\}) \cup \{\hat{r}\}$.
   – Recall $(G \upharpoonright \hat{r}) \Rrightarrow\!\Lleftarrow L'_{\hat{r}}$, and $(G \upharpoonright r) \Rrightarrow\!\Lleftarrow L'_r$ for every $r \in R \setminus \{\hat{r}\}$. Then, $(G \upharpoonright r) \Rrightarrow\!\Lleftarrow L'_r$ for every $r \in (R \setminus \{\hat{r}\}) \cup \{\hat{r}\}$. Then, $(G \upharpoonright r) \Rrightarrow\!\Lleftarrow L'_r$ for every $r \in R$.
   – Recall $\checkmark_R(G)$ and $\mathcal{S} \in [\![\phi(G, \mathtt{true})]\!]$, and $(G \upharpoonright r) \Rrightarrow\!\Lleftarrow L'_r$ for every $r \in R$. Then, by [⋈], $\{G\} \bowtie \{L'_r\}_{r\in R}$. Then, $\mathcal{G} \bowtie \mathcal{L}'$ □
6. – Recall $(\mathcal{G}, \mathcal{S}) \bowtie (\mathcal{L}, \mathcal{S})$. Then, by Defn. 41, $\mathcal{G} = \{G\}$ and $\mathcal{L} = \{L_r\}_{r\in R}$ and $\checkmark_R(G)$ and $\mathcal{S} \in [\![\phi(G, \mathtt{true})]\!]$, and $(G \upharpoonright r) \Rrightarrow\!\Lleftarrow L_r$ for every $r \in R$, for some $G, \{L_r\}_{r\in R}$.
   – Recall $(\mathcal{L}, \mathcal{S}) \xrightarrow{\dot{\gamma}} (\mathcal{L}', \mathcal{S}')$. Then, $(\{L_r\}_{r\in R}, \mathcal{S}) \xrightarrow{\dot{\gamma}} (\mathcal{L}', \mathcal{S}')$. Then, by Defn. 33, $\mathcal{S}' = \mathsf{effect}(\dot{\gamma}, \mathcal{S})$ and $\{L_r\}_{r\in R} \xrightarrow{\xi_1,\gamma} \mathcal{L}'$ and $\mathcal{S} \in [\![\xi_1]\!]$ and $\dot{\gamma} = \mathsf{con}_{\mathcal{S}}(\gamma)$, for some ξ_1, γ.
   – Recall $\{L_r\}_{r\in R} \xrightarrow{\xi_1,\gamma} \mathcal{L}'$. Then, by Defn. 32, $\mathcal{L}' = \{L'_{1r}\}_{r\in R}$ and $\{L_r\}_{r\in R} \xrightarrow{\xi_1,\gamma} \{L'_{1r}\}_{r\in R}$, for some $\{L'_{1r}\}_{r\in R}$.
   – Recall $\{L_r\}_{r\in R} \xrightarrow{\xi_1,\gamma} \{L'_{1r}\}_{r\in R}$. Then, by Lem. 66, $\mathsf{subj}(\gamma) \subseteq R$. Then, $\mathsf{subj}(\gamma) = R \cap \mathsf{subj}(\gamma)$ and $R = (R \setminus \mathsf{subj}(\gamma)) \cup \mathsf{subj}(\gamma)$.
   – Recall $(G \upharpoonright r) \Rrightarrow\!\Lleftarrow L_r$ for every $r \in R$. Then, $(G \upharpoonright r) \Rrightarrow\!\Lleftarrow L_r$ for every $r \in R \cap \mathsf{subj}(\gamma)$. Then, $(G \upharpoonright r) \Rrightarrow\!\Lleftarrow L_r$ for every $r \in \mathsf{subj}(\gamma)$.
   – Recall $\dot{\gamma} \neq \tau$. Then, $\mathsf{con}_{\mathcal{S}}(\gamma) \neq \tau$. Then, by Defn. 19, $\gamma \neq \tau$.
   – Recall $\{L_r\}_{r\in R} \xrightarrow{\xi_1,\gamma} \{L'_{1r}\}_{r\in R}$ and $\gamma \neq \tau$. Then, by Lem. 67, $L_r \xrightarrow{\xi_{1r},\gamma\upharpoonright r} L'_{1r}$ for every $r \in \mathsf{subj}(\gamma)$, and $\xi_1 = \bigwedge\{\xi_{1r}\}_{r\in\mathsf{subj}(\gamma)}$, and $L_r = L'_{1r}$ for every $r \in R \setminus \mathsf{subj}(\gamma)$, for some $\{\xi_{1r}\}_{r\in\mathsf{subj}(\gamma)}$.
   – Recall $(G \upharpoonright r) \Rrightarrow\!\Lleftarrow L_r \xrightarrow{\xi_{1r},\gamma\upharpoonright r} L'_{1r}$, for every $r \in \mathsf{subj}(\gamma)$. Then, by Lem. 95, $(G \upharpoonright r) \xrightarrow{\xi_{1r},\gamma\upharpoonright r} L'_{2r} \Rrightarrow\!\Lleftarrow L'_{1r}$, for every $r \in \mathsf{subj}(\gamma)$, for some $\{L'_{2r}\}_{r\in\mathsf{subj}(\gamma)}$.

- Recall $(G \upharpoonright r) \xrightarrow{\xi_{1r}, \gamma \upharpoonright r} L'_{2r}$ for every $r \in \mathsf{subj}(\gamma)$ and $\checkmark_R(G)$ and $\gamma \neq \tau$. Then, by Lem. 64, $G \xrightarrow{\xi_2, \gamma} G'$, and $L'_{2r} = G' \upharpoonright r$ for every $r \in \mathsf{subj}(\gamma)$, and $\xi_{1r} = \xi_2 \upharpoonright r$ for every $r \in \mathsf{subj}(\gamma)$, for some G', ξ_2.
- Recall $G \xrightarrow{\xi_2, \gamma} G'$. Then, by Lem. 53, $\mathsf{subj}(\xi_2) \subseteq \mathsf{subj}(\gamma)$. Then, by Lem. 26, $[\![\xi_2]\!] = [\![\bigwedge\{\xi_2 \upharpoonright r\}_{r \in \mathsf{subj}(\gamma)}]\!]$. Then, $[\![\xi_2]\!] = [\![\bigwedge\{\xi_{1r}\}_{r \in \mathsf{subj}(\gamma)}]\!]$. Then, $[\![\xi_2]\!] = [\![\xi_1]\!]$.
- Recall $\mathcal{S} \in [\![\xi_1]\!]$. Then, $\mathcal{S} \in [\![\xi_2]\!]$.
- Recall $\mathcal{S} \in [\![\phi(G, \mathtt{true})]\!]$ and $\mathcal{S} \in [\![\xi_2]\!]$. Then, $\mathcal{S} \in [\![\phi(G, \mathtt{true})]\!] \cap [\![\xi_2]\!]$. Then, by Defn. 13, $\mathcal{S} \in [\![\phi(G, \mathtt{true}) \wedge \xi_2]\!]$.
- Recall $\mathcal{S} \in [\![\phi(G, \mathtt{true}) \wedge \xi_2]\!]$. Then, $[\![\phi(G, \mathtt{true}) \wedge \xi_2]\!] \neq \emptyset$.
- Recall $\checkmark_R(G)$ and $[\![\phi(G, \mathtt{true}) \wedge \xi_2]\!] \neq \emptyset$ and $G \xrightarrow{\xi_2, \gamma} G'$. Then, by Lem. 89, $\checkmark_R(G')$.
- Recall $\checkmark_R(G)$ and $G \xrightarrow{\xi_2, \gamma} G'$. Then, by Lem. 91, $[\![\phi(G, \mathtt{true}) \wedge \xi_2]\!] \subseteq [\![\phi(\gamma, \phi(G', \mathtt{true}))]\!]$.
- Recall $\mathcal{S} \in [\![\phi(G, \mathtt{true}) \wedge \xi]\!] \subseteq [\![\phi(\gamma, \phi(G', \mathtt{true}))]\!]$. Then, $\mathcal{S} \in [\![\phi(\gamma, \phi(G', \mathtt{true}))]\!]$. Then, by Lem. 79, $\mathsf{effect}(\mathsf{con}_{\mathcal{S}}(\gamma), \mathcal{S}) \in [\![\phi(G', \mathtt{true})]\!]$. Then, $\mathsf{effect}(\dot{\gamma}, \mathcal{S}) \in [\![\phi(G', \mathtt{true})]\!]$.
- Recall $L'_{2r} \Rrightarrow\!\!\Lleftarrow L'_{1r}$ for every $r \in \mathsf{subj}(\gamma)$. Then, $(G' \upharpoonright r) \Rrightarrow\!\!\Lleftarrow L'_{1r}$ for every $r \in \mathsf{subj}(\gamma)$.
- Recall $(G \upharpoonright r) \Rrightarrow\!\!\Lleftarrow L_r$ for every $r \in R$. Then, $(G \upharpoonright r) \Rrightarrow\!\!\Lleftarrow L_r$ for every $r \in R \setminus \mathsf{subj}(\gamma)$. Then, by Lem. 94, $L_r \Rrightarrow\!\!\Lleftarrow (G \upharpoonright r)$ for every $r \in R \setminus \mathsf{subj}(\gamma)$.
- Recall $G \xrightarrow{\xi_2, \gamma} G'$ and $\checkmark_R(G)$. Then, by Lem. 59, $(G \upharpoonright r) \xrightarrow{\mathtt{true}, \tau} (G' \upharpoonright r)$ for every $r \in R \setminus \mathsf{subj}(\gamma)$.
- Recall $L_r \Rrightarrow\!\!\Lleftarrow (G \upharpoonright r) \xrightarrow{\mathtt{true}, \tau} (G' \upharpoonright r)$ for every $r \in R \setminus \mathsf{subj}(\gamma)$. Then, by Lem. 95, $L_r \Rrightarrow\!\!\Lleftarrow (G' \upharpoonright r)$ for every $r \in R \setminus \mathsf{subj}(\gamma)$. Then, by Lem. 94, $(G' \upharpoonright r) \Rrightarrow\!\!\Lleftarrow L_r$ for every $r \in R \setminus \mathsf{subj}(\gamma)$. Then, $(G' \upharpoonright r) \Rrightarrow\!\!\Lleftarrow L'_{1r}$ for every $r \in R \setminus \mathsf{subj}(\gamma)$.
- Recall $(G' \upharpoonright r) \Rrightarrow\!\!\Lleftarrow L'_{1r}$ for every $r \in \mathsf{subj}(\gamma)$, and $(G' \upharpoonright r) \Rrightarrow\!\!\Lleftarrow L'_{1r}$ for every $r \in R \setminus \mathsf{subj}(\gamma)$. Then, $(G' \upharpoonright r) \Rrightarrow\!\!\Lleftarrow L'_{1r}$ for every $r \in (R \setminus \mathsf{subj}(\gamma)) \cup \mathsf{subj}(\gamma)$. Then, $(G' \upharpoonright r) \Rrightarrow\!\!\Lleftarrow L'_{1r}$ for every $r \in R$.
- Recall $\checkmark_R(G')$ and $\mathsf{effect}(\dot{\gamma}, \mathcal{S}) \in [\![\phi(G', \mathtt{true})]\!]$, and $(G' \upharpoonright r) \Rrightarrow\!\!\Lleftarrow L'_{1r}$ for every $r \in R$. Then, by $[\bowtie]$, $(\{G'\}, \mathsf{effect}(\dot{\gamma}, \mathcal{S})) \bowtie (\{L'_{1r}\}_{r \in R}, \mathsf{effect}(\dot{\gamma}, \mathcal{S}))$. Then, $(\{G'\}, \mathsf{effect}(\dot{\gamma}, \mathcal{S})) \bowtie (\mathcal{L}', \mathcal{S}')$. Then, $\mathcal{G}^{\P} = \{G'\}$ and $\mathcal{S}^{\P} = \mathsf{effect}(\dot{\gamma}, \mathcal{S})$ and $(\mathcal{G}^{\P}, \mathcal{S}^{\P}) \bowtie (\mathcal{L}', \mathcal{S}')$, for some $\mathcal{G}^{\P}, \mathcal{S}^{\P}$.
- Recall $G \xrightarrow{\xi_2, \gamma} G'$. Then, by [$\rightarrow$2-Glob], $\{G\} \xrightarrow{\xi_2, \gamma} \{G'\}$.
- Recall $\{G\} \xrightarrow{\xi_2, \gamma} \{G'\}$ and $\mathcal{S} \in [\![\xi_2]\!]$ and $\dot{\gamma} = \mathsf{con}_{\mathcal{S}}(\gamma)$. Then, by [$\rightarrow$3], $(\{G\}, \mathcal{S}) \xrightarrow{\dot{\gamma}} (\{G'\}, \mathsf{effect}(\dot{\gamma}, \mathcal{S}))$. Then, $\mathcal{G}^{\dagger} = \{G\}$ and $\mathcal{G}^{\ddagger} = \{G'\}$ and $\mathcal{S}^{\dagger} = \mathcal{S}$ and $\mathcal{S}^{\ddagger} = \mathsf{effect}(\dot{\gamma}, \mathcal{S})$ and $(\mathcal{G}^{\dagger}, \mathcal{S}^{\dagger}) \xrightarrow{\dot{\gamma}} (\mathcal{G}^{\ddagger}, \mathcal{S}^{\ddagger})$, for some $\mathcal{G}^{\dagger}, \mathcal{G}^{\ddagger}, \mathcal{S}^{\dagger}, \mathcal{S}^{\ddagger}$.
- By [$\rightarrow^*$3-Base], $(\{G\}, \mathcal{S}) \xrightarrow{\{\tau\}}{}^{*} (\{G\}, \mathcal{S})$ and $(\{G'\}, \mathsf{effect}(\dot{\gamma}, \mathcal{S})) \xrightarrow{\{\tau\}}{}^{*} (\{G'\}, \mathsf{effect}(\dot{\gamma}, \mathcal{S}))$. Then, by [$\Rightarrow$3], $(\{G\}, \mathcal{S}) \Longrightarrow (\{G\}, \mathcal{S})$ and $(\{G'\}, \mathsf{effect}(\dot{\gamma}, \mathcal{S})) \Longrightarrow (\{G'\}, \mathsf{effect}(\dot{\gamma}, \mathcal{S}))$. Then, $(\mathcal{G}, \mathcal{S}) \Longrightarrow (\mathcal{G}^{\dagger}, \mathcal{S}^{\dagger})$ and $(\mathcal{G}^{\ddagger}, \mathcal{S}^{\ddagger}) \Longrightarrow (\mathcal{G}^{\P}, \mathcal{S}^{\P})$. □

Proof (of Thm. 3). Recall $\checkmark_R(\mathcal{G})$ and $\mathcal{S} \in [\![\phi(\mathcal{G}, \mathtt{true})]\!]$. Then, by Lem. 96, $(\mathcal{G}, \mathcal{S}) \bowtie (\mathcal{G} \upharpoonright R, \mathcal{S})$. Then, by Cor. 1, $(\mathcal{G}, \mathcal{S}) \approx (\mathcal{G} \upharpoonright R, \mathcal{S})$. □